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Approximation of a Linear Autonomous Differential Equation with Small Delay

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Abstract: A linear autonomous differential equation with small delay is considered in this paper. It is shown that under a smallness condition the delay differential equation is asymptotically equivalent to a linear ordinary differential equation with constant coefficients. The coefficient matrix of the ordinary differential equation is a solution of an associated matrix equation and it can be written as a limit of a sequence of matrices obtained by successive approximations. The eigenvalues of the approximating matrices converge exponentially to the dominant characteristic roots of the delay differential equation and an explicit estimate for the approximation error is given.

Keywords: delay differential equation; ordinary differential equation; asymptotic equivalence; approximation; eigenvalue

1. Introduction

Let \mathbb{C} and $\mathbb{C}^{n \times n}$ denote the set of complex numbers and the n -dimensional space of complex column vectors, respectively. Given a norm $\|\cdot\|$ on \mathbb{C}^n , the associated induced norm on $\mathbb{C}^{n \times n}$ will be denoted by the same symbol.

We will study the linear autonomous delay differential equation

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \quad (1)$$

where $\tau > 0$, $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ is a nonzero matrix. It is well-known that if $\phi : [-\tau, 0] \rightarrow \mathbb{C}^n$ is a continuous initial function, then Equation (1) has a unique solution $x : [-\tau, \infty) \rightarrow \mathbb{C}^n$ with initial values $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$ (see [1]). The characteristic equation of Equation (1) has the form

$$\det \Delta(\lambda) = 0, \quad \text{where } \Delta(\lambda) = \lambda I - A - Be^{-\lambda\tau}. \quad (2)$$

Throughout the paper, we will assume that

$$\|B\| \tau e^{1+\|A\|\tau} < 1, \quad (3)$$

which may be viewed as a smallness condition on the delay τ . We will show that if (3) holds, then Equation (1) is asymptotically equivalent to the ordinary differential equation

$$\dot{x} = Mx, \quad (4)$$

where $M \in \mathbb{C}^{n \times n}$ is the unique solution of the matrix equation

$$M = A + Be^{-M\tau} \quad (5)$$

such that

$$\|M\| < \mu_0, \quad \text{where } \mu_0 = -\tau^{-1} \ln(\|B\|\tau) > 0. \quad (6)$$

Furthermore, the coefficient matrix M in Equation (4) can be written as a limit of successive approximations

$$M = \lim_{k \rightarrow \infty} M_k, \quad (7)$$

where

$$M_0 = 0 \quad \text{and} \quad M_{k+1} = A + Be^{-M_k \tau} \quad \text{for } k = 0, 1, 2, \dots \quad (8)$$

The convergence in (7) is exponential and we give an estimate for the approximation error $\|M - M_k\|$. It will be shown that those characteristic roots of Equation (1) which lie in the half-plane $\operatorname{Re} \lambda > -\mu_0$ with μ_0 as in (6) coincide with the eigenvalues of matrix M . As a consequence, the above dominant characteristic roots of Equation (1) can be approximated by the eigenvalues of M_k . We give an explicit estimate for the approximation error which shows that the convergence of the eigenvalues of M_k to the dominant characteristic roots of Equation (1) is exponentially fast.

The investigation of differential equations with small delays has received much attention. Some results which are related to our study are discussed in the last section of the paper.

2. Main Results

In this section, we formulate and prove our main results which were indicated in the Introduction.

2.1. Solution of the Matrix Equation and Its Approximation

First we prove the existence and uniqueness of the solution of the matrix Equation (5) satisfying (6).

Theorem 1. *Suppose (3) holds. Then Equation (5) has a unique solution $M \in \mathbb{C}^{n \times n}$ such that (6) holds.*

Before we present the proof of Theorem 1, we establish some lemmas.

Lemma 1. *Let $P, Q \in \mathbb{C}^{n \times n}$ and $\gamma = \max\{\|P\|, \|Q\|\}$. Then*

$$\|P^k - Q^k\| \leq k\gamma^{k-1}\|P - Q\| \quad \text{for } k = 1, 2, \dots \quad (9)$$

Proof. We will prove by induction on k that

$$P^k - Q^k = \sum_{j=0}^{k-1} P^j (P - Q) Q^{k-1-j} \quad (10)$$

for $k = 1, 2, \dots$. Evidently, (10) holds for $k = 1$. Suppose for induction that (10) holds for some positive integer k . Then

$$\begin{aligned} P^{k+1} - Q^{k+1} &= P^k(P - Q) + (P^k - Q^k)Q \\ &= P^k(P - Q) + \left(\sum_{j=0}^{k-1} P^j (P - Q) Q^{k-1-j} \right) Q = \sum_{j=0}^k P^j (P - Q) Q^{k-j}. \end{aligned}$$

Thus, (10) holds for all k . From (10), we find that

$$\|P^k - Q^k\| \leq \sum_{j=0}^{k-1} \|P\|^j \|P - Q\| \|Q\|^{k-1-j} \leq \|P - Q\| \sum_{j=0}^{k-1} \gamma^j \gamma^{k-1-j} = k\gamma^{k-1} \|P - Q\|$$

for $k = 1, 2, \dots$. \square

Using Lemma 1, we can prove the following result about the distance of two matrix exponentials.

Lemma 2. Let $P, Q \in \mathbb{C}^{n \times n}$ and $\gamma = \max\{\|P\|, \|Q\|\}$. Then

$$\|e^P - e^Q\| \leq e^\gamma \|P - Q\|. \tag{11}$$

Proof. By the definition of the matrix exponential, we have

$$e^P - e^Q = \sum_{k=0}^{\infty} \frac{P^k}{k!} - \sum_{k=0}^{\infty} \frac{Q^k}{k!} = \sum_{k=1}^{\infty} \frac{P^k - Q^k}{k!}.$$

From this, by the application of Lemma 1, we find that

$$\|e^P - e^Q\| \leq \sum_{k=1}^{\infty} \frac{\|P^k - Q^k\|}{k!} \leq \|P - Q\| \sum_{k=1}^{\infty} \frac{k\gamma^{k-1}}{k!} = \|P - Q\| \sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{(k-1)!} = e^\gamma \|P - Q\|$$

which proves (11). \square

We will also need some properties of the scalar equation

$$\lambda = a + be^{\lambda\tau}. \tag{12}$$

Lemma 3. Let $a \in [0, \infty)$, $b, \tau \in (0, \infty)$ and suppose that

$$b\tau e^{1+a\tau} < 1. \tag{13}$$

If we let $\lambda_0 = -\tau^{-1} \ln(b\tau)$, then $\lambda_0 > 0$ and Equation (12) has a unique root $\lambda_1 \in (0, \lambda_0)$. Moreover,

$$a + be^{\lambda\tau} < \lambda \quad \text{for } \lambda \in (\lambda_1, \lambda_0] \tag{14}$$

and

$$b\tau e^{\lambda\tau} < 1 \quad \text{for } \lambda < \lambda_0. \tag{15}$$

Proof. By virtue of (13), we have $b\tau < e^{-1-a\tau} < 1$ which implies that $\ln(b\tau) < 0$ and hence $\lambda_0 > 0$. Define

$$f(\lambda) = \lambda - a - be^{\lambda\tau} \quad \text{for } \lambda \in \mathbb{R}.$$

We have

$$f'(\lambda) = 1 - b\tau e^{\lambda\tau} \quad \text{and} \quad f''(\lambda) = -b\tau^2 e^{\lambda\tau} \quad \text{for } \lambda \in \mathbb{R}.$$

It is easily seen that $f'(\lambda) = 0$ if and only if $\lambda = -\tau^{-1} \ln(b\tau) = \lambda_0$. Furthermore, (13) is equivalent to $f(\lambda_0) = -\tau^{-1} \ln(b\tau) - a - \tau^{-1} > 0$. Since $f''(\lambda) < 0$ for $\lambda \in \mathbb{R}$, f' strictly decreases on \mathbb{R} . In particular, $f'(\lambda) > f'(\lambda_0) = 0$ for $\lambda < \lambda_0$. Therefore, (15) holds and f strictly increases on $(-\infty, \lambda_0]$. This, together with $f(0) < 0$ and $f(\lambda_0) > 0$, implies that f and hence Equation (12) have a unique root $\lambda_1 \in (0, \lambda_0)$. Since f strictly increases on $[\lambda_1, \lambda_0]$, we have that $f(\lambda) > f(\lambda_1) = 0$ for $\lambda \in (\lambda_1, \lambda_0]$. Thus, (14) holds. \square

Now we can give a proof of Theorem 1.

Proof of Theorem 1. By Lemma 3, if (3) holds, then the equation

$$\mu = \|A\| + \|B\|e^{\mu\tau} \tag{16}$$

has a unique solution $\mu_1 \in (0, \mu_0)$, where μ_0 is given by (6). Moreover,

$$\|A\| + \|B\|e^{\mu\tau} < \mu \quad \text{for } \mu \in (\mu_1, \mu_0] \quad (17)$$

and

$$\|B\|\tau e^{\mu\tau} < 1 \quad \text{for } \mu < \mu_0. \quad (18)$$

Let $\mu \in [\mu_1, \mu_0)$ be fixed. Define

$$F(M) = A + Be^{-M\tau} \quad \text{for } M \in \mathbb{C}^{n \times n} \quad (19)$$

and

$$S = \{ M \in \mathbb{C}^{n \times n} \mid \|M\| \leq \mu \}. \quad (20)$$

Clearly, S is a nonempty and closed subset of $\mathbb{C}^{n \times n}$. By virtue of (17), we have for $M \in S$,

$$\|F(M)\| \leq \|A\| + \|B\|e^{\|M\|\tau} \leq \|A\| + \|B\|e^{\mu\tau} \leq \mu. \quad (21)$$

Thus, F maps S into itself. Let $M_1, M_2 \in S$. By the application of Lemma 2, we obtain

$$\|F(M_1) - F(M_2)\| = \|B(e^{-M_1\tau} - e^{-M_2\tau})\| \leq \|B\|\|e^{-M_1\tau} - e^{-M_2\tau}\| \leq \|B\|\tau e^{\mu\tau}\|M_1 - M_2\|.$$

In view of (18), $F : S \rightarrow S$ is a contraction and hence there exists a unique $M \in S$ such that $M = F(M)$. Since $\mu \in [\mu_1, \mu_0)$ was arbitrary, this completes the proof. \square

In the next theorem, we show that the unique solution of Equation (5) satisfying (6) can be written as a limit of successive approximations M_k defined by (8) and we give an estimate for the approximation error.

Theorem 2. Suppose (3) holds and let $M \in \mathbb{C}^{n \times n}$ be the solution of Equation (5) satisfying (6). If $\{M_k\}_{k=0}^{\infty}$ is the sequence of matrices defined by (8), then

$$\|M_k\| \leq \mu_1 \quad \text{for } k = 0, 1, 2, \dots, \quad (22)$$

and

$$\|M - M_k\| \leq \mu_1 q^k \quad \text{for } k = 0, 1, 2, \dots, \quad (23)$$

where μ_1 is the unique root of Equation (16) in the interval $(0, \mu_0)$ and $q = \|B\|\tau e^{\mu_1\tau} < 1$ (see (18)).

Proof. Note that $M_{k+1} = F(M_k)$ for $k = 0, 1, 2, \dots$, where F is defined by Equation (19). Taking $\mu = \mu_1$ in the proof of Theorem 1, we find that $\|M\| \leq \mu_1$. Moreover, from (20) and (21), we obtain that $\|M_k\| \leq \mu_1$ for $k = 0, 1, 2, \dots$. From this and Equations (5) and (8), by the application of Lemma 2, we obtain for $k \geq 0$,

$$\|M - M_{k+1}\| = \|B(e^{-M\tau} - e^{-M_k\tau})\| \leq \|B\|\|e^{-M\tau} - e^{-M_k\tau}\| \leq \|B\|\tau e^{\mu_1\tau}\|M - M_k\| = q\|M - M_k\|.$$

From the last inequality, it follows by easy induction on k that

$$\|M - M_k\| \leq q^k \|M - M_0\| = q^k \|M\| \leq q^k \mu_1$$

for $k = 0, 1, 2, \dots$ \square

2.2. Dominant Eigenvalues and Eigensolutions

Let us summarize some facts from the theory of linear autonomous delay differential equations (see [1,2]). By an *eigenvalue* of Equation (1), we mean an eigenvalue of the generator of the solution

semigroup (see [1,2] for details). It is known that $\lambda \in \mathbb{C}$ is an eigenvalue of Equation (1) if and only if λ is a root of the characteristic equation (2). Moreover, for every $\beta \in \mathbb{R}$, Equation (1) has only finite number of eigenvalues with $\text{Re } \lambda > \beta$. By an *entire solution* of Equation (1), we mean a differentiable function $x : (-\infty, \infty) \rightarrow \mathbb{C}^n$ satisfying Equation (1) for all $t \in (-\infty, \infty)$. To each eigenvalue λ of Equation (1), there correspond nontrivial entire solutions of the form $p(t)e^{\lambda t}$, $t \in (-\infty, \infty)$, where $p(t)$ is a \mathbb{C}^n -valued polynomial in t . Such solutions are sometimes called *eigensolutions* corresponding to λ .

The following theorem shows that under the smallness condition (3) the eigenvalues of Equation (1) with $\text{Re } \lambda > -\mu_0$ coincide with eigenvalues of matrix M from Theorem 1 and the corresponding eigensolutions satisfy the ordinary differential Equation (4).

Theorem 3. *Suppose (3) holds so that $\mu_0 = -\tau^{-1} \ln(\|B\|\tau) > 0$, and define*

$$\Lambda = \{ \lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0, \text{Re } \lambda > -\mu_0 \}.$$

Let $M \in \mathbb{C}^{n \times n}$ be the unique solution of Equation (5) satisfying (6). Then $\Lambda = \sigma(M)$, where $\sigma(M)$ denotes the set of eigenvalues of M . Moreover, for every $\lambda \in \Lambda$, Equations (1) and (4) have the same eigensolutions corresponding to λ .

In the sequel, the eigenvalues of Equation (1) with $\text{Re } \lambda > -\mu_0$ will be called *dominant*.

As a preparation for the proof of Theorem 3, we establish three lemmas. First we show that if M is a solution of the matrix Equation (5), then every solution of the ordinary differential Equation (4) is an entire solution of the delay differential Equation (1).

Lemma 4. *Let $M \in \mathbb{C}^{n \times n}$ be a solution of Equation (5). Then every $v \in \mathbb{C}^n$, $x(t) = e^{Mt}v$, $t \in (-\infty, \infty)$, is an entire solution of Equation (1).*

Proof. Since $e^P e^Q = e^{P+Q}$ whenever P and $Q \in \mathbb{C}^{n \times n}$ commute, from Equation (5), we find that

$$\dot{x}(t) = Me^{Mt}v = (A + Be^{-M\tau})e^{Mt}v = Ae^{Mt}v + Be^{-M\tau}e^{Mt}v = Ax(t) + Be^{M(t-\tau)}v = Ax(t) + Bx(t - \tau)$$

for $t \in (-\infty, \infty)$. \square

In the following lemma, we prove the uniqueness of entire solutions of the delay differential Equation (1) with an appropriate exponential growth as $t \rightarrow -\infty$.

Lemma 5. *Suppose (3) holds. If x_1 and x_2 are entire solutions of Equation (1) with $x_1(0) = x_2(0)$ and such that*

$$\sup_{t \leq 0} \|x_j(t)\|e^{\mu_0 t} < \infty, \quad j = 1, 2, \tag{24}$$

with μ_0 as in (6), then $x_1 = x_2$ identically on $(-\infty, \infty)$.

Proof. Define

$$C = \sup_{t \leq 0} \|x_1(t) - x_2(t)\|e^{\mu_0 t}.$$

By virtue of (24), we have that $0 \leq C < \infty$. From Equation (1), we find for $t \leq 0$,

$$x_j(t) = x_j(0) - A \int_t^0 x_j(s) ds - B \int_t^0 x_j(s - \tau) ds, \quad j = 1, 2.$$

From this, taking into account that $x_1(0) = x_2(0)$, we obtain for $t \leq 0$,

$$\begin{aligned} \|x_1(t) - x_2(t)\| &\leq \|A\| \int_t^0 \|x_1(s) - x_2(s)\| ds + \|B\| \int_t^0 \|x_1(s - \tau) - x_2(s - \tau)\| ds \\ &\leq \|A\|C \int_t^0 e^{-\mu_0 s} ds + \|B\|C \int_t^0 e^{-\mu_0(s-\tau)} ds \\ &= C(\|A\| + \|B\|e^{\mu_0\tau}) \int_t^0 e^{-\mu_0 s} ds \leq C \frac{\|A\| + \|B\|e^{\mu_0\tau}}{\mu_0} e^{-\mu_0 t}. \end{aligned}$$

The last inequality implies for $t \leq 0$,

$$\|x_1(t) - x_2(t)\|e^{\mu_0 t} \leq C \frac{\|A\| + \|B\|e^{\mu_0\tau}}{\mu_0}.$$

Hence $C \leq \kappa C$, where

$$\kappa = \frac{\|A\| + \|B\|e^{\mu_0\tau}}{\mu_0}.$$

By virtue of (17), we have that $\kappa < 1$. Hence $C = 0$ and $x_1(t) = x_2(t)$ for $t \leq 0$. The uniqueness theorem ([1] Chapter 2, Theorem 2.3) implies that $x_1(t) = x_2(t)$ for all $t \in (-\infty, \infty)$. \square

Now we show that those entire solutions of Equation (1) which satisfy the growth condition

$$\sup_{t \leq 0} \|x(t)\|e^{\mu_0 t} < \infty \quad \text{with } \mu_0 \text{ as in (6)} \tag{25}$$

coincide with the solutions of the ordinary differential Equation (4).

Lemma 6. Suppose (3) holds. Then, for every $v \in \mathbb{C}^n$, Equation (1) has exactly one entire solution x with $x(0) = v$ and satisfying (25) given by

$$x(t) = e^{Mt}v \quad \text{for } t \in (-\infty, \infty), \tag{26}$$

where $M \in \mathbb{C}^{n \times n}$ is the solution of Equation (5) with property (6).

Proof. By Lemma 4, x defined by Equation (26) is an entire solution of Equation (1). Moreover, from Equations (6) and (26), we find for $t \leq 0$,

$$\|x(t)\| \leq e^{\|M\||t|} \|v\| \leq e^{\mu_0|t|} \|v\| = e^{-\mu_0 t} \|v\|.$$

Hence $\sup_{t \leq 0} \|x(t)\|e^{\mu_0 t} \leq \|v\| < \infty$. Thus, x given by Equation (26) is an entire solution of Equation (1) with $x(0) = v$ and satisfying (25). The uniqueness follows from Lemma 5. \square

Now we can give a proof of Theorem 3.

Proof of Theorem 3. Suppose that $\lambda \in \Lambda$. Since $\det \Delta(\lambda) = 0$, there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\Delta(\lambda)v = 0$ and hence $x(t) = e^{\lambda t}v$, $t \in (-\infty, \infty)$, is an entire solution of Equation (1). Since $\text{Re } \lambda > -\mu_0$, we have for $t \leq 0$,

$$\|x(t)\| = |e^{\lambda t}| \|v\| = e^{t \text{Re } \lambda} \|v\| \leq e^{-\mu_0 t} \|v\|,$$

which implies (25). Thus, $x(t) = e^{\lambda t}v$ is an entire solution of (1) with $x(0) = v$ and satisfying (25). By Lemma 6, we have that $e^{\lambda t}v = e^{Mt}v$ for $t \in (-\infty, \infty)$. Hence

$$\frac{e^{\lambda t} - 1}{t} v = \frac{e^{Mt} - I}{t} v \quad \text{for } t \in \mathbb{R} \setminus \{0\}.$$

Letting $t \rightarrow 0$, we obtain $\lambda v = Mv$. This proves that $\Lambda \subset \sigma(M)$.

Now suppose that $\lambda \in \sigma(M)$. Then there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Mv = \lambda v$. According to Lemma 4, $x(t) = e^{Mt}v = e^{\lambda t}v$ is an entire solution of Equation (1). Hence $\Delta(\lambda)v = 0$ which implies that $\det \Delta(\lambda) = 0$. In order to prove that $\lambda \in \Lambda$, it remains to show that $\operatorname{Re} \lambda > -\mu_0$. It is well-known that $\rho(M) \leq \|M\|$, where $\rho(M) = \sup_{\lambda \in \sigma(M)} |\lambda|$ is the spectral radius of M . This, together with (6), yields

$$|\operatorname{Re} \lambda| \leq |\lambda| \leq \rho(M) \leq \|M\| < \mu_0.$$

Therefore $\operatorname{Re} \lambda > -\mu_0$ which proves that $\sigma(M) \subset \Lambda$.

Let $\lambda \in \Lambda = \sigma(M)$. By Lemma 4, every eigensolution of the ordinary differential equation (4) corresponding to λ is an eigensolution of the delay differential equation (1). Now suppose that x is an eigensolution of the delay differential equation (1) corresponding to λ . Then $x(t) = p(t)e^{\lambda t}$, where $p(t)$ is a \mathbb{C}^n -valued polynomial in t . If m is the order of the polynomial p , then there exists $K > 0$ such that

$$\|p(t)\| \leq K(1 + |t|^m) \quad \text{for } t \in (-\infty, \infty).$$

Since $\operatorname{Re} \lambda > -\mu_0$, we have that $\epsilon = \operatorname{Re} \lambda + \mu_0 > 0$. From this, we find for $t \leq 0$,

$$\|x(t)\| = \|p(t)\|e^{\lambda t} = \|p(t)\|e^{t \operatorname{Re} \lambda} \leq K(1 + |t|^m)e^{t \operatorname{Re} \lambda} = K(1 + |t|^m)e^{\epsilon t}e^{-\mu_0 t}.$$

Hence

$$\|x(t)\|e^{\mu_0 t} \leq K(1 + |t|^m)e^{\epsilon t} \rightarrow 0 \quad \text{as } t \rightarrow -\infty.$$

Thus, x is an entire solution of Equation (1) satisfying the growth condition (25). By Lemma 6, x is a solution of the ordinary differential equation (4). \square

2.3. Asymptotic Equivalence

The following result from the monograph by Diekmann et al. [2] gives an asymptotic description of the solutions of Equation (1) in terms of the eigensolutions.

Proposition 1. ([2] Chapter I, Theorem 5.4) *Let $x : [-\tau, \infty) \rightarrow \mathbb{C}^{n \times n}$ be a solution of Equation (1) corresponding to some continuous initial function $\phi : [-\tau, 0] \rightarrow \mathbb{C}^n$. For any $\gamma \in \mathbb{R}$ such that $\det \Delta(\lambda) = 0$ has no roots on the vertical line $\operatorname{Re} \lambda = \gamma$, we have the asymptotic expansion*

$$x(t) = \sum_{j=1}^l p_j(t)e^{\lambda_j t} + o(e^{\gamma t}) \quad \text{as } t \rightarrow \infty, \tag{27}$$

where $\lambda_1, \lambda_2, \dots, \lambda_l$ are the finitely many roots of the characteristic equation (2) with real part greater than γ and $p_j(t)$ are \mathbb{C}^n -valued polynomials in t of order less than the multiplicity of λ_j as a zero of $\det \Delta(\lambda)$.

Now we can formulate our main result about the asymptotic equivalence of Equations (1) and (4).

Theorem 4. *Suppose that (3) holds so that $\mu_0 = -\tau^{-1} \ln(\|B\|\tau) > 0$. Let $M \in \mathbb{C}^{n \times n}$ be the solution of Equation (5) satisfying (6). Then the following statements are valid.*

(i) *Every solution of the ordinary differential equation (4) is an entire solution of the delay differential equation (1).*

(ii) *For every solution $x : [-\tau, \infty) \rightarrow \mathbb{C}^{n \times n}$ of the delay differential equation (1) corresponding to some continuous initial function $\phi : [-\tau, 0] \rightarrow \mathbb{C}^n$, there exists a solution \tilde{x} of the ordinary differential equation (4) such that*

$$x(t) = \tilde{x}(t) + o(e^{-\mu_0 t}) \quad \text{as } t \rightarrow \infty. \tag{28}$$

Proof. Conclusion (i) follows from Lemma 1. We shall prove conclusion (ii) by applying Proposition 1 with $\gamma = -\mu_0$. We need to verify that Equation (2) has no root on the vertical line $\operatorname{Re} \lambda = -\mu_0$. Suppose for

contradiction that there exists $\lambda \in \mathbb{C}$ such that $\det \Delta(\lambda) = 0$ and $\operatorname{Re} \lambda = -\mu_0$. Then there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\Delta(\lambda)v = 0$ and hence $\lambda v = Av + Be^{-\lambda\tau}v$. From this, we find that

$$\begin{aligned} |\lambda| \|v\| &\leq \|A\| \|v\| + \|B\| \|e^{-\lambda\tau}v\| = \|A\| \|v\| + \|B\| |e^{-\lambda\tau}| \|v\| \\ &= (\|A\| + \|B\| e^{-\tau \operatorname{Re} \lambda}) \|v\| = (\|A\| + \|B\| e^{\mu_0\tau}) \|v\|. \end{aligned}$$

Hence $|\lambda| \leq \|A\| + \|B\| e^{\mu_0\tau}$, which together with (17), yields

$$\mu_0 = |\operatorname{Re} \lambda| \leq |\lambda| \leq \|A\| + \|B\| e^{\mu_0\tau} < \mu_0,$$

a contradiction. Thus, we can apply Proposition 1 with $\gamma = -\mu_0$, which implies that the asymptotic relation (28) holds with

$$\tilde{x}(t) = \sum_{j=1}^l p_j(t) e^{\lambda_j t}, \quad (29)$$

where $\lambda_1, \lambda_2, \dots, \lambda_l$ are those eigenvalues of Equation (1) which have real part greater than $-\mu_0$ and $p_j(t)$ are \mathbb{C}^n -valued polynomials in t . According to Theorem 3, the eigensolutions of Equation (1) corresponding to eigenvalues with real part greater than $-\mu_0$ are solutions of the ordinary differential equation (4). Hence \tilde{x} given by Equation (29) is a solution of Equation (4). \square

2.4. Approximation of the Dominant Eigenvalues

We will need the following result about the distance of the eigenvalues of two matrices in terms of the norm of their difference due to Bhatia, Elsner and Krause [3].

Proposition 2. [3, Theorem 3] *Let $P, Q \in \mathbb{C}^{n \times n}$ and $\gamma = \max\{\|P\|, \|Q\|\}$. Then the eigenvalues of P and Q can be enumerated as $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n in such a way that*

$$\max_{1 \leq j \leq n} |\lambda_j - \mu_j| \leq 4 \cdot 2^{-1/n} n^{1/n} (2\gamma)^{1-1/n} \|P - Q\|^{1/n}. \quad (30)$$

Recall that the dominant eigenvalues of Equation (1) are those roots of Equation (2) which have real part greater than $-\mu_0$. According to Theorem 3, if (3) holds, then the dominant eigenvalues of Equation (1) coincide with the eigenvalues of M , the unique solution of Equation (5) satisfying (6). By Theorem 2, M can be approximated by the sequence of matrices $\{M_k\}_{k=0}^{\infty}$ defined by (8). As a consequence, the dominant eigenvalues of the delay differential equation (1) can be approximated by the eigenvalues of M_k . The explicit estimate (23) for $\|M - M_k\|$, combined with Proposition 2, yields the following result.

Theorem 5. *Suppose (3) holds so that the dominant eigenvalues of Equation (1) coincide with the eigenvalues $\lambda_1, \dots, \lambda_n$ of matrix M from Theorem 1 (see Theorem 3). If $\{M_k\}_{k=0}^{\infty}$ is the sequence of matrices defined by (8), then the eigenvalues $\lambda_1^{[k]}, \dots, \lambda_n^{[k]}$ of M_k can be renumbered such that*

$$\max_{1 \leq j \leq n} |\lambda_j - \lambda_j^{[k]}| \leq 8 \cdot 4^{-1/n} n^{1/n} \mu_1 q^{k/n}, \quad (31)$$

where μ_1 and q have the meaning from Theorem 2.

Since $q < 1$, the explicit error estimate (31) in Theorem 5 shows that under the smallness condition (3) the eigenvalues of M_k converge to the dominant eigenvalues of the delay differential equation (1) at an exponential rate as $k \rightarrow \infty$.

3. Discussion

Let us briefly mention some results which are relevant to our study. For a class of linear differential equations with small delay, Ryabov [4] introduced a family of special solutions and showed that every solution is asymptotic to some special solution as $t \rightarrow \infty$. Ryabov's result was improved by Driver [5], Jarník and Kurzweil [6]. A more precise asymptotic description was given in [7]. For further related results on asymptotic integration and stability of linear differential equations with small delays, see [8] and [9]. Some improvements and a generalization to functional differential equations in Banach spaces were given by Faria and Huang [10]. Inertial and slow manifolds for differential equations with small delays were studied by Chicone [11]. Results on minimal sets of a skew-product semiflow generated by scalar differential equations with small delay can be found in the work of Alonso, Obaya and Sanz [12]. Smith and Thieme [13] showed that nonlinear autonomous differential equations with small delay generate a monotone semiflow with respect to the exponential ordering and the monotonicity has important dynamical consequences. For the effects of small delays on the stability and control, see the paper by Hale and Verduyn Lunel [14].

The results in the above listed papers show that if the delay is small, then there are similarities between the delay differential equation and an associated ordinary differential equation. The description of the associated ordinary differential equation in general requires the knowledge of certain special solutions. Since in most cases the special solutions are not known, the above results are mainly of theoretical interest. In the present paper, in the simple case of linear autonomous differential equations with small delay, we have described the coefficient matrix of the associated ordinary differential equation. Moreover, we have shown that the coefficient matrix can be approximated by a sequence of matrices defined recursively which yields an effective method for the approximation of the dominant eigenvalues.

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