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Inaudibility of *k*-D'Atri Properties

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Abstract: Working on closed Riemannian manifolds the first author and Schueth gave a list of curvature properties which cannot be determined by the eigenvalue spectrum of the Laplace–Beltrami operator. Following Kac, it is said that such properties are inaudible. Here, we add to that list the dimension of the manifold minus three new properties namely k-D'Atri for $k = 3, ..., \dim M - 1$.

Keywords: Laplace operator; isospectral manifolds; geodesic symmetries; D'Atri space; k-D'Atri space; $\mathfrak{GC}$-space

MSC: 58J50; 58J53; 53C25; 53C20; 22E25; 14J70

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1. Introduction

The Inverse Spectral Geometry focus on seeing the unseen [1]. The study of these problems already started in the nineteenth century, inspired by different physical problems. However, the explosion arrived with an affirmative example by M. Kac [2] and a counterexample due to C. Gordon, D. Webb and S. Wolpert in [3], and later with Calderón's problem [4]. The Calderón's problem is also called the Electrical Impedance Tomography Problem in which new advances have been obtained in [5].

These problems can be considered a mix between *Riemannian Geometry*, which studies the geometrical properties on Riemannian manifolds, and *Spectral Geometry*, which focuses on the study of eigenvalue problems. One of the more classical is the closed eigenvalue problem [6].

Let M be a compact and connected manifold without boundary, the solution to this problem is to find all real numbers λ for which there exists a nontrivial solution $f \in C^2(M)$ to the equation

$$\Delta f = \lambda f,$$

where Δ is the Laplace–Beltrami operator acting on functions. The set of real numbers which satisfies the equation is called the eigenvalues of Δ and they form a sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty.$$

The results presented in this paper contribute to Inverse Spectral Geometry from the classical point of view. That is, it contributes to search which geometrical properties can be determined on closed manifolds by the Laplace spectrum on functions. These properties are said to be audible since Kac's paper's. For example, it is well known that the volume of a closed manifold is spectrally determined. However, in [7] was proved that the following properties among others are inaudible: Weak local symmetry, D'Atri property and the type $\mathcal{A}$ property.

D'Atri spaces were introduced by D'Atri and Nickerson in [8] as a generalization of locally symmetric spaces. The fact that the local geodesic symmetries been volume preserving (up to sign) characterize these spaces. This property became equivalent to the fact that the geodesic symmetries preserve the mean curvature of small geodesic spheres. Two dimensional D'Atri spaces are locally symmetric and then, they have constant sectional curvature. O. Kowalski classified the three dimensional spaces in [9] and he proved that all of them are either locally symmetric or locally isometric to a naturally reductive spaces. In dimension 4, the classification of D'Atri spaces is known only in the locally homogeneous case [10]. Moreover, it is still unknown whether all of them are locally homogeneous (i.e., if the pseudo-group of the local isometries acts locally and transitively on it). Another characterization of D'Atri spaces were proved by D'Atri and Nickerson [8] and improved by Szabó [11] using an infinite series of curvature conditions, namely the Ledger conditions. More precisely, M is a D'Atri space if and only if it satisfies the infinite series of odd Ledger conditions. When a Riemannian manifold satisfies the first odd Ledger condition it is a type $\mathcal{A}$ space or in other words, it has the type $\mathcal{A}$ property. Thus type $\mathcal{A}$ spaces contain D'Atri spaces as a subclass.

On the other hand, k-D'Atri spaces of dimension n, $1 \leq k \leq n-1$, are a generalization of D'Atri spaces introduced by Kowalski, Prüfer and Vanhecke in [12]. These spaces are those where the geodesic symmetries preserve the k-th elementary symmetric functions of the eigenvalues of the shape operators of all small geodesic spheres. In fact, D'Atri and 1-D'Atri are equivalent conditions and moreover, Druetta proved in [13] that 2-D'Atri is also equivalent to D'Atri condition.

An open question about k-D'Atri spaces is to determine the interrelation between k-D'Atri spaces for different values of k, $k = 2, \ldots, n-1$, as well as their relation with locally homogeneous spaces. In a different direction, another interesting open question is to determine if the k-D'Atri property can be audible for each value of k. In this paper we solve the last one, given a negative answer for any value of k.

Main Result. *Let M be a Riemannian closed manifold of dimension n, the property of being k-D'Atri for all k, $k = 1, \ldots, n-1$, cannot be heard.*

Particularly, we obtain directly the following corollary.

Corollary 1. *Under the assumption of the main result, the property of being k-D'Atri for each k, $k = 3, \ldots, n-1$ is inaudible.*

Two closed Riemannian manifolds are *isospectral* if they have the same eigenvalue spectrum of the Laplace operator acting on functions, counting multiplicities. Thus, a strategy to find an inaudible property is to search two isospectral manifolds which differ from such property. To prove the main result, we will use Szabó manifolds [14]. For the sake of completeness, these manifolds will be presented in detail in Section 3. The needed preliminaries about k-D'Atri spaces will be shown in Section 2. We present the proof of the main results in the last section.

2. About k-D'Atri Properties

Let M be a Riemannian manifold, a point $m \in M$ and a vector $v \in T_mM$, $\|v\| = 1$. We denote by $\gamma_v(r)$ the geodesic in M which starts in m and has initial vector v. Moreover, for each small $r > 0$, we denote by $S_v(r)$ the shape operator of the geodesic sphere

$$G_m(r) = \left\{ \gamma_w(r) = \exp_m(rw) : \ w \in T_mM, \ \|w\| = 1 \right\}$$

at $\gamma_v(r)$. For each $m \in M$ the local geodesic symmetry s_m is defined by

$$s_m = \exp_m \circ (-Id) \circ \exp_m^{-1}.$$

An elementary symmetric function σ_k of a symmetric endomorphism A on an n-dimensional real vector space is given by its characteristic polynomial

$$\det(\lambda I - A) = \lambda^n - \sigma_1(A)\lambda^{n-1} + \cdots + (-1)^k \sigma_k(A)\lambda^{n-k} + \cdots + (-1)^n \sigma_n(A)$$

where

$$\sigma_k(A) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1}(A) \cdots \lambda_{i_k}(A)$$

with $1 \le i_1 < \cdots < i_k \le n$ and $\{\lambda_1(A), \ldots, \lambda_n(A)\}$ the set of n eigenvalues of A.

Definition 1. *An n-dimensional Riemannian manifold is said to be a k-D'Atri space, $1 \le k \le n-1$, if the geodesic symmetries preserve the k-th elementary symmetric functions of the eigenvalues of the shape operator of all small geodesic spheres. That is, for each small $r > 0$ and each unit vector $v \in T_m M$, M is a k-D'Atri space for some $1 \le k \le n-1$ if and only if*

$$\sigma_k(S_v(r)) = \sigma_k(S_{-v}(r)).$$

All these spaces are relevant examples of a more general one introduced by Gray in [15].

Definition 2. *We say that a Riemannian manifold M is a type $\mathcal{A}$ space if and only if the Ricci tensor is cyclic parallel, this is*

$$(\nabla_X ric)(X, X) = 0$$

for all $X \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection.

Proposition 1 ([16])**.** *If M is a k-D'Atri space then is a type $\mathcal{A}$-space.*

Moreover, when a space has the property of being k-D'Atri for all possible values of k, it has an extra geometrical property.

Proposition 2 ([16])**.** *M is an n-dimensional k-D'Atri space for all $k = 1, \ldots, n-1$ if and only if for any small real $r > 0$ and any unit vector $v \in T_m M$, the eigenvalues of $S_v(r)$ are preserved by the geodesic symmetries s_m for all $m \in M$, that is*

$$ds_m|_{\gamma_v(r)} \circ S_v(r) = S_{-v}(r) \circ ds_m|_{\gamma_v(r)}.$$

This property was introduced by J. Berndt, F. Prüfen and L. Vanhecke in [17] and namely $\mathfrak{GC}$-property.

3. The Riemannian Manifolds $N^{(a,b)}$

Now we are going to expose $N^{(a,b)}$, the Szabó manifolds [14], as a special class of the manifolds $N(j)$ introduced in [18]. To construct $N(j)$ we need:

1. A two step nilpotent Lie algebra $\mathfrak{g}(j) = \mathfrak{v} \oplus \mathfrak{z}$ with an inner product for which $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal, where $\mathfrak{z}$ is central, $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is a linear map and the Lie bracket $[\cdot,\cdot] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ is given by the equation

   $$\langle [X,Y], Z \rangle = \langle j_Z X, Y \rangle, \quad X, Y \in \mathfrak{v},\ Z \in \mathfrak{z}.$$

   The Lie algebra $\mathfrak{g}(j)$ has an associated two-step simply connected nilpotent Lie group $\tilde{G}(j)$ defined by the exponential map, $\exp : \mathfrak{v} \oplus \mathfrak{z} \to \tilde{G}(j)$ by $\exp(X, Z) = (X + Z)$. Its Lie group multiplication is given by the Campbell-Baker-Hausdorff formula as follows

   $$\exp(X, Z) \cdot \exp(Y, W) = \exp\left(X + Y, Z + W + \frac{1}{2}[X, Y]\right).$$

Please note that the inner product on the Lie algebra $\mathfrak{g}(j)$ defines a left-invariant metric on the Lie group $\tilde{G}(j)$, that is a metric for which the left translations by group elements are isometries.

2. We consider the submanifold of $\tilde{G}(j)$ without boundary

$$\tilde{N}(j) = \left\{ \exp(X, \tilde{Z}) \in \tilde{G}(j) : \; X \in S^{\dim \mathfrak{v}-1} \text{ and } \tilde{Z} \in \mathfrak{z} \right\} \cong S^{\dim \mathfrak{v}-1} \times \mathfrak{z}.$$

3. Now, to obtain a closed manifold, we take a lattice $\mathcal{L}$ of full rank in $\mathfrak{z}$ and we consider $G(j) = \tilde{G}(j)/\exp(\mathcal{L})$.
4. Finally, we obtain the closed submanifold

$$N(j) = \left\{ \exp(X, Z) \in G(j) : \; X \in S^{\dim \mathfrak{v}-1} \text{ and } Z \in \mathfrak{z}/\mathcal{L} \right\} \cong S^{\dim \mathfrak{v}-1} \times T^{\dim \mathfrak{z}}.$$

This construction gives us the following diagram

$$\begin{array}{ccccccc} \mathfrak{g}(j) & \stackrel{\exp}{\longrightarrow} & \tilde{G}(j) & \rightsquigarrow & G(j) & = & \tilde{G}(j)/\exp(\mathcal{L}) \\ & & \bigcup & & \bigcup & & \\ & & \tilde{N}(j) & \rightsquigarrow & N(j) & = & \tilde{N}(j)/\exp(\mathcal{L}), \end{array}$$

where $\rightsquigarrow$ denotes a Riemannian covering. Please note that the tangent space of $\tilde{N}(j)$ at some $p = \exp(x, z) \in \tilde{N}(j)$ with $x \in \mathfrak{v}$, $\|x\| = 1$, $z \in \mathfrak{z}$, is given by

$$T_p\tilde{N}(j) = L_{p*}\left\{(X, Z) : \; X \in \mathfrak{v}, \; X \perp x, \; Z \in \mathfrak{z}\right\}.$$

Moreover, $N(j)$ has constant scalar curvature (see [18]).

To get the Szabó manifolds we need to consider the next particular map j. Let $\mathbb{H} = \operatorname{span}\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the algebra of quaternions with the usual multiplication. For $a, b \in \mathbb{N}_0$ with $a + b > 0$, we define $\mathfrak{v}$ as the direct orthogonal sum of $a + b$ copies of $\mathbb{H}$. Let $\mathfrak{z} = \operatorname{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathcal{L} = \operatorname{span}_{\mathbb{Z}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the linear map $j^{(a,b)} : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ defined by

$$j_Z^{(a,b)}(X_1, \ldots, X_a, X_{a+1}, \ldots, X_{a+b}) := (X_1 Z, \ldots, X_a Z, Z X_{a+1}, \ldots, Z X_{a+b}).$$

Finally, we denote $N^{(a,b)} = N(j^{(a,b)})$, respectively $\tilde{N}^{(a,b)} = \tilde{N}(j^{(a,b)})$.

Now, we are interested in finding pairs of isospectral manifolds inside the class of $N^{(a,b)}$. The next result is essential.

Proposition 3 ([18])**.** *If two linear maps $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ have the same eigenvalues counting multiplicities in $\mathbb{C}$, then the closed Riemannian manifolds $N(j)$ and $N(j')$ are isospectral for the Laplace operator on functions.*

Please note that $j^{(a,b)}$ is of Heissenberg type, hence for ${j_Z^{(a,b)}}^2 = -\|Z\|^2 Id_{\mathfrak{v}}$. Thus, their eigenvalues are $\pm i\|Z\|$, each with multiplicity $\dim \mathfrak{v}/2$.

Corollary 2. *Two submanifolds $N^{(a,b)}$ and $N^{(a',b')}$ are isospectral if and only if $a + b = \dim \mathfrak{v}/4 = a' + b'$.*

Moreover, the pair of isospectral manifolds $N^{(a+b,0)}$ and $N^{(a,b)}$, $b \geq 0$, are an optimal pair to study the audibility of k-D'Atri spaces because they also have the following property that proves the inaudibility of the local homogeneous property.

Proposition 4 ([14])**.** *$N^{(a+b,0)}$ are locally homogeneous while $N^{(a,b)}$, $b \geq 0$ are not.*

4. Proof of Main Results

Weakly symmetric spaces were introduced by Selberg in [19]. Szabó in [11] introduced a new definition which was called *ray symmetric spaces*. Then, Berndt and Vanhecke proved in [20] that these two definitions are equivalent. A Riemannian manifold is called weakly symmetric (in the sense of Szabó) if for each $m \in M$ and each nontrivial geodesic γ starting in m, there exists an isometry f of M which fixes m and reverses γ, that is

$$df_m(\dot{\gamma}(0)) = -\dot{\gamma}(0).$$

Related with this kind of spaces, it is well known the following result.

Proposition 5 ([17]). *Every weakly symmetric space is a $\mathfrak{GC}$-space.*

A Riemannian manifold is *weakly-locally symmetric* (see [7]) if for every $m \in M$ there exists $\varepsilon > 0$ such that for any unit speed geodesic γ in M with $\gamma(0) = m$ there exists an isometry of the distance ball $B_\varepsilon(m)$ which fixes m and reverses $\gamma|_{(-\varepsilon,\varepsilon)}$. With this definition we have the following consequences.

Lemma 1 ([7]). *Let M be a complete, simply connected, weakly-locally symmetric Riemannian manifold. Then M is weakly symmetric. In particular, the universal Riemannian covering of any complete, weakly-locally symmetric Riemannian manifold is weakly symmetric.*

Now, let us focus on checking the property of being k-D'Atri on Szabó manifolds.

As is shown in [7], the manifolds $N^{(a+b,0)}$ are weakly locally symmetric for any $a, b \in \mathbb{N}_0$, $a + b > 0$.

Therefore, $\tilde{N}^{(a+b,0)}$ are weakly symmetric by the previous Lemma and they are $\mathfrak{GC}$-spaces by Proposition 5. Finally, using Proposition 2, $\tilde{N}^{(a+b,0)}$ are k-D'Atri spaces for all k, $k = 1, \ldots, n-1$. Now, $N^{(a+b,0)}$ inherits this property because it is a local property and these two Riemannian manifolds are locally isometric. Thus, $N^{(a+b,0)}$ are k-D'Atri for all k.

On the other hand, it is known that $\tilde{N}^{(a,b)}$ are not type $\mathcal{A}$-spaces by [7].

Therefore, using Proposition 2, $\tilde{N}^{(a,b)}$ are not k-D'Atri for any k. Moreover, $N^{(a,b)}$ neither satisfy the property of being k-D'Atri for any k because $\tilde{N}^{(a,b)}$ is its universal Riemannian covering and the property is local.

Then, we have two isospectral manifolds, $N^{(a+b,0)}$ and $N^{(a,b)}$, one of them is k-D'Atri for all $k = 1, \ldots, n-1$ and the other is not k-D'Atri for any possible value of k.

The proof of the Corollary 1 is now immediate from the fact that if $N^{(a+b,0)}$ is k-D'Atri for all k, it is in particular for each k.

5. Conclusions and Applications

Inverse spectral geometry is based on determining the shape and properties of unknown objects using the least amount of information, for example, with only the spectrum of a determined operator.

Following Kac [2], it is said that the properties which can be recovered by the spectrum of the Laplace–Beltrami operator are audible.

From a more applied point of view, inverse problems have to do with moving from effect to cause. Therefore, the treatment of these problems is both mathematical and computational. Given a certain measurement data from an unknown object of interest, the point is to design a computational algorithm that takes the data as input and produces, for example, an image of the unknown object. There are some operators whose applications are already a reality, such as the Dirichlet-to-Neumann operator for which already exists an experimental team developing its applications in Electrical Impedance Tomography with promising advances in the detection of breast cancer.

The main result proved in this paper provides us the fact that one cannot determine by the eigenvalues of the Laplace–Beltrami operator if a Riemannian closed manifold is k-D'Atri or not, for each possible value of k.

Therefore, a computational algorithm cannot be designed to determine these properties. This will avoid the costs of creating an applied study in relation to the property of being k-D'Atri.

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