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Some Results on (Generalized) Fuzzy Multi-H_v-Ideals of H_v-Rings

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Received: 11 October 2019 ; Accepted: 1 November 2019 ; Published: 6 November 2019

Abstract: The concept of fuzzy multiset is well established in dealing with many real life problems. It is possible to find various applications of algebraic hypercompositional structures in natural, technical and social sciences, where symmetry, or the lack of symmetry, is clearly specified and laid out. In this paper, we use fuzzy multisets to introduce the concept of fuzzy multi-H_v-ideals as a generalization of fuzzy H_v-ideals. Moreover, we introduce the concept of generalized fuzzy multi-H_v-ideals as a generalization of generalized fuzzy H_v-ideals. Finally, we investigate the properties of these new concepts and present different examples.

Keywords: H_v-structures; H_v-ring; fundamental equivalence relation; H_v-ideal; multiset; fuzzy multiset; fuzzy multi-H_v-ideal.

MSC: 20N25, 20N20.

1. Introduction

Symmetry is one of the central concepts of science, especially theoretical physics, mathematics and geometry of the 20th century. A given phenomenon or object is symmetrical if it is possible to introduce or consider a certain symmetry operation by which the phenomenon or object becomes in a certain sense identical to itself. The notion of symmetry has fascinated thinkers since antiquity (e.g., Pythagoreans). Later, in the so-called Erlangen program, Felix Klein tied a group of symmetry to each geometry. Mathematically, these symmetry operations are most often described by the term "group". We distinguish continuous symmetry, which are described mathematically mainly by the term "Lie groups", and discrete symmetry, which are described mainly by the term "discrete group". In mathematics, a symmetric relation is one in which variables can be exchanged or index permutations can be made without changing the relation (understood as a geometric object). The natural generalization of classical group theory is the approach of algebraic hyperstructures, introduced by F. Marty [1] during the eighth Congress of Scandinavian Mathematicians that was held in 1934. Marty generalized the notion of a group (which is a non-empty set with a binary operation satisfying some axioms and the operation of two elements is an element) to that of a hypergroup. A hypergroup is a non-empty set equipped with an associative and reproductive hyperoperation, where the composition of any two elements in it is a non-empty set. Since then, researchers started studying different kinds of hyperstructures such as: hyperrings, hypermodules, hypervector spaces, and many others by considering both parts: theoretical part as well as their applications to different subjects of science. Later in 1990, Th. Vougiouklis introduced weak hyperstructures (or H_v-structures) as a generalization of the concept of algebraic hyperstructures (hypergroups, hyperrings, hypermodules). The name "weak hyperstructures" is due to having some axioms of classical algebraic hyperstructures are replaced by their corresponding weak axioms in weak hyperstructures. Many researchers such as Corsini [2], Corsini and Leoreanu [3], Davvaz [4,5], Davvaz and Leoreanu-Fotea [6], Davvaz and Cristea [7] and Vougiouklis [8] wrote books related to (weak) hyperstructure theory and their applications. An overview about hyperstructure theory was published by Hoskova and Chvalina in [9].

On the other hand, fuzzy mathematics is an almost new branch in mathematics which was introduced in 1965 by Zadeh (see [10]). It is an extension of the classical notion of set and it is related to fuzzy set theory and fuzzy logic. Fuzzy sets are sets whose elements have degrees of membership that vary between 0 and 1 both inclusive. In classical set theory, the elements' membership in a certain set is usually identified by the condition that an element either belongs to the set or does not belong to it. By contrast, fuzzy set theory enables the gradual evaluation of the membership of elements in a set with values ranging between 0 and 1. If the membership function of a fuzzy set takes only the values 0, 1 then we go back to the classical notion of a set. As a generalization of fuzzy sets, Yager [11] introduced the concept of Fuzzy Multiset and investigated a calculus for them. Fuzzy Multiset permits the occurrence of an element more than once and each occurrence may have the same or different membership values.

In [12], Onasanya and Hoskova-Mayerova introduced multi-fuzzy groups induced by multisets. In [13,14], the authors studied fuzzy multi-polygroups and fuzzy multi-hypergroups. Moreover, Davvaz [15] and Davvaz et al. [16] discussed fuzzy H_v-ideals and generalized fuzzy H_v-ideals and investigated their properties. Our paper generalizes the work in [12,13,15,17] to combine H_v-rings and fuzzy multisets. More specifically, it is concerned about fuzzy multi-H_v-ideals and generalized fuzzy multi-H_v-ideals and it is constructed subsequently: Our motivation is described in Introduction, Section 2 presents basic notions with respect to (weak) hyperstructures and fuzzy multisets that are used throughout the paper. Section 3 defines and studies the properties of fuzzy multi-H_v-ideals and their relation to H_v-ideals. Finally, Section 4 defines generalized fuzzy multi-H_v-ideals and studies their properties.

2. Basic Definitions

In this section, we present some preliminary definitions and results related to hyperstructure theory [3,4,6] and fuzzy multisets [18] that are used throughout the paper.

2.1. (Weak) Hyperstructure Theory

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if X and Y are two non-empty subsets of H and $h \in H$, then we define:

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} x \circ y, \; h \circ X = \{h\} \circ X \text{ and } X \circ h = X \circ \{h\}.$$

H_v-structures were introduced by T. Vougiouklis, and studied in detail in [8,19,20], as a generalization of the ordinary algebraic hyperstructures. The equalities presented in some axioms of classical algebraic hyperstructures are substituted by non-empty intersection in H_v-structures. A hypergroupoid $(H, \circ)$ is called a *quasi-hypergroup* if $a \circ H = H \circ a = H$ for all $a \in H$. And it is called an *H_v-semigroup* if $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$. A hypergroupoid $(H, \circ)$ is called an *H_v-group* if it is a quasi-hypergroup and an H_v-semigroup. A multivalued system $(R, +, \cdot)$ is an *H_v-ring* if (1) $(R, +)$ is an H_v-group; (2) $(R, \cdot)$ is is an H_v-semigroup; (3) "$\cdot$" is weak distributive with respect to $+$.

Let $\{R_\alpha : \alpha \in \Gamma\}$ be a collection of H_v-rings (See [7]) and $\prod_{\alpha \in \Gamma} R_\alpha = \{< x_\alpha >: x_\alpha \in R_\alpha\}$. Then $(\prod_{\alpha \in \Gamma} R_\alpha, \oplus, \otimes)$ is an H_v-ring, where

$$< x_\alpha > \oplus < y_\alpha > = \{< z_\alpha >: z_\alpha \in x_\alpha + y_\alpha, \alpha \in \Gamma\},$$

$$< x_\alpha > \otimes < y_\alpha > = \{< z_\alpha >: z_\alpha \in x_\alpha \cdot y_\alpha, \alpha \in \Gamma\}.$$

A subset S of an H_v-ring $(R, +, \cdot)$ is called an *H_v-subring* if $(S, +, \cdot)$ is an H_v-ring. To prove that $(S, +, \cdot)$ is an H_v-subring of $(R, +, \cdot)$, it suffices to show that $x + S = S + x = S$ and $x \cdot y \subseteq S$ for all $x, y \in R$. An H_v-subring S of $(R, +, \cdot)$ is called an *H_v-ideal* of R if $R \cdot S \subseteq S$ and $S \cdot R \subseteq S$.

Let $(R, +, \star)$ and $(S, +', \star')$ be two H_v-rings. Then $f : R \to S$ is said to be *strong homomorphism* if $f(x + y) = f(x) +_1 f(y)$ and $f(x \star y) = f(x) \star' f(y)$ for all $x, y \in R$. $(R, +, \star)$ and $(S, +', \star')$ are called *isomorphic H_v-rings*, and written as $R \cong S$, if there exists a bijective function $f : R \to S$ that is also a strong homomorphism.

Fundamental relations are used as a tool to connect and relate the classes of hyperstuctures and algebraic structures together. In [8], Vougiouklis defined the notion of fundamental relation on H_v-rings. Koskas [21] introduced the fundamental relation $\beta^\star$ on hypergroups and later in 1990, Vougiouklis [8] introduced the fundamental relation $\gamma^\star$ on hyperrings. These fundamental relations $\beta^\star$ (for hypergroups (H_v-groups)) and $\gamma^\star$ (for hyperrings (H_v-rings)) are defined as the smallest strongly regular equivalence relations so that the quotient would be group and ring respectively. Many authors studied fundamental relations such as: Antampoufis and Hoskova-Mayerova [22], Corsini [2], Cristea and Norouzi [23–26], Davvaz [16], Freni [27], etc..

For all $n > 1$, we define the relation γ on an H_v-ring $(R, +, \cdot)$ as follows:

$$a\gamma b \iff \{a, b\} \subseteq u, \; u \text{ is any finite sum of finite products of elements in } R.$$

Clearly, the relation γ is reflexive and symmetric. The $\gamma^\star$, the transitive closure of γ, is called the *fundamental equivalence relation* on R and $(R/\gamma^\star, \oplus, \odot)$ is its *fundamental ring*, where for all $a, b \in R$,

$$\gamma^\star(a) \oplus \gamma^\star(b) = \gamma^\star(c) \text{ for all } c \in \gamma^\star(a) + \gamma^\star(b),$$

$$\gamma^\star(a) \odot \gamma^\star(b) = \gamma^\star(c) \text{ for all } c \in \gamma^\star(a) \cdot \gamma^\star(b).$$

2.2. Fuzzy multisets

A multiset (or bag) is a set containing repeated elements. [28,29] A fuzzy multiset is a generalization of fuzzy set and it was introduced by Yager in [11] under the name *fuzzy bag*. In these fuzzy bags the count of the number of elements itself becomes a crisp bag.

Definition 1 ([10]). *Let U be any non-empty set. A fuzzy set on U is characterized by a membership function $\mu_A(x)$ that assigns any element in U a grade of membership in A. The fuzzy set may be represented by the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in U\}$, where $\mu_A(x) \in [0, 1]$.*

Definition 2 ([30]). *Let X be a non-empty set and Q be the set of all crisp multisets drawn from the interval $[0, 1]$. A fuzzy multiset A drawn from X is represented by a function $CM_A : X \to Q$.*

In the above definition, the value $CM_A(x)$ is a crisp multiset drawn from $[0, 1]$. For each $x \in X$, $CM_A(x)$ is defined as the decreasingly ordered sequence of elements and it is denoted by:

$$\{\mu^1_A(x), \mu^2_A(x), \ldots, \mu^p_A(x)\} : \mu^1_A(x) \geq \mu^2_A(x) \geq \ldots \geq \mu^p_A(x).$$

A fuzzy set on a set X can be considered as a special case of fuzzy multiset where $CM_A(x) = \{\mu^1_A(x)\}$ for all $x \in X$.

Example 1. *Let $X = \{a, b, c, d\}$. Then $A = \{(0.7, 0.5)/b, (0.7, 0.2, 0.1, 0.1)/c, (0.3, 0.1)/d\}$ and $B = \{(1, 1)/a, (0.7, 0.6, 0.5, 0.1)/b, (0.3, 0.1)/c, (0.5, 0.4, 0.1)/d\}$ are fuzzy multisets of X.*

In Example 1, by $(0.7, 0.5)/b$ we mean that $CM_A(b) = (0.7, 0.5)$.

Definition 3 ([31]). *Let X, Y be non-empty sets, $f : X \to Y$ be a mapping, and A a fuzzy multiset of X and B a fuzzy multiset of Y. Then*

1. *The image of A under f is denoted by $f(A)$ or*

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{f(x)=y} CM_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

2. *The inverse image of B under f is denoted by $f^{-1}(B)$ where $CM_{f^{-1}(B)}(x) = CM_B(f(x))$.*

Example 2. *Let X be a non-empty set, S be a non-empty subset of X, and A be a fuzzy multiset of S. By considering the inclusion map $f : S \to X$, $f(x) = x$ for all $x \in S$, we get that*

$$CM_{f(A)}(x) = \begin{cases} CM_A(x) & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

is a fuzzy multiset of X.

3. Fuzzy Multi-H_v-Ideal

In this section, we introduce for the first time the notion of fuzzy multi-H_v-ideal as a generalization of fuzzy H_v-ideal, present several examples and results related to this new concept. The results in [15] related to fuzzy H_v-ideals can be considered as a special case of the results of this section.

Definition 4. *Let $(R, +, \cdot)$ be an H_v-ring. A fuzzy multiset A (with fuzzy count function CM_A) over R is a fuzzy multi-H_v-ideal of R if for all $x, y \in R$, the following conditions hold.*

1. $CM_A(x) \wedge CM_A(y) \leq \inf\{CM_A(z) : z \in x + y\}$;
2. *for every $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and $CM_A(x) \wedge CM_A(a) \leq CM_A(y)$;*
3. *for every $x, a \in R$ there exists $z \in H$ such that $x \in z + a$ and $CM_A(x) \wedge CM_A(a) \leq CM_A(z)$;*
4. $CM_A(x) \vee CM_A(y) \leq CM_A(z)$ *for all $z \in x \cdot y$.*

Remark 1. *Let $(R, +, \cdot)$ be an H_v-ring with "+" a commutative hyperoperation and A be a fuzzy multiset over R. To prove that A is a fuzzy multi-H_v-ideal of R, it suffices to prove Conditions 1, 2, and 4 or Conditions 1, 3, and 4 of Definition 4. This is clear as in the case of commutative H_v-group, Conditions 2 and 3 are equivalent to each other.*

Example 3. *Let $(R, +, \cdot)$ be an H_v-ring with a fixed element $a \in R$ and A be a fuzzy multiset of R defined as $CM_A(x) = CM_A(a)$ for all $x \in R$. Then A is a fuzzy multi-H_v-ideal of R (the constant fuzzy multi-H_v-ideal.).*

Remark 2. *Let $(R, +, \cdot)$ be an H_v-ring. Then we can define at least one fuzzy multi-H_v-ideal of R which is mainly the one that is defined in Example 3.*

We present some examples on non-constant fuzzy multi-H_v-ideals.

Example 4. *Let $(R_1, +_1, \cdot_1)$ be the H_v-ring defined as follows:*

$+_1$	0	1
0	0	R_1
1	R_1	1

$\cdot_1$	0	1
0	0	0
1	0	R_1

It is clear that $A = \{(0.8, 0.6, 0.6, 0.1)/0, (0.5, 0.4, 0.4)/1\}$ *is a fuzzy multi-*H_v*-ideal of* R_1.

Example 5. *Let* $(R_2, +_2, \cdot_2)$ *be the* H_v*-ring defined by the following tables:*

$+_2$	a	b	c
a	a	b	c
b	b	b	R_2
c	c	R_2	c

$\cdot_2$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

It is clear that $A = \{(0.9, 0.7, 0.6, 0.6, 0.1)/a, (0.8, 0.4, 0.2)/b, (0.8, 0.4, 0.2)/c\}$ *is a fuzzy multi-*H_v*-ideal of* R_2.

Example 6. *Let* $(R_3, +_3, \cdot_3)$ *be the* H_v*-ring defined by the following tables:*

$+_3$	d	e	f
d	d	e	f
e	e	$\{e, f\}$	d
f	f	e	d

$\cdot_3$	d	e	f
d	d	d	d
e	d	e	f
f	d	f	d

It is clear that both: $A = \{(0.9, 0.7, 0.6, 0.6, 0.1)/d, (0.9, 0.7, 0.6, 0.6, 0.1)/f\}$ *and* $B = \{(0.9, 0.8, 0.8, 0.1)/d\}$ *are fuzzy multi-*H_v*-ideals of* R_3.

Proposition 1. *Let* $(R, +)$ *be an* H_v*-group and "*$\cdot$*" be any hyperoperation on* R *with* $\{x, y\} \subseteq x \cdot y$ *for all* $x, y \in R$. *Then* A *is a fuzzy multi–*H_v*-ideal of the* H_v*-ring* $(R, +, \cdot)$ *if and only if* A *is the constant fuzzy multi-*H_v*-ideal of* R.

Proof. It is clear that if A is the fuzzy multiset described in Example 3 then A is a fuzzy multi-H_v-ideal of R. Let A be a fuzzy multi-H_v-ideal of R and $a \in R$. Having $x, a \in x \cdot a$ for all $x \in R$ and Condition 4 of Definition 4 implies that both $CM(x)$ and $CM(a)$ are greater than or equal $CM(x) \vee CM(a)$. Thus, $CM_A(x) = CM_A(a)$ for all $x \in R$. □

Example 7. *Let $(R, +, \cdot)$ be the H_v-ring defined by the following tables:*

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,2\}$
1	$\{0,1\}$	1	$\{1,2\}$
2	R	$\{1,2\}$	$\{1,2\}$

Using Proposition 1, we get that the constant fuzzy multi-H_v-ideal of R is the only fuzzy multi-H_v-ideal of R.

Notation 1. *Let $(R, +, \cdot)$ be an H_v-ring, A be a fuzzy multiset of R and $CM_A(x) = (\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x))$. Then*

- $CM_A(x) = 0$ *if* $\mu_A^1(x) = 0$,
- $CM_A(x) > 0$ *if* $\mu_A^1(x) > 0$,
- $CM_A(x) = \underline{1}$ *if* $CM_A(x) = (\underbrace{1, \ldots, 1}_{s\ times})$ *where*

$$s = \max\{k \in \mathbb{N} : CM_A(y) = (\mu_A^1(y), \mu_A^2(y), \ldots, \mu_A^k(y)), \mu_A^k(y) \neq 0, y \in R\}.$$

Definition 5. *Let $(R, +, \cdot)$ be an H_v-ring and A be a fuzzy multiset of R. Then $A_\star = \{x \in R : CM_A(x) > 0\}$ and $A^\star = \{x \in R : CM_A(x) = \underline{1}\}$.*

Proposition 2. *Let $(R, +, \cdot)$ be an H_v-ring and A be a fuzzy multi-H_v-ideal of R. Then $A_\star$ is either the empty set or an H_v-ideal of R.*

Proof. Let $a \in A_\star \neq \emptyset$. First, we show that $a + A_\star = A_\star + a = A_\star$. We prove $a + A_\star = A_\star$ and $A_\star + a = A_\star$ is done similarly. For all $x \in A_\star$ and $z \in a + x$, we have $CM_A(z) \geq CM_A(a) \wedge CM_A(x) > 0$. The latter implies that $z \in A_\star$ and hence, $A_\star + a \subseteq A_\star$. Moreover, for all $x \in A_\star$, Condition 2 of Definition 4 implies that there exist $y \in R$ such that $x \in a + y$ and $CM_A(y) \geq CM_A(x) \wedge CM_A(a) > 0$. The latter implies that $y \in A_\star$ and $x \in a + A_\star$. Thus, $A_\star \subseteq a + A_\star$. Now, we prove that $R \cdot A_\star \subseteq A_\star$ and $A_\star \cdot R \subseteq A_\star$. We prove that $R \cdot A_\star \subseteq A_\star$ and $A_\star \cdot R \subseteq A_\star$ is done similarly. Let $r \in R$ and $x \in A_\star$. Then for all $z \in r \cdot x$, Condition 4 of Definition 4 implies that $CM(z) \geq CM(r) \vee CM(x) > 0$. Thus, $z \in A_\star$. □

Proposition 3. *Let $(R, +, \cdot)$ be an H_v-ring and A be a fuzzy multi-H_v-ideal of R. Then $A^\star$ is either the empty set or an H_v-ideal of R.*

Proof. Let $a \in A^\star \neq \emptyset$. First, we show that $a + A^\star = A^\star + a = A^\star$. We prove $a + A^\star = A^\star$ and $A^\star + a = A^\star$ is done similarly. For all $x \in A^\star$ and $z \in a + x$, we have $CM_A(z) \geq CM_A(a) \wedge CM_A(x) = \underline{1}$. The latter implies that $z \in A^\star$ and hence, $A^\star + a \subseteq A^\star$. Moreover, for all $x \in A^\star$, Condition 2 of Definition 4 implies that there exist $y \in R$ such that $x \in a + y$ and $CM_A(y) \geq CM_A(x) \wedge CM_A(a) = \underline{1}$. The latter implies that $y \in A^\star$ and $x \in a + A^\star$. Thus, $A^\star \subseteq a + A^\star$. Now, we prove that $R \cdot A^\star \subseteq A^\star$ and $A^\star \cdot R \subseteq A^\star$. We prove that $R \cdot A^\star \subseteq A^\star$ and $A^\star \cdot R \subseteq A^\star$ is done similarly. Let $r \in R$ and $x \in A^\star$. Then for all $z \in r \cdot x$, Condition 4 of Definition 4 implies that $CM(z) \geq CM(r) \vee CM(x) = \underline{1}$. Thus, $z \in A^\star$. □

Example 8. *Let $(R_3, +_3, \cdot_3)$ be the H_v-ring presented in Example 6. Having $A = \{(0.9, 0.7, 0.6, 0.6, 0.1)/d, (0.9, 0.7, 0.6, 0.6, 0.1)/f\}$, $B = \{(0.9, 0.8, 0.8, 0.1)/d\}$ fuzzy multi-H_v-ideals of R_3, we get that $A_\star = \{d, f\}$ and $B_\star = \{d\}$ are H_v-ideals of R_3. Also, $A^\star = B^\star = \emptyset$.*

Notation 2. *Let $(R, +, \cdot)$ be an H_v-ring, A be a fuzzy multiset of R and $CM_A(x) = (\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x))$. We say that $CM_A(x) \geq (t_1, \ldots, t_k)$ if $p \geq k$ and $\mu_A^i(x) \geq t_i$ for all $i = 1, \ldots, k$. If $CM_A(x) \ngeq (t_1, \ldots, t_k)$ and $(t_1, \ldots, t_k) \ngeq CM_A(x)$ then we say that $CM_A(x)$ and $(t_1, \ldots, t_k)$ are not comparable.*

Theorem 1. *Let $(R, +, \cdot)$ be an H_v-ring, A a fuzzy multiset of R with fuzzy count function CM and $t = (t_1, \ldots, t_k)$ where $t_i \in [0, 1]$ for $i = 1, \ldots, k$ and $t_1 \geq t_2 \geq \ldots \geq t_k$. Then A is a fuzzy multi-H_v-ideal of R if and only if CM_t is either the empty set or an H_v-ideal of R.*

Proof. Let CM_t be an H_v-ideal of R and $x, y \in R$. By setting $t_0 = CM(x) \wedge CM(y)$, we get that $x, y \in CM_{t_0}$. Having CM_{t_0} an H_v-ideal of R implies that for all $z \in x + y$, $CM(z) \geq t_0 = CM(x) \wedge CM(y)$. We prove Condition 2 of Definition 4 and Condition 3 is done similarly. Let $a, x \in R$ and $t_0 = CM(x) \wedge CM(a)$. Then $a, x \in CM_{t_0}$. Having CM_{t_0} an H_v-ideal of R implies that $a + CM_{t_0} = CM_{t_0}$. The latter implies that there exist $y \in CM_{t_0}$ such that $x \in a + y$. Thus, $CM(y) \geq t_0 = CM(x) \wedge CM(a)$. We prove now Condition 4 of Definition 4. Let $x, y \in R$ and $z \in x \cdot y$. By setting $t_1 = CM(x)$ and $t_2 = CM(y)$, we get that $x \in CM_{t_1}$ and $y \in CM_{t_2}$. Having $CM_{t_1} \cdot R \subseteq CM_{t_1}$ and $R \cdot CM_{t_2} \subseteq CM_{t_2}$ implies that $z \in CM_{t_1}$ and $z \in CM_{t_2}$. Thus, $CM(z) \geq t_1 \vee t_2 \geq CM(x) \vee CM(y)$.

Conversely, let A be a fuzzy multi-H_v-ideal of R and $CM_t \neq \emptyset$. We need to show that $CM_t = a + CM_t = CM_t + a$ for all $a \in CM_t$. We prove that $CM_t = a + CM_t$ and $CM_t = CM_t + a$ is done similarly. Let $x \in CM_t$. Then $CM(z) \geq CM(x) \wedge CM(a) \geq t$ for all $z \in a + x$. The latter implies that $z \in CM_t$. Thus, $a + CM_t \subseteq CM_t$. Let $x \in CM_t$. Since A is a fuzzy multi-H_v-ideal of R, it follows that there exist $y \in R$ such that $x \in a + y$ and $CM(y) \geq CM(x) \wedge CM(a) \geq t$. The latter implies that $y \in CM_t$ and hence, $CM_t \subseteq a + CM_t$. We prove now that $R \cdot CM_t \subseteq CM_t$ and $CM_t \cdot R \subseteq R$ is done similarly. Let $y \in CM_t$ and $x \in R$. For all $z \in x \cdot y$, Condition 4 of Definition 4 implies that $CM(z) \geq CM(x) \vee CM(y) \geq t$. Thus, $z \in CM_t$. □

Corollary 1. *Let $(R, +, \cdot)$ be an H_v-ring. If R has no proper H_v-ideals then every fuzzy multi-H_v-ideal of R is the constant fuzzy multi-H_v-ideal.*

Proof. Let A be a fuzzy multi-H_v-ideal of R and suppose, to get contradiction, that A is not the constant fuzzy multi-H_v-ideal. Then there exist $x, y \in R$ with $CM(x) \neq CM(y)$. We have three cases for $CM(x) \neq CM(y)$: $CM(x) < CM(y)$, $CM(x) > CM(y)$, and $CM(x)$ and $CM(y)$ are not comparable. If $CM(x) < CM(y)$ then $y \in CM_t$ and $x \notin CM_t$ for $t = CM(y)$. If $CM(x) > CM(y)$ or $CM(x)$ and $CM(y)$ are not comparable, then $x \in CM_t$ and $y \notin CM_t$ for $t = CM(x)$. Using Theorem 1, we get that $CM_t (\neq R)$ is an H_v-ideal of R. □

Proposition 4. *Let $(R, +, \cdot)$ be an H_v-ring and S be an H_v-ideal of R. Then $S = CM_t$ for some $t = (t_1, \ldots, t_k)$ where $t_i \in [0, 1]$ for $i = 1, \ldots, k$ and $t_1 \geq t_2 \geq \ldots \geq t_k$.*

Proof. Let $t = (t_1, \ldots, t_k)$ where $t_i \in [0, 1]$ for $i = 1, \ldots, k$ and define the fuzzy multiset A of R as follows:

$$CM(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $S = CM_t$. We still need to prove that CM is a fuzzy multi-H_v-ideal of R. Using Theorem 1, it suffices to show that $CM_\alpha \neq \emptyset$ is an H_v-ideal of R for all $\alpha = (a_1, \ldots, a_s)$ with $a_i \in [0,1]$ and $a_1 \geq \ldots \geq a_s$ for $i = 1, \ldots, s$. One can easily see that

$$CM_\alpha = \begin{cases} R & \text{if } \alpha = 0 \\ S & \text{if } 0 < \alpha \leq t \\ \emptyset & \text{if } (\alpha > t) \text{ or } (\alpha \text{ and } t \text{ are not comparable}). \end{cases}$$

Thus, CM_α is either the empty set or an H_v-ideal of R. □

Next, we deal with some operations on fuzzy multi-H_v-ideals.

Definition 6. *Let $(R, +, \cdot)$ be an H_v-ring and A, B be fuzzy multisets of R. Then $A \circ B$ is defined by the following fuzzy count function.*

$$CM_{A \circ B}(x) = \vee\{CM_A(y) \wedge CM_B(z) : x \in y + z\}.$$

Theorem 2. *Let $(R, +, \cdot)$ be an H_v-ring and A be a fuzzy multiset of H. If A is a fuzzy multi-H_v-ideal of R then $A \circ A = A$.*

Proof. Let $z \in R$. Then $CM_A(z) \geq CM_A(x) \wedge CM_A(y)$ for all $z \in x + y$. The latter implies that $CM_A(z) \geq \vee\{CM_A(x) \wedge CM_B(y) : z \in x + y\} \geq CM_{A \circ A}(z)$. Thus, $A \circ A \subseteq A$. Having $(R, +, \cdot)$ an H_v-ring and A a fuzzy multi-H_v-idear of R implies that for every $x \in R$ there exist $y \in R$ such that $x \in x + y$ and $CM_A(y) \geq CM_A(x)$. Moreover, we have $CM_{A \circ A}(x) = \vee\{CM_A(y) \wedge CM_B(z) : x \in y + z\} \geq CM_A(x) \wedge CM_A(y) = CM_A(x)$. Thus, $A \subseteq A \circ A$. □

Definition 7. *Let R be a non-empty set and A be a fuzzy multiset of R. We define A', the complement of A, to be the fuzzy multiset defined as: For all $x \in R$,*

$$CM_{A'}(x) = \underline{1} - CM_A(x).$$

Example 9. *Let $R = \{a, b, c\}$ be a set and A be a fuzzy multiset with fuzzy count function CM defined as: $CM(a) = 0, CM(b) = (1,1,1), CM(c) = (0.5, 0.3, 0.1)$. Then $A' = \{(1,1,1)/a, (0.9, 0.7, 0.5)/c\}$.*

Remark 3. *Let $(R, +, \cdot)$ be an H_v-ring and A be the constant fuzzy multi-H_v-ideal of R defined in Example 3. Then A' is also a fuzzy multi-H_v-ideal of R.*

Remark 4. *Let $(R, +, \cdot)$ be an H_v-ring and A be a fuzzy multi-H_v-ideal of R. Then A' is not necessary a fuzzy multi-H_v-ideal of R.*

We illustrate Remark 4 by the following example.

Example 10. *Let the triple $(R_3, +_3, \cdot_3)$ be the H_v-ring defined in Example 6 and $B = \{(0.9, 0.8, 0.8, 0.1)/d\}$ be a fuzzy multi-H_v-ideals of R_3.*

Then $B' = \{(0.9, 0.2, 0.2, 0.1)/d, (1,1,1,1)/e, (1,1,1,1)/f\}$ is not a fuzzy multi-H_v-ideals of R_3. This is clear as $d \in d \cdot e$ and $CM_{B'}(d) \ngeq CM_{B'}(d) \vee CM_{B'}(e) = (1,1,1,1)$.

Proposition 5. *Let $(R_\alpha, +_\alpha, \cdot_\alpha)$ be an H_v-ring with a fuzzy multiset A_α for all $\alpha \in \Gamma$. If A_α is a fuzzy multi-H_v-ideal of R_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} A_\alpha$ is a fuzzy multi-H_v-ideal of the $\prod_{\alpha\in\Gamma} R_\alpha$. Where $CM_{\prod_{\alpha\in\Gamma} A_\alpha}(< x_\alpha >) = \inf_{\alpha\in\Gamma} CM_{A_\alpha}(x_\alpha)$.*

Proof. The proof is straightforward. □

We present an example when $|\Gamma| = 2$.

Example 11. *Let* $(R_1, +_1, \cdot_1)$ *be the* H_v*-ring presented in Example 4 and*

$$A = \{(0.8, 0.6, 0.6, 0.1)/0, (0.5, 0.4, 0.4)/1\}$$

be a fuzzy multi-H_v*-ideal of* R_1*. Then* $A \times A$ *given by:*

$$\{(0.8, 0.6, 0.6, 0.1)/(0,0), (0.5, 0.4, 0.4)/(0,1), (0.5, 0.4, 0.4)/(1,0), (0.5, 0.4, 0.4)/(1,1)\}$$

is a fuzzy multi-H_v*-ideal of* $R_1 \times R_1$*.*

The next two propositions discuss the strong homomorphic image and pre-image of a fuzzy multi-H_v-ideal.

Proposition 6. *Let* $(R_1, +_1, \cdot_1), (R_2, +_2, \cdot_2)$ *be* H_v*-rings, A be a fuzzy multiset of* R_1 *and* $f : R_1 \to R_2$ *be a surjective strong homomorphism. If A is a fuzzy multi-*H_v*-ideal of* R_1 *then* $f(A)$ *is a fuzzy multi-*H_v*-ideal of* R_2*.*

Proof. Let $y_1, y_2 \in R_2$ and $y_3 \in y_1 +_2 y_2$. Since $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$, it follows that there exist $x_1, x_2 \in R_1$ such that $CM_A(x_1) = \bigvee_{f(x)=y_1} CM_A(x)$ and $CM_A(x_2) = \bigvee_{f(x)=y_2} CM_A(x)$. Having f a homomorphism implies that $y_3 \in f(x_1) +_2 f(x_2) = f(x_1 +_1 x_2)$. The latter implies that there exists $x_3 \in x_1 + x_2$ such that $y_3 = f(x_3)$. Since A is a fuzzy multi-H_v-ideal of R_1, it follows that $CM_{f(A)}(y_3) \geq CM_A(x_3) \geq CM_A(x_1) \wedge CM_A(x_2) = CM_{f(A)}(y_1) \wedge CM_{f(A)}(y_2)$. We prove now Condition 2 of Definition 4 and Condition 3 is done similarly. Let $y, b \in R_2$. Since $f^{-1}(y) \neq \emptyset$ and $f^{-1}(b) \neq \emptyset$ then there exist $x_1, a \in R_1$ such that $CM_A(x_1) = \bigvee_{f(x)=y} CM_A(x)$ and $CM_A(a) = \bigvee_{f(x)=b} CM_A(x)$. Having A a fuzzy multi-H_v-ideal of R_1 implies that there exist $x_2 \in R_1$ with $x_1 \in a +_1 x_2$ and $CM_A(x_2) \geq CM_A(x_1) \wedge CM_A(a)$. Since f is a strong homomorphism, it follows that $y = f(x_1) \in f(x_2) +_2 b$ and $CM_{f(A)}(f(x_2)) \geq CM_A(x_2) \geq CM_A(x_1) \wedge CM_A(a) = CM_{f(A)}(y) \wedge CM_{f(A)}(b)$. We prove now Condition 4 of Definition 4 for $f(A)$. Let $y_1, y_2 \in R_2$ and $y_3 \in y_1 \cdot_2 y_2$. Since $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$, it follows that there exist $x_1, x_2 \in R_1$ such that $CM_A(x_1) = \bigvee_{f(x)=y_1} CM_A(x)$ and $CM_A(x_2) = \bigvee_{f(x)=y_2} CM_A(x)$. Having f a strong homomorphism implies that $y_3 \in f(x_1) \cdot_2 f(x_2) = f(x_1 \cdot_1 x_2)$. The latter implies that there exists $x_3 \in x_1 \cdot_1 x_2$ such that $y_3 = f(x_3)$. Since A is a fuzzy multi-H_v-ideal of R_1, it follows that $CM_{f(A)}(y_3) \geq CM_A(x_3) \geq CM_A(x_1) \vee CM_A(x_2) = CM_{f(A)}(y_1) \vee CM_{f(A)}(y_2)$. □

Proposition 7. *Let* $(R_1, +_1, \cdot_1), (R_2, +_2, \cdot_2)$ *be* H_v*-rings, B be a fuzzy multiset of* R_2 *and* $f : R_1 \to R_2$ *be a surjective strong homomorphism. If B is a fuzzy multi-*H_v*-ideal of* R_2 *then* $f^{-1}(B)$ *is a fuzzy multi-*H_v*-ideal of* R_1*.*

Proof. Let $x_1, x_2 \in R_1$ and $x_3 \in x_1 +_1 x_2$. Then $CM_{f^{-1}(B)}(x_3) = CM_B(f(x_3))$. Having $f(x_3) \in f(x_1 +_1 x_2) = f(x_1) +_2 f(x_2)$ implies that $CM_{f^{-1}(B)}(x_3) = CM_B(f(x_3)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2)) = CM_{f^{-1}(B)}(x_1) \wedge CM_{f^{-1}(B)}(x_2)$. We prove now Condition 2 of Definition 4 and Condition 3 is done similarly. Let $x, a \in R_1$. Having $y = f(x), b = f(a) \in R_2$ and B a fuzzy multi-hypergroup of R_2 implies that there exist $z \in R_2$ such that $y \in b +_2 z$ and $CM_B(z) \geq CM_B(y) \wedge CM_B(b)$. Since f is a surjective strong homomorphism, it follows that there exist $w \in R_1$ such that $f(w) = z$ and $x \in a +_1 w$. We get now that $CM_{f^{-1}(B)}(w) = CM_B(z) \geq CM_B(y) \wedge CM_B(b) = CM_{f^{-1}(B)}(x) \wedge CM_{f^{-1}(B)}(a)$. To prove Condition 4 for $f^{-1}(B)$, let $x_3 \in x_1 \cdot_1 x_2$. Then $f(x_3) \in f(x_1) \cdot_2 f(x_2)$. Having $CM_{f^{-1}(B)}(x_3) = CM_B(f(x_3)) \geq CM_B(f(x_1)) \vee CM_B(f(x_2)) = CM_{f^{-1}(B)}(x_1) \vee CM_{f^{-1}(B)}(x_2)$ completes the proof. □

Corollary 2. *Let $(R,+,\cdot)$ be an H_v-ring with fundamental relation $\gamma^\star$ and A be a fuzzy multiset of R. If A is a fuzzy multi-H_v-ideal of R then B is a fuzzy multi-H_v-ideal of $(R/\gamma^\star,\oplus,\odot)$. Where*

$$CM_B(\gamma^\star(x)) = \bigvee_{\alpha\in\gamma^\star(x)} CM_A(\alpha).$$

Proof. Let A be a fuzzy multi-H_v-ideal of R and $f: R\to R/\gamma^\star$ be the map defined by $f(x)=\gamma^\star(x)$. Then f is a surjective homomorphism. Proposition 6 asserts that $f(A)$ is a fuzzy multi-H_v-ideal of $R/\gamma^\star$ where

$$CM_{f(A)}(\gamma^\star(x)) = \bigvee_{f(\alpha)=\gamma^\star(x)} CM_A(\alpha) = \bigvee_{\alpha\in\gamma^\star(x)} CM_A(\alpha) = CM_B(\gamma^\star(x)).$$

Therefore, B is a fuzzy multi-H_v-ideal of $(R/\gamma^\star,\oplus,\odot)$. □

Definition 8. *Let $(R,+,\cdot)$ be a ring. A fuzzy multiset A (with fuzzy count function CM_A) over R is a fuzzy multi-ideal of R if for all $x,y\in R$, the following conditions hold.*

1. *$CM_A(x)\wedge CM_A(y)\leq CM_A(x+y)$ for all $x,y\in R$;*
2. *$CM_A(-x)\geq CM_A(x)$ for all $x\in R$;*
3. *$CM_A(x)\vee CM_A(y)\leq CM_A(x\cdot y)$ for all $x,y\in R$.*

Proposition 8. *Let $(R,+,\cdot)$ be an H_v-ring with fundamental relation $\gamma^\star$ and A be a fuzzy multiset of R. If A is a fuzzy multi-H_v-ideal of R then B is a fuzzy multi-ideal of the ring $(R/\gamma^\star,\oplus,\odot)$. Where*

$$CM_B(\gamma^\star(x)) = \bigvee_{\alpha\in\gamma^\star(x)} CM_A(\alpha).$$

Proof. Corollary 2 asserts that Conditions 1 and 3 of Definition 8 are satisfied. We need to prove Condition 2. Having $(R/\gamma^\star,\oplus,\odot)$ a ring implies that there exist a zero element, say $\overline{0}$ such that $\overline{0}\oplus\gamma^\star(x)=\gamma^\star(x)\oplus\overline{0}=\gamma^\star(x)$ and $\overline{0}\odot\gamma^\star(x)=\gamma^\star(x)\odot\overline{0}=\overline{0}$ for all $\gamma^\star(x)\in R/\gamma^\star$. Having B a fuzzy multi-H_v-ideal of $(R/\gamma^\star,\oplus,\odot)$ implies that $CM_B(\overline{0})\geq CM_B(\gamma^\star(x))$ for all $\gamma^\star(x)\in R/\gamma^\star$. Since $(R/\gamma^\star,\oplus,\odot)$ a ring, it follows that for every $\gamma^\star(x)\in R/\gamma^\star$ there exist $-\gamma^\star(x)\in R/\gamma^\star$ with $-\gamma^\star(x)\oplus\gamma^\star(x)=\overline{0}$. Having B a fuzzy multi-H_v-ideal of $(R/\gamma^\star,\oplus,\odot)$ and using Condition 2 of Definition 4 implies that for $\gamma^\star(x)$ and $\overline{0}$ there exists $\gamma^\star(y)$ such that $\overline{0}\in\gamma^\star(x)\oplus\gamma^\star(y)$ and $CM_B(\gamma^\star(y))\geq CM_B(\overline{0})\wedge CM_B(\gamma^\star(x))=CM_B(\gamma^\star(x))$. It is clear that $\gamma^\star(y)=-\gamma^\star(x)$. □

Example 12. *Let $(R_3,+_3,\cdot_3)$ be the H_v-ring presented in Example 6. One can easily see that the fundamental ring $R_3/\gamma^\star=\{\gamma^\star(d),\gamma^\star(e)\}$ and is isomorphic to the ring of integers under standard addition and multiplication modulo 2. Using Proposition 8, we get that $\{(0.9,0.7,0.6,0.6,0.1)/\gamma^\star(d)\}$ is a fuzzy multi-ideal of $R_3/\gamma^\star$.*

4. Generalized Fuzzy Multi-H_v-Ideal

In this section, we generalize the notion of fuzzy multi-H_v-ideal defined in Section 3 to generalized fuzzy multi-H_v-ideal, investigate its properties, and present some examples.

Notation 3. *Let A be a fuzzy multiset of a non-empty set R with a fuzzy count function CM. We say that:*

1. *$x_t\in CM$ when $CM(x)\geq t$,*
2. *$x_t\in qCM$ when $CM(x)+t\geq \underline{1}$,*
3. *$x_t\in\vee qCM$ when $x_t\in CM$ or $x_t\in qCM$,*

4. $\underline{0.5} = (\underbrace{0.5, \ldots, 0.5}_{s\ times})$ *where*

$$s = \max\{k \in \mathbb{N} : CM_A(y) = (\mu_A^1(y), \mu_A^2(y), \ldots, \mu_A^k(y)), \mu_A^k(y) \neq 0, y \in R\}.$$

Definition 9. *Let $(R, +, \cdot)$ be an H_v-ring. A fuzzy multiset A (with fuzzy count function CM) over R is an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R if for all $x, y \in R$, $0 \leq t, r \leq \underline{1}$, the following conditions hold.*

1. $x_t \in CM$, $y_r \in CM$ *implies* $z_{t \wedge r} \in \vee qCM$ *for all* $z \in x + y$;
2. $x_t \in CM$, $a_r \in CM$ *implies* $y_{t \wedge r} \in \vee qCM$ *for some* $y \in R$ *with* $x \in a + y$;
3. $x_t \in CM$, $a_r \in CM$ *implies* $z_{t \wedge r} \in \vee qCM$ *for some* $z \in R$ *with* $x \in z + a$;
4. $y_t \in CM$, $x \in R$ *implies* $z_t \in \vee qCM$ *for all* $z \in x \cdot y$
   ($x_t \in CM$, $y \in R$ *implies* $z_t \in \vee qCM$ *for all* $z \in x \cdot y$).

Remark 5. *Let $(R, +, \cdot)$ be an H_v-ring and A a fuzzy multiset of R. If A is a fuzzy multi-H_v-ideal of R then A is an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R.*

Example 13. *Let $(R, +, \cdot)$ be any H_v-ring. Then the constant fuzzy multiset of R is an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R.*

Example 14. *Let $(R_1, +_1, \cdot_1)$ be the H_v-ring presented in Example 4. Having $A = \{(0.8, 0.6, 0.6, 0.1)/0, (0.5, 0.4, 0.4)/1\}$ is a fuzzy multi-H_v-ideal of R_1 implies that $A = \{(0.8, 0.6, 0.6, 0.1)/0, (0.5, 0.4, 0.4)/1\}$ is an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R_1.*

The converse of Remark 5 does not always hold. We illustrate this idea by the following example.

Example 15. *Let $(R, +, \cdot)$ be the H_v-ring defined by the following tables:*

$+$	a	b	c	d
a	a	b	c	d
b	b	$\{a, b\}$	d	c
c	c	d	$\{a, c\}$	b
d	d	c	b	$\{a, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

One can easily see that

$$A = \{(0.7, 0.6, 0.5)/a, (0.9, 0.8, 0.8)/b, (0.9, 0.8, 0.8)/c, (0.9, 0.8, 0.8)/d\}$$

is an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R but not a fuzzy multi-H_v-ideal of R. This is clear as $a \in a \cdot b$ but $CM_A(a) \ngeq CM_A(a) \vee CM_A(b)$.

Proposition 9. *Let $t = (t_1, \ldots, t_k), s = (s_1, \ldots, s_p)$ with $t_1 \geq \ldots \geq t_k$ and $s_1 \geq \ldots \geq s_p$. If $t < s$ then there exists $r = (r_1, \ldots, r_m)$ such that $t < r < s$.*

Proof. We have the following cases:
Case $k < p$. Take $r = (s_1, \dots, s_p, \frac{s_{p+1}}{2})$.
Case $k = p$. Then there exists $i \in \{1, \dots, k\}$ with $t_i < s_i$. Since s_i, t_i are real numbers, it follows that there exists a real number r_i with $t_i < r_i < s_i$. By taking $r = (t_1, \dots, t_{i-1}, r_i, r_i \wedge s_{i+1}, \dots, r_i \wedge s_k)$, we get that $t < r < s$. □

Theorem 3. *Let $(R, +, \cdot)$ be an H_v-ring, A a fuzzy multiset of R with fuzzy count function CM, and for all $x \in R$, $CM(x)$ and $\underline{0.5}$ are comparable. If A is an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R then the following conditions hold:*

(a) $CM(x) \wedge CM(y) \wedge \underline{0.5} \leq CM(z)$ for all $z \in x + y$;

(b) For all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and

$$CM(a) \wedge CM(x) \wedge \underline{0.5} \leq CM(y).$$

(c) For all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and

$$CM(a) \wedge CM(x) \wedge \underline{0.5} \leq CM(z).$$

(d) For all $z \in x \cdot y$, $CM(y) \wedge \underline{0.5} \leq CM(z)$ and $CM(x) \wedge \underline{0.5} \leq CM(z)$.

Proof. It suffices to show that $(1) \to (a)$, $(2) \to (b)$, $(3) \to (c)$, and $(4) \to (d)$.

(1) → (a): Let $x, y \in R$. Since each of $CM(x), CM(y)$ are comparable with $\underline{0.5}$, we can consider the cases: $CM(x) \wedge CM(y) < \underline{0.5}$ and $CM(x) \wedge CM(y) \geq \underline{0.5}$.

For the case $CM(x) \wedge CM(y) < \underline{0.5}$, suppose that there exists $z \in x + y$ with $CM(z) < CM(x) \wedge CM(y) \wedge \underline{0.5}$. We get that $CM(z) < CM(x) \wedge CM(y)$. Proposition 9 asserts that there exists r with $CM(z) < r < CM(x) \wedge CM(y)$. The latter implies that $x_r, y_r \in CM$ and $z \notin CM$. Moreover, having $CM(z) + r < \underline{0.5} + r \leq \underline{1}$ implies that $z \notin qCM_r$. We get that $z_r \notin \vee qCM$ which contradicts (1).

For the case $CM(x) \wedge CM(y) \geq \underline{0.5}$, suppose that there exists $z \in x + y$ with $CM(z) < CM(x) \wedge CM(y) \wedge \underline{0.5}$. We get that $x_{\underline{0.5}}, y_{\underline{0.5}} \in CM$ and $CM(z) < \underline{0.5}$. It is clear that $z_{0.5} \notin \vee qCM$ which contradicts (1).

(2) → (b): Let $x, a \in R$. Since each of $CM(x), CM(a)$ are comparable with $\underline{0.5}$, we can consider the cases: $CM(x) \wedge CM(a) < \underline{0.5}$ and $CM(x) \wedge CM(a) \geq \underline{0.5}$.

For the case $CM(x) \wedge CM(a) < \underline{0.5}$, suppose that for all $y \in R$ with $x \in a + y$ we have $CM(x) \wedge CM(a) = CM(x) \wedge CM(a) \wedge \underline{0.5} > CM(y)$. Proposition 9 asserts that there exists r with $CM(y) < r < CM(x) \wedge CM(a)$. It is clear that $x_r, y_r \in CM$ and $y_r \notin \vee qCM$. The latter contradicts (2).

(3) → (c): This case is done in a similar manner to that of **(2) → (b)**.

(4) → (d): Let $x, y \in R$. Since $CM(y)$ is comparable with $\underline{0.5}$, we can consider the cases: $CM(y) < \underline{0.5}$ and $CM(y) \geq \underline{0.5}$.

For the case $CM(y) < \underline{0.5}$, suppose that there exists $z \in x \cdot y$ with $CM(z) < CM(y) \wedge \underline{0.5} < CM(y)$. Proposition 9 asserts that there exists r with $CM(z) < r < CM(y)$. Then $y_r \in CM$ and $z_r \notin \vee qCM$ which contradicts (4).

For the case $CM(y) \geq \underline{0.5}$, suppose that there exists $z \in x \cdot y$ with $CM(z) < CM(y) \wedge \underline{0.5} \leq \underline{0.5}$. Then $y_{\underline{0.5}} \in CM$ and $z_{\underline{0.5}} \notin \vee qCM$ which contradicts (4). □

Remark 6. *Theorem 3 can be used only when $CM(x)$ and $\underline{0.5}$ are comparable. Otherwise, we should use Definition 9.*

Note that according to Remark 6, we can not use Theorem 3 to the fuzzy multiset given in Example 5 as $CM_A(0) = (0.9, 0.7, 0.6, 0.6, 0.1)$ is not comparable with $\underline{0.5} = (0.5, 0.5, 0.5, 0.5, 0.5)$.

Remark 7. *In case of fuzzy H_v-ideal of an H_v-ring, the conditions of Theorem 3 are necessary and sufficient for a fuzzy set to be a fuzzy H_v-ideal (see [16]). Whereas in our case (fuzzy multiset), the converse of Theorem 3 is not always true. (See Example 16.)*

Example 16. *Let $(R,+,\cdot)$ be the H_v-ring defined in Example 15 and let A be the fuzzy multiset of R with count function CM defined by: $CM(a) = (0.7, 0.6, 0.5)$, $CM(b) = (0.7, 0.5, 0.5)$, $CM(c) = CM(d) = (0.6, 0.6, 0.6)$. Having $\underline{0.5} = (0.5, 0.5, 0.5)$, it is easy to see that Conditions (a), (b), (c), and (d) of Theorem 3 are satisfied. But A is not an $(\in, \in \vee q)$-fuzzy multi-H_v-ideal of R. By taking $t = (0.6, 0.6, 0.3)$, we get that $c_t \in CM$. Having $b \in b \cdot c$, $CM(b) \not\geq t$ and $CM(b) + t = (1.3, 1.1, 0.8) \not\geq \underline{1}$ implies that Condition 4 of Definition 9 is not satisfied.*

5. Conclusions

This paper has introduced algebraic hyperstructures of fuzzy multisets, for the first time, in the forms of fuzzy multi-H_v-ideals and generalized fuzzy multi-H_v-ideals. Several interesting properties related to the new defined notions were investigated and operations on fuzzy multi-H_v-ideals were defined and discussed. It is well known that the concept of fuzzy multiset is well established in dealing with many real life problems. As a result, we can deal with real life problems involving the concept of fuzzy multiset with a different perspective.

Author Contributions: Conceptualization, M.A.T., S.H.-M. and B.D.; methodology, M.A.T. and S.H.-M.; formal analysis and investigation, M.A.T. and S.H.-M.; resources, M.A.T. and S.H.-M.; writing–original draft preparation, M.A.T.; writing–review and editing, M.A.T. and S.H.-M.; supervision, B.D.; project administration and funding acquisition, S.H.-M.

Funding: This research was supported and the APC was funded within the project DZRO K217, supported by the Ministry of Defence in the Czech Republic.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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