Concept Paper

About Chirality in Minkowski Spacetime

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Abstract: In this paper, we show that Lorentz boosts are direct isometries according to the recent mathematical definitions of direct and indirect isometries and of chirality, working for any metric space. Here, these definitions are extended to the Minkowski spacetime. We also show that the composition of parity inversion and time reversal is an indirect isometry, which is the opposite of what could be expected in Euclidean spaces. It is expected that the extended mathematical definition of chirality presented here can contribute to the unification of several definitions of chirality in space and in spacetime, and that it helps clarify the ubiquitous concept of chirality.

1. Introduction

The issue of chirality is of great importance in many fields of science, including real-life applications and the arts [1–8]. The term, "chirality" was introduced in 1894 by Lord Kelvin [9]: "I call any geometrical figure, or group of points, chiral, and say that it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself". Although it is still useful and it is the main one used in chemistry [10], this definition is physical (if not intuitive), rather than mathematical, and it implicitly assumes the existence of an Euclidean space. There were attempts to enhance it [11] and extend it to the classical spacetime [12,13]. Alas, none of these definitions are mathematical, and the one of Barron was severely criticized [14]. It was noticed that, in chemistry, the use of the term "chirality" often engenders confusion and ambiguity [15]. Even worse, in the proceedings of a four-week-long school organized in 2000 by the Clay Mathematics Institute, two distinct approaches were found to understand mirror symmetry, without a clear connection between physical and mathematical methods of proof [16], and even the notion of what one means by "proof" of mirror symmetry differs between the two fields.

A definition of chirality should rely first on a definition of symmetry. However, there are several mathematical definitions of symmetry in the literature, depending on which field it applies, such as geometry, the probability theory, functional analysis, and graph theory. Then, a unifying one was published in 2007 [17], giving rise to a mathematical definition of chirality, which recovers the one of Lord Kelvin and extends it for any metric space [18] (a preliminary version was published in [19]).

We further show that these two definitions (symmetry and chirality) work in the Minkowski spacetime, despite the fact that the Minkowski metric used in special relativity is not a true metric, because it is not positive definite.

2. Results

For clarity, we summarize the symmetry definition given in [17] as follows. An *object* is a function having its input argument in a metric space, E equipped with a distance function δ, and an isometry is an element of the group of all bijections of E onto E preserving δ (at least, the identity is so). An object is symmetric when it is invariant upon an isometry which is not the identity.

Proposition 1. *The symmetry definition in [17], summarized above, is extended to the Minkowski space.*

The chirality definition given in [18] (and based on [17]), is summarized as follows. An isometry is direct when it can be written as the composition of squared isometries (at least the identity is so); if not, it is an indirect isometry. An object is *chiral* when it has no symmetry due to an indirect isometry.

Proposition 2. *The chirality definition in [18], summarized above, is extended to the Minkowski space.*

Proposition 1 is valid because none of the properties of δ were used in [17] and because the isometries of Minkowski spacetime are known to be a group acting on this spacetime: it is the Poincaré group. These isometries preserve inter-event intervals. Proposition 2 is valid because Proposition 1 is valid, and under the condition that all its isometries of the Minkowski space can be classified as being direct or indirect. We perform here this classification.

The Poincaré group is the semi-direct product of the Lorentz group and the translation group in space and time [20,21]. The translation group is an abelian Lie group, and it is a normal subgroup of the Poincaré group. The Poincaré group has four components [21], and the Lorentz group also has four components [20]. Lorentz transformations are linear isometries, leaving the origin fixed. The Lorentz group contains the spatial orthogonal transformations (i.e., the rotations and their compositions with the spatial parity inversion), time reversal, and the boosts, which connects two uniformly moving bodies. Translations and rotations are direct isometries. Time reversal and compositions of spatial rotations with the parity inversion are indirect isometries. Hereafter, we show that boosts are direct isometries and that the composition of spatial indirect isometries with time reversal is an indirect isometry. Then, we complete the classification of all Minkowski isometries as being either direct or indirect—an object in the Minkowski space is chiral if no indirect isometry leaves it invariant.

3. Methods

Conventionally, we retain the signature (+,+,+,-) for the Minkowski metric. Vectors in $\mathbb{R}^3$ are column vectors. A point in the spacetime is a 4-vector. Its spatial component x is a vector of $\mathbb{R}^3$, and its fourth component is the time, denoted ct, where c is the light speed. A single quote at the right of a vector or a matrix indicates a transposition. The identity matrix operating on spatial vectors is $\mathbf{I}$, and the identity matrix acting on 4-vectors is $\mathbf{I_4}$. We denote by v the module of the velocity of a frame moving in the direction of the unit vector u (so, $u'u = 1$). We set $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Because $v \in [0; c[$, we have $\beta \in [0; 1[$ and $\gamma > 1$.

The matrix B, associated to a Lorentz boost in the direction u, can be written as below (this is immediately deduced from Equation (5) in [22], or from Equation (1) in [23]):

$$B = \begin{pmatrix} \mathbf{I} + (\gamma - 1)uu' & -\gamma\beta u \\ -\gamma\beta u' & \gamma \end{pmatrix} \tag{1}$$

Let w be any fixed unit vector of $\mathbb{R}^3$ orthogonal to u, and $u \times w$ the cross product of u and w. We define the rotation matrix R_u, which depends only on u:

$$R_u = \begin{pmatrix} u/\sqrt{2} & w & u \times w & u/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} \tag{2}$$

We establish first the following Lemma:

Lemma 1. *R_u contains the eigenvectors of B, with the respective eigenvalues $\frac{1+\beta}{1-\beta}$, 1, 1 and $\frac{1-\beta}{1+\beta}$; $B = R_u D R'_u$, D being the diagonal matrix which contains the respective eigenvalues of B; $det(B) = 1$.*

Proof. Remember that $\gamma = (1 - \beta^2)^{-1/2}$ and check that $BR_u = R_u D$. $\square$

Now comes the following theorem:

Theorem 1. *Any Lorentz boost with parameter β in a given direction u is the square of two boosts in the direction u, each one of parameter $\lambda = (1 - \sqrt{1 - \beta^2})/\beta$. It is so that $\lambda \in [0; \beta[$.*

Proof. From Lemma 1, $B = R_u D R'_u$, where the rotation R_u depends only on u, not on β. Thus, any boost in the direction u with relative velocity λ can be factorized as $R_u \Delta R'_u$, where the respective diagonal elements of Δ are written $\frac{1+\lambda}{1-\lambda}$, 1, 1, and $\frac{1-\lambda}{1+\lambda}$. We look for a boost B_λ, such that $B = B_\lambda^2$. So, $B = R_u \Delta R'_u R_u \Delta R'_u$, that is, $B = R_u \Delta^2 R'_u$. Equating the two expressions of B, we get $\Delta^2 = D$. The unique solution is such that $(\frac{1+\lambda}{1-\lambda})^2 = \frac{1+\beta}{1-\beta}$, from which $\lambda^2 - \frac{2}{\beta} + 1 = 0$. Only one solution λ does not exceed 1: it is $\lambda = (1 - \sqrt{1 - \beta^2})/\beta$. It can be easily checked that $\lambda \in [0; \beta[$. $\square$

Corollary 1. *A boost is a direct symmetry of the Lorentz group.*

Proof. Apply Theorem 1: this matches the definition of direct isometries given in [18]. $\square$

Now we look at the composition of spatial indirect isometries with time reversal, and we prove the following Theorem:

Theorem 2. *The composition of any spatial indirect isometry with time reversal is an indirect isometry of the Lorentz group.*

Proof. We consider an indirect spatial isometry $P_Q = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$, where the orthonormal matrix Q is such that $Q'Q = I$ and $det(Q) = -1$. The time reversal is $T = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$. The operators P_Q and T commute, and $P_Q \circ T = \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix}$, with $det(P_Q \circ T) = +1$. Let us focus on its lower right diagonal element, -1, and assume that $P_Q \circ T$ is a direct isometry of the Lorentz group. If so, it is a product of squared elements of the group. We require that this product has its lower right diagonal element equal to -1. If T is not part of the product, this latter one would contain only boosts and spatial isometries. Without the presence of T, all group elements in the product have their lower right diagonal element greater or equal to $+1$, and the product cannot get a negative sign at this place (even the product of two boosts cannot do that, because it is the product of one boost by a spatial rotation [22]). An odd number of T is needed in the product to generate this negative sign. Notice that the number of indirect spatial isometries is even in the product, because each square necessarily contains an even number of them, and we would get a negative determinant for the product, a contradiction. Thus, $P_Q \circ T$ is an indirect isometry of the Lorentz group. $\square$

The commutative composition of parity reversal with time reversal is $P_{-I} \circ T = \begin{pmatrix} -I & 0 \\ 0 & -1 \end{pmatrix}$. For clarity, P_{-I} is denoted P, and $P_{-I} \circ T$ is denoted PT.

Corollary 2. *Each of the operators T, P, and PT is a mirror in the Minkowski space. The subgroup of the Lorentz group generated by the operators I_4, T, P, and PT is isomorphic to the Klein four-group.*

Proof. Apply Theorem 2. The fact that T, P, and PT are mirrors is an immediate consequence of Definition 3 in [18]. The composition of these mirrors are: $P \circ T = T \circ P = PT$, $PT \circ P = P \circ PT = T$, and $PT \circ T = T \circ PT = P$: this matches the definition of the abelian Klein four-group [24]. $\square$

Thus, the improper antichronous Lorentz transformation PT is a mirror in Minkowski spacetime. This result may be thought of as being counterintuitive because $det(PT) = +1$, but we recall that we work in $\mathbb{R}^4$ equipped with the Minkowski metric, not with the Euclidean metric.

4. Discussion and Conclusions

We remember that: (i) an isometry is a direct one if it can be written as the composition of squared isometries (thus the composition of direct isometries is a direct isometry), and (ii) any composition of isometries involving one indirect isometry is an indirect isometry. So far, in the Lorentz group, we have identified:

- Direct isometries: spatial rotations and boosts.
- Mirrors (involutions): parity inversion P, time reversal T, and their commutative composition, PT.

The compositions of the symmetry operators above generate the full Lorentz group. The spatial translations and the time translations being direct isometries of the Poincaré group, we are now able to classify any isometry of the Poincaré group as being direct or indirect.Thus, we have defined chirality in the Minkowski spacetime—an object is chiral if no indirect symmetry leaves it invariant. If it is not chiral, it is achiral.

We outline that our point is not to redefine symmetries: they are well-known, and the definition we used corresponds to what is usually encountered in physics. The situation is different for chirality. The word *chirality*, (together with *chiral* and *achiral*), has different definitions, depending on the context in which it was used. When time is discarded, the definitions are mainly based on Lord Kelvin's one, and applied both to rigid objects and to flexible objects, such as molecules conformers (it is a complex topic, far outside the scope of this paper). In spacetime, Lord Kelvin's definition no longer works. In particle physics, *chirality* is related to *helicity*, that is, it depends on the sign of the projection of the spin vector onto the momentum vector, but there are other uses, such as chiral superfields [25], *chiral algebra*, and *chiral homology*. We do not aim to unify all these terminologies. However, to get some compatibility between concepts of chirality in space and in spacetime, in our opinion, the very first step was to extend our definition of chirality to the spacetime of Minkowski, as we did here. This is mainly because there is a consensus about the validity of special relativity. Extensions to these other fields are outside the scope of this paper, and are to be considered further. Another extension is quantitative chirality, that is, measuring how less or more chiral objects are by themselves, not by their interactions [26]. They were considered only in the Euclidean case [27] (see also [28] and Section 2.9 in [26]), and the underlying mathematical model was extended to a theory of docking [29]. However, extending this quantitative chirality theory to objects in spacetime is difficult.

Conflicts of Interest: The author declares no conflict of interest.

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