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Integrability Properties of Cubic Liénard Oscillators with Linear Damping

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Abstract: We consider a family of cubic Liénard oscillators with linear damping. Particular cases of this family of equations are abundant in various applications, including physics and biology. There are several approaches for studying integrability of the considered family of equations such as Lie point symmetries, algebraic integrability, linearizability conditions via various transformations and so on. Here we study integrability of these oscillators from two different points of view, namely, linearizability via nonlocal transformations and the Darboux theory of integrability. With the help of these approaches we find two completely integrable cases of the studied equation. Moreover, we demonstrate that the equations under consideration have a generalized Darboux first integral of a certain form if and only if they are linearizable.

Keywords: Liénard equations; integrability conditions; nonlocal transformations; Darboux polynomials

1. Introduction

In this work we study the following family of Liénard equations

$$y_{zz} + (a_1 y + a_0) y_z + b_3 y^3 + b_2 y^2 + b_1 y + b_0 = 0, \tag{1}$$

where $a_1 \neq 0$, $b_3 \neq 0$ and a_0, $b_i, i = 0, 1, 2$ are arbitrary parameters.

This family of equations has a lot of applications applications. For example, the travelling wave reduction of the Burgers–Huxley equation ([1–3]) belongs to family (1). The generalized modified Emden equation [4,5] is also a particular case of (1). While various particular analytical solutions of (1) have been studied (see, e.g., [1–3,6] and references therein), to the best of our knowledge, a complete analysis of the integrability of (1) has not been carried out yet.

Therefore, the main aim of this work is to study various aspects of the integrability of (1) and their interconnections and find new completely integrable cases of (1). There are different approaches for studying integrability of nonlinear oscillators like point symmetries, algebraic integrability, local and nonlocal equivalence problems (see, e.g., [6–10]) and references therein). In this work we apply two of them to study (1). First, we consider linearizability conditions for (1) via different classes of nonlocal transformations. We show that such conditions allow us to find Liuvillian integrable subfamilies of (1) or subfamilies admitting a non-autonomous first integral. We also demonstrate that the general solution for the linearizable cases can be constructed in the parametric form. Second, we use the Darboux theory of integrability, which is a powerful tool for constructing and classifying first integrals of ordinary differential equations. The main objects in the Darboux theory are invariant algebraic curves (or Darboux polynomials) and exponential factors [11,12]. The knowledge of the complete set of these invariants allows one to derive necessary and sufficient conditions of Darboux and Liouvillian integrability.

Notice that, without the loss of generality, one can assume that $a_1 = 1$ and $a_0 = 0$ in (1). This can be done via scalings and shifts in the dependent variable. Thus, further, we consider the equation

$$y_{zz} + yy_z + b_3y^3 + b_2y^2 + b_1y + b_0 = 0. \tag{2}$$

The rest of this work is organised as follows. In the next section we consider linearizability conditions for (2) via nonlocal transformations and provide new integrable cases of (2). In Section 3 we present the general structure of irreducible Darboux polynomials related to Equation (2). In addition, in Section 3 we classify cubic Liénard oscillators with linear damping possessing generalised Darboux first integrals of a special form. In the last section we briefly summarise and discuss our results.

2. Linearization via Nonlocal Transformations

In this section we consider equivalence criteria between (2) and the damped harmonic oscillator

$$w_{\zeta\zeta} + \beta w_\zeta + \alpha w = 0, \tag{3}$$

with equivalence transformations given by

$$w = F(z, y), \quad d\zeta = G(z, y)dz. \tag{4}$$

Here we suppose that $\beta \neq 0$ and α are arbitrary parameters and $F_yG \neq 0$.

Transformations (4) are often called the generalised Sundman transformations (see, e.g., [6,13–16] and references therein). Linearization of second-order differential equations via (4) was considered for the first time in [13], where the Laguerre normal form of (3) was used. On the other hand, in [14] it was shown that it is not sufficient to consider the Laguerre form of (3) and transformation of second-order differential equations into (3) was studied. However, in [14] only restricted case of transformations (4), namely $F_z = 0$, was considered. Below, we find linearization conditions for (2) without this restriction and show that they lead to a new non-trivial integrable case of (2).

Equivalence between (2) and (3) via (4) not only allows us to find the general solution of the former in the parametric form, but also to construct a first integral for (2). Indeed, although obvious first integral for (3) is nonautonomous, this equation has a less well-known autonomous first integral

$$I = \left(2w_\zeta + (\beta + \sqrt{\beta^2 - 4\alpha})w\right)^{\frac{\sqrt{\beta^2-4\alpha}}{\beta}+1} \left(2w_\zeta + (\beta - \sqrt{\beta^2 - 4\alpha})w\right)^{\frac{\sqrt{\beta^2-4\alpha}}{\beta}-1}. \tag{5}$$

This first integral can be used in combination with (4) to provide a first integral for (2). Let us remark that we assume that $\beta^2 - 4\alpha \neq 0$, since otherwise first integral (5) degenerates. This particular case should be considered elsewhere.

It is worth noting that whether we obtain an autonomous or non-autonomous first integral for (2) from (5) depends on transformations (4) and the existence of an autonomous first integral leads to a Liuvillian integrable equation from family (2). It can be easily shown that (4) keep (5) autonomous if and only if $F_z = G_z = 0$. Thus, further, we consider two cases of transformations (4) separately. First we study the case of $F_z = G_z = 0$ and then we consider the general case of (4).

First we proceed to case of (2) with an autonomous first integral:

Theorem 1. *Equation (2) can be transformed into (3) via (4) with $F_z = G_z = 0$ if and only if*

$$b_0 = b_2 = 0, \quad b_3 = \frac{\alpha}{2\beta^2}, \tag{6}$$

b_1 is arbitrary and

$$F = \alpha y^2 + 2\beta^2 b_1, \quad G = \frac{y}{\beta}. \tag{7}$$

Proof. Let us start with the necessary conditions. Substituting transformations (4) with $F_z = G_z = 0$ into (3) we get

$$y_{zz} + f y_z + g = 0, \tag{8}$$

if the following relation holds

$$GF_{yy} - F_y G_y = 0. \tag{9}$$

Here

$$f = \beta G, \quad g = \frac{\alpha F G^2}{F_y}. \tag{10}$$

Thus, Equation (2) can be transformed into (3) via (4) with $F_z = G_z = 0$ if it is of the form (8). Requiring that $f = y$ and $g = b_3 y^3 + b_2 y^2 + b_1 y + b_0$ and taking into account condition (9) we obtain an overdeterminated system of ordinary differential equations for F and G. The compatibility conditions for this system lead to (6). This completes the proof. □

As a consequence we have that the Liénard equation

$$y_{zz} + y y_z + \frac{\alpha}{2\beta^2} y^3 + b_1 y = 0, \tag{11}$$

is Liuvillian integrable with the following first integral

$$I = \left(4\alpha\beta y_z + (\beta + \sqrt{\beta^2 - 4\alpha})(\alpha y^2 + 2\beta^2 b_1)\right)^{\frac{\sqrt{\beta^2-4\alpha}}{\beta}+1} \times \left(4\alpha\beta y_z + (\beta - \sqrt{\beta^2 - 4\alpha})(\alpha y^2 + 2\beta^2 b_1)\right)^{\frac{\sqrt{\beta^2-4\alpha}}{\beta}-1}. \tag{12}$$

Let us remark that the general solution of (11) can be obtained in the parametric form with the help of (4) and the general solution of (3). On the other hand, since we have autonomous first integral (12), one does not need to explicitly present the general solution of (11).

Now we consider the non-autonomous case of transformations (4), i.e., we assume that $|F_z|^2 + |G_z|^2 \neq 0$. This results in the following family of equations from (2) admitting a first integral:

Theorem 2. *Equation (2) can be transformed into (4) via (2) if and only if*

$$b_3 = \frac{\alpha}{2\beta^2}, \quad b_2 = \frac{(9\alpha - 2\beta^2)\nu}{\beta^2} \quad b_1 = \frac{2\nu^2(27\alpha - 8\beta^2)}{\beta^2}, \quad b_0 = \frac{12\nu^3(9\alpha - 2\beta^2)}{\beta^2}, \tag{13}$$

where ν is arbitrary constant and

$$F = (6\nu + y)^2 e^{-4\nu z}, \quad G = \frac{y}{\beta} + \frac{6\nu}{\beta}. \tag{14}$$

Proof. The proof is similar to those of Theorem 1 and, therefore, is omitted. □

Consequently, the following family of equations

$$y_{zz} + y y_z + \frac{\alpha}{2\beta^2} y^3 + \frac{(9\alpha - 2\beta^2)\nu}{\beta^2} y^2 + \frac{2\nu^2(27\alpha - 8\beta^2)}{\beta^2} y + \frac{12\nu^3(9\alpha - 2\beta^2)}{\beta^2} = 0, \tag{15}$$

has the first integral

$$I = \mathrm{e}^{\frac{-8\nu\sqrt{\beta^2-4\alpha}}{\beta}z}\left[\beta(4y_z+y^2+4\nu y-12\nu^2)+\sqrt{\beta^2-4\alpha}(6\nu+y)^2\right]^{\frac{\sqrt{\beta^2-4\alpha}}{\beta}+1}\times \\ \left[\beta(4y_z+y^2+4\nu y-12\nu^2)-\sqrt{\beta^2-4\alpha}(6\nu+y)^2\right]^{\frac{\sqrt{\beta^2-4\alpha}}{\beta}-1}. \tag{16}$$

The general solution of (15) can be obtained in the parametric form with the help of transformations (4) with (14) as follows

$$y = \pm e^{2\nu z}\sqrt{w} - 6\nu, \quad z = \frac{1}{2\nu}\ln\left\{\pm 2\nu\int\frac{\beta d\zeta}{\sqrt{w}}\right\}, \tag{17}$$

where w is the general solution of (3).

Let us briefly discuss the results of this section. We believe that Theorem 2 provides a new integrable case of (2) since linearization problem for (2) via (4) with $F_z \neq 0$ has not been considered previously. Although linearization via an autonomous case of (4) was studied (see, e.g., [14,15]), the corresponding integral of (3) has not been used for the construction of a first integral for (2) and, thus, we also believe that Liuvillian integrable case of (2) is presented for the first time.

3. Darboux Polynomials

In this section our aim is to study the problem of finding Darboux polynomials for differential equations given by (1). Rewriting these equations as an equivalent dynamical system, we obtain

$$\begin{cases} y_z = w, \\ w_z = -yw - (b_3y^3 + b_2y^2 + b_1y + b_0). \end{cases} \tag{18}$$

All the parameters are supposed to be form the field $\mathbb{C}$ and $b_3 \neq 0$. Let us note that cubic Liénard dynamical systems with linear dumping belong to Liénard dynamical systems of type $(m, m+1)$, which are degenerate in a certain sense. These degeneracy results form the complicated local structure of their solutions. In what follows we shall discuss this fact in detail.

A polynomial $F(y,w) \in \mathbb{C}[y,w] \setminus \mathbb{C}$ is called a Darboux polynomial of system (18) if it satisfies the following equation

$$wF_y - (yw + b_3y^3 + b_2y^2 + b_1y + b_0)F_w = \lambda(y,w)F, \tag{19}$$

where $\lambda(y,w) \in \mathbb{C}[y,w]$ is refereed to as the cofactor. The degree of $\lambda(y,w)$ is at most two. By $\mathbb{C}[y,w]$ we denote the ring of polynomials in variables y and w with coefficients in the field $\mathbb{C}$. The zero set of the Darboux polynomial $F(y,w)$ defines an invariant algebraic curve of the corresponding dynamical system. It is known that Darboux polynomials are of great importance if one studies the integrability problem and wants to derive all independent first integrals that are Darboux or Liouvillian functions [11,12].

An effective method of finding and classifying Darboux polynomials is the method of Puiseux series introduced in articles [10,17]. The main idea of this method is to use the representation of Darboux polynomials in the field of Puiseux series. It is known that any solutions of the equation $F(y,w) = 0$ viewed as an implicit equation can be locally represented by means of Puiseux series [18]. A Puiseux series around the point $y = \infty$ can be defined as

$$w(y) = \sum_{l=0}^{+\infty} c_l y^{\frac{l_0-l}{n_0}}, \tag{20}$$

where $l_0 \in \mathbb{Z}$ and $n_0 \in \mathbb{N}$. The set of all the Puiseux series forms an algebraically closed field, which we shall denote by $\mathbb{C}_\infty\{y\}$. Privileging the variable w with respect to the variable y, we see that the function $w(y)$ satisfies the following first-order ordinary differential equation

$$ww_y + yw + b_3y^3 + b_2y^2 + b_1y + b_0 = 0. \tag{21}$$

It is straightforward to prove ([10], Lemma 2.1) that Darboux polynomials of dynamical system (18) capture Puiseux series satisfying this differential equation. Let us introduce the operator of projection $\{S(y,w)\}_+$ that gives the polynomial part of the expression $S(y,w)$ being a polynomial in y with coefficients from the field $\mathbb{C}_\infty\{y\}$.

The structure of Darboux polynomials related to dynamical system (18) essentially depends on the properties of the following quadratic equation

$$p^2 - \delta p + 2\delta = 0, \tag{22}$$

where the parameter δ is given by

$$\delta = \frac{8b_3 - 1}{b_3}. \tag{23}$$

In what follows by p_1 and p_2 we shall denote the roots of Equation (22). In addition, $\mathbb{Q}^+$ will stand for positive rational numbers.

Theorem 3. *Let $F(y,w)$ with $F(y,w) \in \mathbb{C}[y,w] \setminus \mathbb{C}$ and $F_y \not\equiv 0$ be an irreducible Darboux polynomial of dynamical system (18). The following possibilities take place:*

1. *if p_1, $p_2 \notin \mathbb{Q}^+ \cup \{0\}$, then the polynomial $F(y,w)$ is of degree at most two (with respect to y) and*

$$\begin{gathered} F(y,w) = \left\{ \left\{ w - w^{(1)}(y) \right\}^{s_1} \left\{ w - w^{(2)}(y) \right\}^{s_2} \right\}_+, \\ w^{(k)}(y) = \sum_{m=0}^{\infty} c_m^{(k)} y^{2-m}, \quad c_0^{(k)} = \frac{1}{p_k - 4}, \quad k = 1,2, \end{gathered} \tag{24}$$

   where s_1 and s_2 are either 0 or 1 independently, $s_1 + s_2 > 0$ and all the Puiseux series $w^{(j)}(y)$, which are in fact Laurent series, possess uniquely determined coefficients;

2. *if $p_k \in \mathbb{Q}^+$, $p_l \notin \mathbb{Q}^+$, where either $k = 1$, $l = 2$ or $k = 2$, $l = 1$, then the polynomial $F(y,w)$ takes the form*

$$\begin{gathered} F(y,w) = \left\{ \prod_{j=1}^{N_k} \left\{ w - w_j^{(k)}(y) \right\} \left\{ w - w^{(l)}(y) \right\}^{s_l} \right\}_+, \\ w_j^{(k)}(y) = \sum_{m=0}^{\infty} c_{m,j}^{(k)} y^{2-\frac{m}{n_k}}, \quad w^{(l)}(y) = \sum_{m=0}^{\infty} c_m^{(2)} y^{2-m}, \\ c_{0,j}^{(k)} = \frac{1}{p_k - 4}, \quad c_0^{(l)} = \frac{1}{p_l - 4}, \end{gathered} \tag{25}$$

   where s_l is either 0 or 1, the Puiseux series $w^{(l)}(y)$, which is in fact Laurent series, possesses uniquely determined coefficients and the Puiseux series $w_j^{(k)}(y)$ possess pairwise distinct coefficients $c_{n_k p_k, j}^{(k)}$, the number n_k is defined as $p_k = q_k/n_k$, where q_k, $n_k \in \mathbb{N}$, $(q_k, n_k) = 1$;

3. *if $p_1 \in \mathbb{Q}^+$, $p_2 \in \mathbb{Q}^+$, then the polynomial $F(y,w)$ takes the form*

$$\begin{gathered} F(y,w) = \left\{ \prod_{j=1}^{N_1} \left\{ w - w_j^{(1)}(y) \right\} \prod_{j=1}^{N_2} \left\{ w - w_j^{(2)}(y) \right\} \right\}_+, \\ w_j^{(1)}(y) = \sum_{m=0}^{\infty} c_{m,j}^{(1)} y^{2-\frac{m}{n_1}}, \; w_j^{(2)}(y) = \sum_{m=0}^{\infty} c_{m,j}^{(2)} y^{2-\frac{m}{n_2}}, \; c_{0,j}^{(k)} = \frac{1}{p_k - 4}, \; k = 1,2, \end{gathered} \tag{26}$$

where $N_1 + N_2 > 0$, the Puiseux series $w_j^{(k)}(y)$ possess pairwise distinct coefficients $c_{n_k p_k, j}^{(k)}$, the number n_k is defined as $p_k = q_k/n_k$, where $q_k, n_k \in \mathbb{N}$, $(q_k, n_k) = 1$, $k = 1, 2$;

4. *if $p_1 = p_2 = 0$, then the polynomial $F(y, w)$ takes the form*

$$F(y,w) = w + \frac{1}{4}y^2 + 4b_2 y + 2(b_1 + 16b_2^2) \tag{27}$$

and exists provided that $b_0 = -8b_2(b_1 + 16b_2^2)$.

Proof. Substituting $\lambda(y,w) = \lambda_0(y)w^l$, $F(y,w) = \mu(y)w^N$ with l, $N \in \mathbb{N} \cup \{0\}$, $0 \le l \le 2$ into Equation (19) and balancing the highest-order terms, we conclude that $\mu(y) \in \mathbb{C}$, $l = 0$, and $N \in \mathbb{N}$. This means that cofactors of Darboux polynomials do not depend on w and there are no Darboux polynomials independent on w. In addition, we observe that the highest-order coefficient (with respect to w) of $F(y,w)$ is a constant. Without loss of generality we set $\mu = 1$. Let us suppose that $F(y,w)$ with $F(y,w) \in \mathbb{C}[y,w] \setminus \mathbb{C}$ and $F_y \not\equiv 0$ is an irreducible Darboux polynomial of dynamical system (18).

Now let us perform the classification of Puiseux series near the point $y = \infty$ that satisfy Equation (21). For this aim we shall use the Painlevé methods and the power geometry [19,20]. There exists only one dominant balance producing asymptotics near the point $y = \infty$. This balance and power solution of the corresponding ordinary differential equation take the form

$$ww_y + yw + b_3 y^3 = 0: \quad w^{(k)}(y) = c_0^{(k)} y^2, \quad k = 1, 2, \tag{28}$$

where the coefficients $c_0^{(1,2)}$ satisfy the following equation $2c_0^2 + c_0 + b_3 = 0$. Calculating the Gâteaux derivative of the balance at its power solutions yields the following equation for the Fuchs indices $pc_0 - 4c_0 - 1 = 0$. Expressing c_0 from this equation and substituting the result into the equation $2c_0^2 + c_0 + b_3 = 0$, we get (22). It is known that starting from power asymptotics it is possible to derive asymptotic series possessing these asymptotics as leading-order terms [19,20].

If Equation (22) does not have positive rational solutions, then the Puiseux series related to asymptotics (28) possess uniquely determined coefficients. Since the number of distinct Puiseux series near the point $y = \infty$ satisfying Equation (21) is finite, it follows from the results of article ([10], Theorem 1.4) that the degrees (with respect to w) of irreducible Darboux polynomials is bounded by 2. Using the fact the filed $\mathbb{C}_\infty\{y\}$ is algebraically closed, we represent the polynomials as given in (24).

Further, if one of the solutions of Equation (22) defining the Fuchs indices is a positive rational number and another one is not, then the Puiseux series related to the former case possesses an arbitrary coefficient provided that the compatibility condition for this Fuchs index is satisfied. Another Puiseux series possesses uniquely determined coefficients. As a result, we obtain factorization (25). Since the Darboux polynomial is irreducible, the coefficients $c_{n_k p_k, j}^{(k)}$ corresponding to the positive rational Fuchs index should be pairwise distinct. The number n_k can be obtained from the relation $p_k = q_k/n_k$, where $q_k, n_k \in \mathbb{N}$, $(q_k, n_k) = 1$.

If both solutions of Equation (22) are positive rational numbers, then the Puiseux series have arbitrary coefficients and exist whenever the corresponding compatibility conditions for the Fuchs indices hold. We get factorization (26). Since the Darboux polynomials are irreducible, we conclude that the coefficients with the same upper index $c_{n_k p_k, j}^{(k)}$, $k = 1, 2$ should be pairwise distinct. The numbers n_k, $k = 1, 2$ are found similarly to the previous case.

Finally, we are left with the case $c_0^{(1)} = c_0^{(2)} = -1/4$. In such a situation the left hand side in Equation (22) is identically zero and $b_3 = 1/8$. Making the substitution $w(y) = -y^2/4 + h(y)$ in ordinary differential Equation (21), we get

$$\left(h - \frac{1}{4}y^2\right) h_y + \frac{1}{4}yh + b_2 y^2 + b_1 y + b_0 = 0. \tag{29}$$

Carrying out the classification of Puiseux series near the point $y = \infty$ solving this equation, we obtain the unique series

$$w(y) = \sum_{m=0}^{\infty} c_m y^{2-m}, \quad c_0 = -\frac{1}{4} \tag{30}$$

provided that $b_2 \neq 0$. In the case $b_2 = 0$, $b_1 \neq 0$ the corresponding Puiseux series is also of the form (30) but with $c_1 = 0$. Let us consider the case $b_2 = 0$ and $b_1 = 0$. In such a situation series (30) has an arbitrary coefficient c_2. Let us introduce the new variable

$$u = w + \frac{1}{4}y^2 - c_1 y. \tag{31}$$

There exists the one-to-one correspondence between the Darboux polynomial $F(y, w)$ and the polynomial $G(y, u) = F(y, u - y^2/4 + c_1 y)$. If $F(y, w)$ is irreducible in the ring $\mathbb{C}[y, w]$, then so is $G(y, u)$ in the ring $\mathbb{C}[y, u]$. Using our results, we can represent $G(y, u)$ over the field $\mathbb{C}_\infty\{y\}$ as

$$G(y, u) = \left\{ \prod_{j=1}^{N} (u - c_{2,j}) \right\}. \tag{32}$$

Analysing this representation, we conclude that $G(y, u)$ is irreducible provided that $G(y, u) = u - c_2$. Consequently, irreducible Darboux polynomials of dynamical system (18) in the case $c_0^{(1)} = c_0^{(2)} = -1/4$ exist provided that the corresponding series terminates at the zero term. In all these situations dynamical system (18) possesses Darboux polynomials whenever Equation (21) admits a polynomial solution. Substituting $w(y) = -y^2/4 + c_1 y + c_2$ into the equation in question, we find only one polynomial solution $w(y) = -y^2/4 - 4b_2 x - 2(b_1 + 16b_2^2)$ existing under the condition $b_0 = -8b_2(b_1 + 16b_2^2)$. This completes the proof. □

Remark 1. *If $p_k \in \mathbb{N}$ in the case of representation (25) and the compatibility condition for the Puiseux series $w_j^{(k)}(y)$ to exist is not satisfied, then the irreducible Darboux polynomial (if exists) is of the form $F(y, w) = w - c_0^{(l)} y^2 - c_1^{(l)} y - c_2^{(l)}$. If a similar situation occurs for representation (26), then either $N_1 = 0$ or $N_2 = 0$ and the product in expression (26) involving the corresponding series is absent. Further, if p_1, $p_2 \in \mathbb{N}$ and the compatibility condition for both Puiseux series are not satisfied, then there are no Darboux polynomials.*

Our next step is to study the integrability problem for cubic Liénard equations with linear dumping. Our goal is to relate the results that can be obtained via the generalized Sundman transformations and the results arising in the framework of Darboux theory of integrability. We shall consider the generic case, i.e., let us suppose that Equation (22) does not have solutions in $\mathbb{Q}^+$. In addition, we introduce the parameter σ according to the rule

$$b_3 = \frac{1 - \sigma^2}{8}, \quad \sigma \in \mathbb{C}, \quad \sigma \neq \pm 1. \tag{33}$$

Theorem 4. *Let $\sigma \neq \pm n/(n - 4m)$ with n, $m \in \mathbb{N}$ and $n \neq 4m$, then dynamical system (18) possesses the generalized Darboux first integral of the form*

$$I(y, w, z) = \prod_{j=1}^{N} F_j^{\alpha_j}(y, w) \exp(\alpha_0 z), \quad \alpha_0, \ldots, \alpha_N \in \mathbb{C}, \quad N \in \mathbb{N}, \quad \sum_{j=1}^{N} |\alpha_j| > 0 \tag{34}$$

if and only if one of the following set of conditions is satisfied

$$
\begin{aligned}
&(I): \quad b_0 = 0, \quad b_2 = 0; \\
&(II): \quad b_0 = \frac{192 b_2^3}{(9\sigma^2-1)^2}, \quad b_1 = -\frac{8(27\sigma^2+5)b_2^2}{(9\sigma^2-1)^2}.
\end{aligned} \tag{35}
$$

In case (I) *a first integral takes the form*

$$I(y,w,z) = F_1(y,w)^{\sigma+1} F_2(y,w)^{\sigma-1}, \tag{36}$$

where $F_1(y,w)$ *and* $F_2(y,w)$ *are the Darboux polynomials*

$$F_1(y,w) = w + \frac{1}{4}(1+\sigma)\,y^2 - \frac{2b_1}{\sigma-1}, \quad F_2(y,w) = w + \frac{1}{4}(1-\sigma)\,y^2 + \frac{2b_1}{\sigma+1}. \tag{37}$$

In case (II) *a first integral reads as*

$$I(y,w,z) = F_1(y,w)^{\sigma+1} F_2(y,w)^{\sigma-1} \exp\left[\frac{32\sigma b_2 z}{9\sigma^2-1}\right], \tag{38}$$

where $F_1(y,w)$ *and* $F_2(y,w)$ *are the Darboux polynomials*

$$
\begin{aligned}
F_1(y,w) &= w + \frac{1}{4}(1+\sigma)\,y^2 - \frac{4b_2}{(3\sigma-1)}y + \frac{48b_2^2}{(3\sigma-1)(3\sigma+1)^2}, \\
F_2(y,w) &= w + \frac{1}{4}(1-\sigma)\,y^2 + \frac{4b_2}{(3\sigma+1)}y - \frac{48b_2^2}{(3\sigma+1)(3\sigma-1)^2}.
\end{aligned} \tag{39}
$$

Proof. We begin the proof by recalling that a function of the form (34) is a first integral of a polynomial dynamical system in the plane if and only if $F_1(y,w), \ldots, F_N(y,w)$ are irreducible Darboux polynomials of this system such that their cofactors $\lambda_1(y,w), \ldots, \lambda_N(y,w)$ satisfy the condition [21]

$$\sum_{j=1}^{N} \alpha_j \lambda_j(y,w) + \alpha_0 = 0. \tag{40}$$

Let us note that Equation (22) under normalisation (33) reads as

$$\{p(\sigma+1) - 4\sigma\}\{p(\sigma-1) - 4\sigma\} = 0. \tag{41}$$

This equation does not have positive rational solutions provided that the following conditions $\sigma \neq \pm n/(n-4m)$ with $n, m \in \mathbb{N}$ and $n \neq 4m$ are valid. Consequently, it follows from Theorem 3 that the degree with respect to w of irreducible Darboux polynomials of dynamical system (18) is either 1 or 2. Moreover, if there exists an irreducible Darboux polynomial of degree 2 with respect to w, then it is unique. In addition, there can arise at most two distinct irreducible Darboux polynomials of degree 1 with respect to w. Thus, the number N in expression (34) is either 1 or 2. Calculating the polynomial part in expression (24) with $s_1 = 1, s_2 = 0$ and $s_1 = 0, s_2 = 1$ and $s_1 = 1, s_2 = 1$, we find the Darboux polynomials. Substituting the results into (19), we get restrictions on the parameters of the original system and explicit expressions of the cofactors. If $s_1 = 1, s_2 = 1$, then $N = 1$ and $\lambda_1 = -y + A_0$. Thus we see that condition (40) is not satisfied. Further, this condition is also not valid provided that only one irreducible Darboux polynomial of degree 1 with respect to w exists. This follows from the fact that the cofactor is either of the form $\lambda_1 = (\sigma-1)y/2 + A_0$ or of the form $\lambda_1 = -(\sigma+1)y/2 + A_0$. Finally, we suppose that there exist two distinct irreducible Darboux polynomials of degree 1 with respect to w. In this situation (40) reads as

$$\alpha_1 \frac{\sigma - 1}{2} y + \alpha_1 A_0^{(1)} - \alpha_2 \frac{\sigma + 1}{2} y + \alpha_2 A_0^{(2)} + \alpha_0 = 0. \tag{42}$$

The parameters α_1 and α_2 can be taken in the form: $\alpha_1 = \sigma + 1$ and $\alpha_2 = \sigma - 1$. The value of α_0 we find from relation (42). The result is given in (36) and (38). Recall that restrictions (35) on the parameters of the original system arose when we considered the existence of the corresponding Darboux polynomials. The explicit expressions of the Darboux polynomials are presented in (37) and (39). □

Remark 2. *Let us note that the parameters $\alpha_0, \alpha_1, \ldots, \alpha_N$ are defined up to a constant non-zero multiplier C. The corresponding first integrals with different values of C are functionally dependent.*

Remark 3. *If Equation (41) with $\sigma \neq \pm 1/3$ possesses a positive rational solution, then functions (36) and (38) are still first integrals of dynamical system (18). However, there can exist generalised Darboux first integrals (34) of a more complicated structure.*

4. Conclusions and Discussion

In this section we briefly summarise and discuss our results. Let us remark that linearizability conditions presented in Theorems 1, 2 coincide with those of Theorem 4 and, therefore, the following statement holds:

Theorem 5. *Equation (2) is equivalent to (3) via transformations (4) if and only if it has a generalized Darboux first integral of the form (34).*

Proof. If we let $\frac{\alpha}{\beta^2} = \frac{1-\sigma^2}{4}$ and take into account the expression for b_2 from (13), then formulas (6), (13) transform into (33), (35) and integrals (12), (16) into integrals (36), (38). This completes the proof. □

As far as linearizbility conditions presented in Section 2 are concerned, to the best of our knowledge, Theorem 2 provides a new criterion of linearizability of (2). Although conditions presented in Theorem 1 can be obtained from known ones (see, e.g., [14,15]), we believe that first integral (12) is presented here for the first time. Moreover, not only one can provide a first integral for linearizable cases of (2), but one can also construct the general solution of the corresponding equation from (2) with the help of (4) and the general solution of (3). Another novel result of the present article is the general structure of irreducible Darboux polynomials related to Equation (2). In addition, the necessary and sufficient conditions enabling existence of the generalised Darboux first integrals (34) in generic cases also seem to be new.

Finally, let us mention that we have classified generalised Darboux first integrals of the form (34) for generic values of the parameters appearing in the cubic Liénard oscillators with linear damping. However, there may exist some exceptional cases with interesting integrability properties. Consequently, these exceptional cases are also worth studying.

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