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Extremal Problems of Some Family of Holomorphic Functions of Several Complex Variables

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Abstract: Many authors, e.g., Bavrin, Jakubowski, Liczberski, Pfaltzgraff, Sitarski, Suffridge, and Stankiewicz, have discussed some families of holomorphic functions of several complex variables described by some geometrical or analytical conditions. We consider a family of holomorphic functions of several complex variables described in n-circular domain of the space $\mathbb{C}^n$. We investigate relations between this family and some of type of Bavrin's families. We give estimates of G-balance of k-homogeneous polynomial, a distortion type theorem and a sufficient condition for functions belonging to this family. Furthermore, we present some examples of functions from the considered class.

Keywords: holomorphic functions; n-circular domains in C^n; Minkowski function; Bavrins families; growth and distortion theorems

MSC: 2005; 32A30; 30C45

1. Introduction

A domain, $G \subset \mathbb{C}^n$, $n \geq 2$, containing the origin is called complete n-circular, if $z\Lambda = (z_1\lambda_1, ..., z_n\lambda_n) \in G$, for each $z = (z_1, ..., z_n) \in G$ and every $\Lambda = (\lambda_1, ..., \lambda_n) \in \overline{E^n}$, where E is the disc $\{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$.
In the paper, we assume that G is a bounded complete n-circular domain.

Let us consider the Minkowski function $\mu_G\colon \mathbb{C}^n \to [0, \infty)$

$$\mu_G(z) = inf\{t > 0\colon \frac{1}{t}z \in G\}, \quad z \in \mathbb{C}^n.$$

We shall use the continuity of μ_G and the following facts as well:

(i) $G = \{z \in \mathbb{C}^n\colon \mu_G(z) < 1\}$,
(ii) $\partial G = \{z \in \mathbb{C}^n : \mu_G(z) = 1\}$.

Remember that μ_G is a seminorm in $\mathbb{C}^n$ for complete n-circular domain G and is a norm in $\mathbb{C}^n$ in the case if G is also convex. Taking this fact into account, we will use a generalization $\mu_G(Q_{f,k})$ of the norm of k-homogeneous polynomials $Q_{f,k}$ (see [1]). In view of the k-homogeneity of $Q_{f,k}$, the formula for ∂G and the maximum principle for modulus of holomorphic functions of several variables, we can put for $k \in \mathbb{N}$

$$\mu_G(Q_{f,k}) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|Q_{f,k}(w)|}{(\mu_G(w))^k} = \sup_{v \in \partial G} |Q_{f,k}(v)| = \sup_{u \in G} |Q_{f,k}(u)|. \tag{1}$$

For every $k \in \mathbb{N}$, the quantity $\mu_G(Q_{f,k})$ has the following basic property

$$|Q_{f,k}| \leq \mu_G(Q_{f,k})(\mu_G(w))^k, \quad w \in \mathbb{C}^n,$$

which generalizes the well-known inequality

$$|Q_{f,k}| \le ||Q_{f,k}|| \cdot ||w||^k, \quad w \in \mathbb{C}^n.$$

The quantities $\mu_G(Q_{f,k})$ in [2] are called G-balance of the k-homogeneous polynomial $Q_{f,k}$. Let $\mathcal{H}_G$ denote the family of holomorphic functions $f\colon G \to \mathbb{C}$ and let $\mathcal{L} : \mathcal{H}_G \to \mathcal{H}_G$ be the Temljakov linear operator (see [3]), which is defined by

$$\mathcal{L}f(z) = f(z) + \mathcal{D}f(z)(z)\mathcal{L}f(z) = f(z) + \mathcal{D}f(z)(z), \quad z \in G, \tag{2}$$

where $Df(z)$ is the Frechet derivative of f at the point z.

Bavrin (see [4,5]) considered the subclasses $\mathcal{V}_G(\gamma)$, $\mathcal{M}_G(\gamma)$, $\mathcal{N}_G$, and $\mathcal{R}_G$ of the class $\mathcal{H}_G(1) = \{f \in \mathcal{H}_G : f(0) = 1\}$. We say that $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to

- $\mathcal{V}_G(\gamma)$, $0 \le \gamma < 1$, if $f(z) \ne 0$ for $z \in G$ and

$$Re\mathcal{L}f(z) > \gamma, \quad z \in G. \tag{3}$$

- $\mathcal{M}_G(\gamma)$, $0 \le \gamma < 1$, if $f(z) \ne 0$, for $z \in G$ and

$$Re\frac{\mathcal{L}f(z)}{f(z)} > \gamma, \quad z \in G. \tag{4}$$

- $\mathcal{N}_G$, if $\mathcal{L}f(z) \ne 0$, for $z \in G$ and

$$Re\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f(z)} > 0, \quad z \in G. \tag{5}$$

- $\mathcal{R}_G$, if there exists a function $\phi \in \mathcal{N}_G$ such that

$$Re\frac{\mathcal{L}f(z)}{\mathcal{L}\phi(z)} > 0, \quad z \in G. \tag{6}$$

In particular, $\mathcal{V}_G(0) = \mathcal{V}_G$ and $\mathcal{M}_G(0) = \mathcal{M}_G$.

In the case $n = 2$, Bavrin (see [4]) gave the following geometrical interpretation for functions from $\mathcal{M}_G$. A function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{M}_{\mathcal{G}}$ if two conditions are maintained:

(i) The function $z_1 f(z_1, \alpha z_1)$ of one variable z_1 is starlike univalent in the disc, which is the projection onto the plane $z_2 = 0$ of the intersection of the domain $\mathcal{G}$, and every analytic plane $\left\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = \alpha z_1\right\}, \alpha \in \mathbb{C};$.
(ii) The function $z_2 f(0, z_2)$ is starlike univalent on the intersection $\mathcal{G}$ and the plane $z_1 = 0$..

In connection with this interpretation, we may say that the family $\mathcal{M}_{\mathcal{G}}$ corresponds to the well-known class $\mathcal{S}^*$ of the normalized univalent starlike functions $F : E \to \mathbb{C}$. In the same way, we can say that the family $\mathcal{N}_G$ ($\mathcal{R}_G$) corresponds to the class $\mathcal{S}^c$ ($\mathcal{S}^{cc}$) (see [6]) of normalized holomorphic univalent convex (close-to-convex) functions.

Note that the class $\mathcal{M}_G$ has been used in research of some linear invariant families of locally biholomorphic mappings in $\mathbf{C}^n$ (see [7]).

Here, we consider a subfamily $\mathcal{K}_G^-(\gamma)$ of the family $\mathcal{H}_G(1)$.

We say that $f \in \mathcal{H}_G(1)$ belongs to $\mathcal{K}_G^-(\gamma)$, $0 \le \gamma < 1$, if there exists such a function, $h \in \mathcal{M}_G(\frac{1}{2})$, that satisfies the condition

$$\mathrm{Re}\frac{\mathcal{L}f(z)}{h(z)h(-z)} > \gamma, \quad z \in G. \tag{7}$$

The family $\mathcal{K}_G^-(\gamma)$ corresponds to the class $\mathcal{K}_s(\gamma)$ of functions of one complex variable introduced by Kowalczyk and Leś-Bomba (see [8]) defined as follows.

Let $F : E \to C$, $F(0) = 0$, $F'(0) = 1$, be a holomorphic function in E. We say that $F \in \mathcal{K}_s(\gamma)$, $0 \le \gamma < 1$, if there exists a function $H \in \mathcal{S}^*(\frac{1}{2})$ such that

$$\mathrm{Re}\frac{-\zeta^2 F'(\zeta)}{H(\zeta)H(-\zeta)} > \gamma, \quad \zeta \in E. \tag{8}$$

The family $\mathcal{K}_G^-(\gamma)$ is in a way associated with the family $\mathcal{K}_G^-$ considered by Leś-Bomba and Liczberski in [1]. In particular when $\gamma = 0$, we have $\mathcal{K}_G^-(0) = \mathcal{K}_G^-$.

We say that $f \in \mathcal{H}_G(1)$ belongs to family $\mathcal{K}_G^-$ if there exists such a function $h \in \mathcal{M}_G(\frac{1}{2})$ that satisfies the condition

$$\mathrm{Re}\frac{\mathcal{L}f(z)}{h(z)h(-z)} > 0, \quad z \in G. \tag{9}$$

While presenting the properties of the family $\mathcal{K}_G^-(\gamma)$, we will use the number $\Delta(G)$-characteristic, which is assigned to each bounded complete n-circular domain, G, by the following formula (see [4]),

$$\Delta = \Delta(G) = \sup_{z=(z_1,\dots,z_n)\in G} \left| \sum_{j=1}^{n} z_j \right|. \tag{10}$$

Now, we present two examples of functions from this family:

Example 1. *Let* $\alpha \in \mathbb{R}$, $A = \{z \in G : \sum_{j=1}^n z_j = 0\}$, $0 \le \gamma < 1$ *and*

$$f_1(z) = \begin{cases} \frac{(1-\gamma)\Delta}{e^{i\alpha}\sum_{j=1}^n z_j} \log \frac{\Delta + e^{i\alpha}\sum_{j=1}^n z_j}{\sqrt{\Delta^2 + (e^{i\alpha}\sum_{j=1}^n z_j)^2}} + \frac{\gamma\Delta}{e^{i\alpha}\sum_{j=1}^n z_j} \arctan \frac{e^{i\alpha}\sum_{j=1}^n z_j}{\Delta}, & z \in G \setminus A, \\ 1, & z \in A, \end{cases} \tag{11}$$

where $\log 1 = 0$ *and* $\sqrt{1} = 1$. *Then, the function* f_1 *belongs to the family* $\mathcal{K}_G^-(\gamma)$.

Indeed, this function belongs to $\mathcal{H}_G(1)$ (has a holomorphic extension defined on G, A is a nowheredense and closed subset of G (see [9])) and f_1 for $z \in G \setminus A$ expands into a series of homogenous polynomials

$$f_1(z) = 1 - \frac{e^{i\alpha}\sum_{j=1}^n z_j}{\Delta} + \frac{1}{3}(1 - 2\gamma)\left(\frac{e^{i\alpha}\sum_{j=1}^n z_j}{\Delta}\right)^2 - \text{(higher order terms)}.$$

Moreover,

$$\mathcal{L}f_1(z) = \Delta^2 \frac{\Delta - (1-2\gamma)e^{i\alpha}\sum_{j=1}^n z_j}{\left(\Delta + e^{i\alpha}\sum_{j=1}^n z_j\right)\left(\Delta^2 + \left(e^{i\alpha}\sum_{j=1}^n z_j\right)^2\right)}, \quad z \in G.$$

Let us consider the function $h_1 \in \mathcal{M}_G(\frac{1}{2})$ (see [1]) of the form

$$h_1(z) = \frac{\Delta}{\sqrt{\Delta^2 + \left(e^{i\alpha}\sum_{j=1}^n z_j\right)^2}}, \quad z \in G.$$

Therefore

$$\mathrm{Re}\frac{\mathcal{L}f_1(z)}{h_1(z)h_1(-z)} = \mathrm{Re}\frac{\Delta - (1-2\gamma)e^{i\alpha}\sum_{j=1}^n z_j}{\Delta + e^{i\alpha}\sum_{j=1}^n z_j} > \gamma, \quad z \in G,$$

the condition (7) holds and $f_1 \in \mathcal{K}_G^-(\gamma)$.

Example 2. *Let* $\alpha \in \mathbb{R}$, $A = \{z \in G : \sum_{j=1}^n z_j = 0\}$, $0 \le \gamma < 1$ *and*

$$f_2(z) = \begin{cases} \frac{\Delta\gamma}{2e^{i\alpha}\sum_{j=1}^n z_j} \log \frac{\Delta + e^{i\alpha}\sum_{j=1}^n z_j}{\Delta - e^{i\alpha}\sum_{j=1}^n z_j} + \frac{(1-\gamma)\Delta}{\Delta - e^{i\alpha}\sum_{j=1}^n z_j}, & z \in G \setminus A, \\ 1, & z \in A, \end{cases} \tag{12}$$

where $\log 1 = 0$ *and* $\sqrt{1} = 1$. *Then, the function* f_2 *belongs to the family* $\mathcal{K}_G^-(\gamma)$.

Indeed, this function belongs to $\mathcal{H}_G(1)$ and for $z \in G \setminus A$ it expands into a series of homogenous polynomials

$$f_2(z) = 1 + (1-\gamma)\frac{e^{i\alpha}\sum_{j=1}^n z_j}{\Delta} + (1 - \frac{2}{3}\gamma)\left(\frac{e^{i\alpha}\sum_{j=1}^n z_j}{\Delta}\right)^2 + \text{(higher order terms)}.$$

Moreover,

$$\mathcal{L}f_2(z) = \Delta^2 \frac{\Delta + (1-2\gamma)e^{i\alpha}\sum_{j=1}^n z_j}{\left(\Delta - e^{i\alpha}\sum_{j=1}^n z_j\right)\left(\Delta^2 - \left(e^{i\alpha}\sum_{j=1}^n z_j\right)^2\right)}, \quad z \in G.$$

Let us consider the function $h_2 \in \mathcal{M}_G(\frac{1}{2})$ (see [1]) of the form

$$h_2(z) = \frac{\Delta}{\Delta - e^{i\alpha}\sum_{j=1}^n z_j}, \quad z \in G.$$

Therefore

$$\mathrm{Re}\frac{\mathcal{L}f_2(z)}{h_2(z)h_2(-z)} = \mathrm{Re}\frac{\Delta + (1-2\gamma)e^{i\alpha}\sum_{j=1}^n z_j}{\Delta - e^{i\alpha}\sum_{j=1}^n z_j} > \gamma, \quad z \in G,$$

the condition (7) holds and $f_2 \in \mathcal{K}_G^-(\gamma)$.

2. Main results

The relation between the class $\mathcal{K}_G^-(\gamma)$ and another type of Bavrin's families is the following.

Theorem 1. *Let* $0 \le \gamma < 1$ *and let* $G \subset \mathbb{C}^n$ *be a bounded complete n-circular domain. Then, the following inclusions hold,*

$$\mathcal{V}_G(\gamma) \subsetneq \mathcal{K}_G^-(\gamma) \subset \mathcal{K}_G^-.$$

Proof. Firstly, we show that $\mathcal{V}_G(\gamma) \subset \mathcal{K}_G^-(\gamma)$. Let us assume that $f \in \mathcal{V}_G(\gamma)$. Then condition (3) is satisfied. Let us note that the function $h = 1$ belongs to the family $\mathcal{M}_G(\frac{1}{2})$ and

$$\mathrm{Re}\frac{\mathcal{L}f(z)}{h(z)h(-z)} = \mathrm{Re}\mathcal{L}f(z) > \gamma, \quad z \in G.$$

This means that f belongs to the family $\mathcal{K}_G^-(\gamma)$.

Now, we show that $\mathcal{V}_G(\gamma) \neq \mathcal{K}_G^-(\gamma)$. Let us consider the function $f_2 \in \mathcal{K}_G^-(\gamma)$ (see Example 2) with $\alpha = 0$. Then there exists a point, $\overset{\circ}{z} \in G$, such that for $\Delta = \Delta(G)$ we have

$$\sum_{j=1}^n \overset{\circ}{z}_j = \Delta \cdot \left(\frac{2}{3} - \frac{1}{2}i\right).$$

Let us fix $\gamma = \frac{1}{2}$. For such $\overset{\circ}{z}$ we have

$$\mathrm{Re}\mathcal{L}f(\overset{\circ}{z}) = -\frac{3024}{18421} < 0,$$

therefore f is not in $\mathcal{V}_G$, so $\mathcal{K}_G^-(\gamma) \setminus \mathcal{V}_G \neq \emptyset$.

Let $f \in \mathcal{K}_G^-(\gamma)$. Therefore, for $0 \le \gamma < 1$ the condition (7) occurs, but since $\gamma \ge 0$ the condition (9) also applies. Thus $f \in \mathcal{K}_G^-$. $\square$

In the paper, [1] it has been proved that $\mathcal{K}_G^- \subsetneq \mathcal{R}_G$, so we have

Corollary 1. *Let $0 \le \gamma < 1$ and let $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain. Then the following inclusion holds*

$$\mathcal{K}_G^-(\gamma) \subsetneq \mathcal{R}_G.$$

We will now present estimates of G-balance of k-homogenous polynomial $Q_{f,k}$ in the family $\mathcal{K}_G^-(\gamma)$.

Theorem 2. *Let $0 \le \gamma < 1$ and let $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain and let $f \in \mathcal{K}_G^-(\gamma)$. If the expansion of the holomorphic function f into a series of homogeneous polynomials is of the form*

$$f(z) = \sum_{k=0}^{\infty} Q_{f,k}(z), \quad z \in G, \quad Q_{f,0} = 1 \tag{13}$$

then

$$\mu_G(Q_{f,k}(z)) \le \begin{cases} 1 - \frac{k}{k+1}\gamma, & for \quad k \quad even, \\ 1 - \gamma, & for \quad k \quad odd. \end{cases} \tag{14}$$

The estimates are sharp.

Proof. Let $f \in \mathcal{K}_G^-(\gamma)$ and $h \in \mathcal{M}_G(\frac{1}{2})$ be associated with f by condition (7). Then, $g(z) = h(z)h(-z) \in \mathcal{M}_G$ and g is even (see [1]). Therefore, from condition (7), there exists function $g \in \mathcal{M}_G$ and function $p \in \mathcal{C}_G(\gamma) = \{p \in \mathcal{H}_G(1) : Re(z) > \gamma, \quad 0 \le \gamma < 1, \quad z \in G\}$, such that

$$\mathcal{L}f(z) = g(z)p(z). \tag{15}$$

Assuming that

$$p(z) = \sum_{k=0}^{\infty} Q_{p,k}(z), \quad z \in G, \quad Q_{p,0} = 1 \tag{16}$$

and

$$g(z) = \sum_{k=0}^{\infty} Q_{g,k}(z), \quad z \in G, \quad Q_{g,0} = 1, \tag{17}$$

therefore, in view of the fact that

$$\mathcal{L}f(z) = \sum_{k=0}^{\infty} (k+1)Q_{f,k}(z), \quad z \in G, \tag{18}$$

and the uniqueness theorems for expansions into series of homogenous polynomials, we obtain for $k \in \mathbf{N}$,

$$(k+1)Q_{f,k}(z) = \sum_{l=0}^{k} Q_{p,k-l}(z)Q_{g,l}(z), \quad z \in G. \tag{19}$$

We have (see [10])

$$|Q_{p,\nu}| \le 2(1-\gamma), \quad z \in G, \quad \nu \in \mathbb{N}, \tag{20}$$

and we have (see [1])

$$|Q_{g,2m}(z)| \leq 1, \quad z \in G, \quad m \in \mathbb{N}, \tag{21}$$

because $Q_{g,2m-1} = 0$ for $m \in \mathbb{N}$ (g in Equation (15) is even).

While, identity (19) implies that for $z \in G$ and $k \in \mathbf{N}$, we obtain

$$\begin{aligned}
(k+1)|Q_{f,k}(z)| &\leq \left|\sum_{l=0}^{k} Q_{p,k-l}(z)Q_{g,l}(z)\right| \\
&\leq |Q_{p,0}(z)||Q_{g,k}| + \left|\sum_{l=0}^{k-1} Q_{p,k-l}(z)Q_{g,l}(z)\right| \\
&\leq \begin{cases} 1+2(1-\gamma)\cdot\frac{k}{2}, & for \quad k \quad even, \\ 0+2(1-\gamma)\cdot\frac{k+1}{2}, & for \quad k \quad odd. \end{cases}
\end{aligned}$$

Then

$$\sup_{z\in G} |(Q_{f,k}(z))| \leq \begin{cases} 1-\frac{k}{k+1}\gamma, & for \quad k \quad even, \\ 1-\gamma, & for \quad k \quad odd. \end{cases} \tag{22}$$

Having (1), we obtain (14). It remains to show the sharpness of (14). Let us observe that the function $f_2 \in \mathcal{K}_G^-(\gamma)$ of the form (12) is the extremal function. Indeed, as the homogenous polynomials $Q_{f,k}$ in its development (13), $k \in \mathbb{N}$ and $0 \leq \gamma < 1$ have the form

$$Q_{f,k}(z) = \begin{cases} \left(1-\frac{k}{k+1}\gamma\right)\left(\frac{e^{i\alpha}}{\Delta}\sum_{j=1}^{n} z_j\right)^k, & for \quad k \quad even, \\ (1-\gamma)\left(\frac{e^{i\alpha}}{\Delta}\sum_{j=1}^{n} z_j\right)^k, & for \quad k \quad odd. \end{cases} \tag{23}$$

Moreover, if $k \in \mathbb{N}$ is even, using (10), we have

$$\mu_G(Q_{f,k}) = \sup_{z\in\partial G} |Q_{f,k}(z)| = \sup_{z\in G} \left|\left(1-\frac{k}{k+1}\gamma\right)\left(\frac{e^{i\alpha}}{\Delta}\sum_{j=1}^{n} z_j\right)^k\right| = 1-\frac{k}{k+1}\gamma.$$

In the same way we get for a natural odd k that

$$\mu_G(Q_{f,k}) = 1-\gamma.$$

□

Now, we prove a sufficient condition for functions belonging to the investigated class $\mathcal{K}_G^-(\gamma)$.

Theorem 3. *Let $h \in \mathcal{M}_G(\frac{1}{2})$, $0 \leq \gamma < 1$ and $g(z) = h(z)h(-z)$ for $z \in G$ expands, as in (17). If the expansion of the holomorphic function, f, into a series of homogeneous polynomials is of the form (13) and the function f satisfies the condition*

$$2\sum_{k=1}^{\infty}(k+1)|Q_{f,k}(z)| + (|1-2\gamma|+1)\sum_{k=1}^{\infty}|Q_{g,2k}(z)| < 2(1-\gamma), \quad k \in \mathbb{N}, z \in G, \tag{24}$$

then f belongs to $\mathcal{K}_G^-(\gamma)$ and it is generated by h.

Proof. Let $0 \le \gamma < 1$. If the expansion of f into a series of homogenous polynomials has the form (13), then $\mathcal{L}f(z)$ has the form (18). Let

$$\begin{aligned}\Lambda &= |\mathcal{L}f(z) - h(z)h(-z)| - |\mathcal{L}f(z) + (1-2\gamma)h(z)h(-z)| \\ &= \left|\sum_{k=1}^{\infty}(k+1)Q_{f,k}(z) - \sum_{k=1}^{\infty} Q_{g,2k}(z)\right| \\ &\quad - \left|2(1-\gamma) + \sum_{k=1}^{\infty}(k+1)Q_{f,k}(z) + (1-2\gamma)\sum_{k=1}^{\infty} Q_{g,2k}(z)\right|. \end{aligned} \tag{25}$$

Therefore, for $z \in G$ from (24), we have the inequalities

$$\begin{aligned}\Lambda &\le \sum_{k=1}^{\infty}(k+1)|Q_{f,k}(z)| + \sum_{k=1}^{\infty}|Q_{g,2k}(z)| \\ &\quad - \left(|2(1-\gamma)| - \left|\sum_{k=1}^{\infty}(k+1)Q_{f,k}(z) + (1-2\gamma)\sum_{k=1}^{\infty} Q_{g,2k}(z)\right|\right) \\ &\le \sum_{k=1}^{\infty}(k+1)|Q_{f,k}(z)| + \sum_{k=1}^{\infty}|Q_{g,2k}(z)| \\ &\quad -2(1-\gamma) + \sum_{k=1}^{\infty}(k+1)|Q_{f,k}(z)| + |1-2\gamma|\sum_{k=1}^{\infty}|Q_{g,2k}(z)| \\ &= -2(1-\gamma) + \sum_{k=1}^{\infty}2(k+1)|Q_{f,k}(z)| + (|1-2\gamma|+1)\sum_{k=1}^{\infty}|Q_{g,2k}(z)| < 0. \end{aligned}$$

Thus, we obtain

$$|\mathcal{L}f(z) - h(z)h(-z)| < |\mathcal{L}f(z) + (1-2\gamma)h(z)h(-z)|, \quad z \in G,$$

which is equivalent to the inequality

$$\left|\frac{\mathcal{L}f(z)}{h(z)h(-z)} - 1\right| < \left|\frac{\mathcal{L}f(z)}{h(z)h(-z)} + 1 - 2\gamma\right|, \quad z \in G,$$

and consequently we have (7). Thus, $f \in \mathcal{K}_G^-(\gamma)$, which completes the proof. □

Below, we provide a distortion type theorem for the considered family of functions.

Theorem 4. *Let $0 \le \gamma < 1$ and $0 \le r < 1$, and let $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain. If $f \in \mathcal{K}_G^-(\gamma)$, then*

$$\frac{1-(1-2\gamma)r}{(1+r)(1+r^2)} \le |\mathcal{L}f(z)| \le \frac{1+(1-2\gamma)r}{(1-r)(1-r^2)}, \quad \mu_G(z) \le r, \tag{26}$$

$$\frac{1-\gamma}{r}\log\frac{1+r}{\sqrt{1+r^2}} + \frac{\gamma}{r}\arctan r \le |f(z)| \le \frac{\gamma}{2r}\log\frac{1+r}{1-r} + \frac{1-\gamma}{1-r}, \quad \mu_G(z) \le r. \tag{27}$$

The both lower estimates and the upper estimation in (27) are sharp.

Proof. Let $0 \le \gamma < 1$. In the case $r = 0$, the estimates (26) and (27) hold (in (27) for $r = 0$; it is understood as a limit when r tends to 0).

Let $r \in (0,1)$ and $\mu_G(z) = r$. First, we put

$$H(\zeta) = \zeta h\left(\zeta\frac{z}{\mu_G(z)}\right), \quad \zeta \in E,$$

where $h \in \mathcal{M}_G(\frac{1}{2})$. The function H is holomorphic, $H(0) = 0$,

$$H'(\zeta) = \mathcal{L}h\left(\zeta \frac{z}{\mu_G(z)}\right), \quad \zeta \in E$$

and $H'(0) = 1$. As a result, by (4) for $\gamma = \frac{1}{2}$, we obtain

$$\mathrm{Re}\frac{\zeta H'(\zeta)}{H(\zeta)} = \mathrm{Re}\frac{\mathcal{L}h\left(\zeta \frac{z}{\mu_G(z)}\right)}{h\left(\zeta \frac{z}{\mu_G(z)}\right)} > \frac{1}{2}, \quad \zeta \in E.$$

Therefore, $H \in S^*\left(\frac{1}{2}\right)$. Let us fix

$$F(\zeta) = \zeta f\left(\zeta \frac{z}{\mu_G(z)}\right), \quad \zeta \in E,$$

where $f \in \mathcal{K}_G^-(\gamma)$ is connected with h by (7). The function F is holomorphic, $F(0) = 0$, and

$$F'(\zeta) = \mathcal{L}f\left(\zeta \frac{z}{\mu_G(z)}\right), \quad \zeta \in E.$$

Therefore,

$$\mathrm{Re}\frac{-\zeta^2 F'(\zeta)}{H(\zeta)H(-\zeta)} = \mathrm{Re}\frac{\mathcal{L}f\left(\zeta \frac{z}{\mu_G(z)}\right)}{h\left(\zeta \frac{z}{\mu_G(z)}\right) h\left(-\zeta \frac{z}{\mu_G(z)}\right)} > \gamma, \quad \zeta \in E,$$

thus F belongs to $\mathcal{K}_s(\gamma)$ (F satisfies the condition (8)). Therefore, by the growth and distortion theorem for functions of the family $\mathcal{K}_s(\gamma)$ (see [8]) we conclude that

$$\frac{1-(1-2\gamma)r}{(1+r)(1+r^2)} \le \left|\mathcal{L}f\left(\zeta \frac{z}{\mu_G(z)}\right)\right| \le \frac{1+(1-2\gamma)r}{(1-r)(1-r^2)}, \quad \zeta \in E$$

$$\frac{1-\gamma}{r}\log\frac{1+r}{\sqrt{1+r^2}} + \frac{\gamma}{r}\arctan r \le \left|\zeta f\left(\zeta \frac{z}{\mu_G(z)}\right)\right| \le \frac{\gamma}{2r}\log\frac{1+r}{1-r} + \frac{1-\gamma}{1-r}, \quad \zeta \in E.$$

Putting $\zeta = \mu_G(z) = r \in (0,1)$, by the maximum principle, we obtain the estimates (26) and (27). Now, we will prove the second part of the theorem.

We show that for every $r \in [0,1)$ there exists a point $\overset{\circ}{z}$, $\mu_G(\overset{\circ}{z}) = r$, such that the function $f_1 \in \mathcal{K}_G^-(\gamma)$ with an appropriate α gives the equality in the lower part of inequalities (26) and (27).

It is obvious that $\Delta = \max_{z \in \partial G} |\sum_{j=1}^n z_j|$, so there exists a point $\overset{*}{z} \in \partial G$ for which $\left|\sum_{j=1}^n \overset{*}{z}_j\right| = \Delta$. By virtue of the properties of the domain G and the Minkowski function μ_G there exists a point $\overset{\circ}{z} = (\overset{\circ}{z}_1, ..., \overset{\circ}{z}_n) \in G \setminus \{0\}$, $\mu_G(\overset{\circ}{z}) = r \in (0,1)$ for which $\overset{*}{z} = (\mu_G(\overset{\circ}{z}))^{-1}\, \overset{\circ}{z}$. Therefore, we have

$$\left|\sum_{j=1}^n \overset{\circ}{z}_j\right| = \Delta \mu_G(\overset{\circ}{z}). \tag{28}$$

For $\overset{\circ}{z} \in G \setminus \{0\}$, let us choose a function f_1 with $\alpha = \alpha_0 \in \mathbb{R}$, such that

$$e^{i\alpha_0} \sum_{j=1}^n \overset{\circ}{z}_j = \left|\sum_{j=1}^n \overset{\circ}{z}_j\right|. \tag{29}$$

Therefore,

$$|\mathcal{L}f_1(\overset{\circ}{z})| = \frac{1-(1-2\gamma)\mu_G(\overset{\circ}{z})}{(1+\mu_G(\overset{\circ}{z}))(1+(\mu_G(\overset{\circ}{z}))^2)} = \frac{1-(1-2\gamma)r}{(1+r)(1+r^2)},$$

and

$$|f_1(\overset{\circ}{z})| = \frac{1-\gamma}{\mu_G(\overset{\circ}{z})} \log \frac{1+\mu_G(\overset{\circ}{z})}{\sqrt{1+(\mu_G(\overset{\circ}{z}))^2}} + \frac{\gamma}{\mu_G(\overset{\circ}{z})} \arctan \mu_G(\overset{\circ}{z}) = \frac{1-\gamma}{r} \log \frac{1+r}{\sqrt{1+r^2}} + \frac{\gamma}{r} \arctan r,$$

which makes the lower estimations (26) and (27) sharp for $z \in G$, such that $\mu_G(z) \le r$, i.e., for $z \in \overline{rG}$, $r \in (0,1)$.

Similarly, we can show that the function $f_2 \in \mathcal{K}_G^{-}(\gamma)$ gives the equality in the upper part of the inequality (27). The upper estimation in (26) is not sharp. □

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References

1. Leś-Bomba, E.; Liczberski, P. On some family of holomorphic functions of several complex variables. *Sci. Bull. Chem. Sec. Math. Comput. Sci.* **2007**, *2*, 7–16.
2. Długosz, R.; Leś, E. Embedding theorems and extremal problems for holomorphic functions on circular domains of $\mathbb{C}^n$. *Complex Var. Elliptic Equ.* **2014**, *59*, 883–899. [CrossRef]
3. Temljakov, A.A. Integral representation of functions of two complex variables. *Izv. Akad. Nauk. SSSR Ser. Mat.* **1957**, *21*, 89–92. (In Russian)
4. Bavrin, I.I. *A Class of Regular Bounded Functions in the Case of Several Complex Variables and Extremal Problems in that Class*; Moskow Obl. Ped. Inst.: Moscow, Russia, 1976; pp. 1–69. (In Russian)
5. Bavrin, I.I. Generalization of functions, regular in the disc and close-to-starlike, to the case of several complex variables. *Anal. Math.* **1979**, *5*, 257–267. (In Russian)
6. Duren, P.L. *Univalent Functions*; Sringer: New York, NY, USA, 1983.
7. Pfaltzgraff, J.A.; Suffridge, T.J. An extension theorem and linear invariant families generated by starlike maps. *Ann. UMCS Sect. Math.* **1999**, *53*, 193–207.
8. Kowalczyk, J.; Leś-Bomba, E. On a subclass of close-to-convex functions. *Appl. Math. Lett.* **2010**, *23*, 1147–1151. [CrossRef]
9. Chądzyski, J. *Wstęp do Analizy Zespolonej, cze II. Funkcje Holomorficzne Wielu Zmiennych Zespolonych*; Wydawnictwo UŁ, Łódź Universitas: Kraków, Poland, 2006. (In Polish)
10. Dziok, J. *Wstęp do Geometrycznej Teorii Funkcji Analitycznych*; Wydawnictwo Uniwersytetu Rzeszowskiego: Rzeszów, Poland, 2008. (In Polish)