Article

Necessary Optimality Conditions in Isoperimetric Constrained Optimal Control Problems

Silviu-Aurelian Urziceanu

Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania; urziceanusilviu@gmail.com

check for
updates

Abstract: In this paper, we focus on a new class of optimal control problems governed by a simple integral cost functional and isoperimetric-type constraints (constant level sets of some simple integral functionals). By using the notions of a variational differential system and adjoint equation, necessary optimality conditions are established for a feasible solution in the considered optimization problem. More precisely, under simplified hypotheses and using a modified Legendrian duality, we establish a maximum principle for the considered optimization problem.

1. Introduction

Currently, the optimal control theory is under continuous development. This theory is based on the optimization of some functionals with ordinary differential equations/partial differential equations (in short, ODE/PDE) constraints, all depending on the control functions (for variational control problems with first-order PDE constraints, the reader is directed to Mititelu and Treanţă [1], Treanţă [2,3], and Treanţă and Arana-Jiménez [4,5]). There are three approaches: variational calculus, the maximum principle, and dynamic programming. One of the most important approaches is the second (see Pontriaguine et al. [6]), providing necessary optimality conditions. Also, in additional conditions, from this principle one can derive the Euler-Lagrange or Hamilton equations (see Treanţă [7,8]). Noether-type theorems between symmetries and conservation laws for dynamical systems (governed by autonomous second-order Lagrangians) have been investigated in Treanţă [9]. The sufficient optimality conditions are more complicated and they have most frequently been formulated in simplified versions (see, for instance, Hestenes [10], Maurer and Pickenhain [11], Rosenblueth and Sanchez Licea [12,13], and Treanţă [14]). In [15], Agrachev et al. provided sufficient conditions for a bang-bang extremal (to be a strong local minimizer) in a Mayer control problem, where the state space and the end point constraints are finite-dimensional smooth submanifolds and the controls take values in a polyhedron and appear linearly. Caputo (see [16,17]) has developed fundamental identities linking the optimal solution functions and optimal value functions for reciprocal pairs of isoperimetric control problems. In Aronsson and Barron [18], Aronsson–Euler equations have been obtained for L^∞ variational problems with holonomic, nonholonomic, isoperimetric, and isosupremic constraints on the minimizer.

Motivated and inspired by the aforementioned research works, the main aim of this paper is to formulate and prove necessary optimality conditions for a feasible solution in isoperimetric-type

constrained control problems. More precisely, by using the concepts of a *variational differential system* , *adjoint equation and a modified Legendrian duality*, under simplified hypotheses, we establish a maximum principle for the considered optimal control problem.

The optimal control problems with isoperimetric constraints are governed by some basic elements: (1) the control $u(t)$ (arising in the objective functional and isoperimetric constraints) so that any change in the control $u(t)$ involves a change in the state $x(t)$, (2) the state equation (the isoperimetric constraints) provides the dependence between the control and the state, (3) the functional to be extremized, called the objective functional or the cost functional, depending on the control and the state. The goal is to find an admissible control that generates a satisfactory state and extremizes the value of the objective functional (the admissible control having this property is called optimal control in the considered optimization problem). For other ideas that are connected to this subject, the reader is directed to Evans [19], Kalaba and Spingarn [20,21], Lee and Markus [22], Barbu et al. [23], van Brunt [24], and Treanţă [25,26].

2. Problem Formulation and Auxiliary Results

In this section, we investigate the following optimization problem governed by simple integral objective functional and isoperimetric-type constraints:

$$(CP) \qquad \max_{(x,u)} \int_0^{t_0} X\left(t, x(t), u(t)\right) dt \qquad (1)$$

such that

$$\int_0^{t_0} X^a\left(t, x(t), \dot{x}(t), u(t)\right) dt = l^a, \quad a = \overline{1, r}, \ r \leq n, \qquad (2)$$

$$x(0) = x_0, \quad x(t_0) = x_{t_0}, \quad t \in [0, t_0]. \qquad (3)$$

Our working hypotheses and notations are: $t \in [0, t_0] \subset \mathbb{R}$ is the *time*; the space $\mathbb{X}$ of all functions $x \colon [0, t_0] \to \mathbb{R}^n$, where $x(t) = (x^i(t))$, $i = \overline{1, n}$, is a C^2-class function (called the *state variable*), and the space $\mathbb{U}$ of all functions $u \colon [0, t_0] \to \mathbb{R}^k$, where $u(t) = (u^\alpha(t))$, $\alpha = \overline{1, k}$, is a piecewise continuous function (called the *control variable*), satisfying (2) and (3), which define the *feasible set* associated with (CP); $X^a\left(t, x(t), \dot{x}(t), u(t)\right)$, $a = \overline{1, r}$, $r \leq n$, are C^1-class functions; the objective functional (*non-autonomous Lagrange functional*) $X\left(t, x(t), u(t)\right)$ is a C^1-class function.

Further, in order to study the previous optimal control problem, let us introduce the *auxiliary variables* $y^a(t)$, $a = \overline{1, r}$, as follows

$$y^a(t) = \int_0^t X^a\left(s, x(s), \dot{x}(s), u(s)\right) ds,$$

$$y^a(0) = 0, \quad y^a(t_0) = l^a, \quad a = \overline{1, r}, \ r \leq n,$$

or, equivalently,

$$\dot{y}^a(t) = X^a\left(t, x(t), \dot{x}(t), u(t)\right), \quad y^a(t_0) = l^a, \quad a = \overline{1, r}, \ r \leq n. \qquad (4)$$

In this way, the isoperimetric-type constraints are substituted by constraints of ODE type.

In the following, Einstein's summation is assumed. By considering the *Lagrange multiplier*, $p(t) = (p_a(t))$, $a = \overline{1, r}$, also called *co-state variable (co-state vector)*, we introduce a new Lagrangian

$$\mathcal{L}\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t), u(t), p(t)\right) = X\left(t, x(t), u(t)\right)$$

$$+ p_a(t)\left[X^a\left(t, x(t), \dot{x}(t), u(t)\right) - \dot{y}^a(t)\right],$$

changing the aforementioned isoperimetric-type constrained optimization problem into a new optimal control problem (without isoperimetric-type constraints):

$$(CP_1) \qquad \max_{(x,u;y,p)} \int_0^{t_0} \mathcal{L}\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t), u(t), p(t)\right) dt$$

such that

$$p(t) \in \mathcal{P}, \quad t \in [0, t_0],$$

$$x(0) = x_0, \quad y(0) = 0, \quad x(t_0) = x_{t_0}, \quad y(t_0) = l,$$

where $\mathcal{P}$ represents the set of all co-state functions (introduced in the next section).

Now, by considering the *control Hamiltonian*

$$\mathcal{H}\left(t, x(t), \dot{x}(t), u(t), p(t)\right) = X\left(t, x(t), u(t)\right) + p_a(t) X^a\left(t, x(t), \dot{x}(t), u(t)\right),$$

or, equivalently,

$$\mathcal{H} = \mathcal{L} + p_a \dot{y}^a \quad \text{(modified Legendrian duality),}$$

we can rewrite the aforementioned optimal control problem under the next equivalent form:

$$(CP_1) \qquad \max_{(x,u;y,p)} \int_0^{t_0} \left[\mathcal{H}\left(t, x(t), \dot{x}(t), u(t), p(t)\right) - p_a(t) \dot{y}^a(t)\right] dt$$

such that

$$p(t) \in \mathcal{P}, \quad t \in [0, t_0],$$

$$x(0) = x_0, \quad y(0) = 0, \quad x(t_0) = x_{t_0}, \quad y(t_0) = l.$$

In the following, let us consider the relations formulated in (4). Now, we fix the control variable $u(t)$ and the associated solution $y(t)$ of (4). Denote by $y(t, \eta)$ a differentiable variation for $y(t)$, fulfilling

$$\dot{y}^a(t, \eta) = X^a\left(t, x(t, \eta), \dot{x}(t, \eta), u(t)\right),$$

$$y(t, 0) = y(t), \quad a = \overline{1, r}, \ r \leq n.$$

By a derivation with respect to η, for $\eta = 0$, we get the following *variational differential system*

$$\dot{y}^a_\eta(t, 0) = X^a_{x^i}\left(t, x(t), \dot{x}(t), u(t)\right) x^i_\eta(t, 0) + X^a_{\dot{x}^i}\left(t, x(t), \dot{x}(t), u(t)\right) \dot{x}^i_\eta(t, 0).$$

By considering $y^a_\eta(t, 0) = v^a(t)$, $x^i_\eta(t, 0) = \omega^i(t)$, $\dot{x}^i_\eta(t, 0) = \tilde{\omega}^i(t)$, we introduce the following *adjoint differential equation*

$$\dot{p}_a(t) v^a(t) = -p_a(t) \left[X^a_{x^i}\left(t, x(t), \dot{x}(t), u(t)\right) \omega^i(t) + X^a_{\dot{x}^i}\left(t, x(t), \dot{x}(t), u(t)\right) \tilde{\omega}^i(t)\right]$$

associated with the previous variational differential system in the sense that $p_a(t) v^a(t)$ is a first integral, that is,

$$\frac{d}{dt}\left[p_a(t) v^a(t)\right] = 0.$$

3. Main Result: Necessary Optimality Conditions for (*CP*)

In this section, we formulate and prove the main result of this paper. More concretely, under simplified hypotheses, we establish necessary conditions of optimality for the considered optimal control problem (*CP*) involving isoperimetric-type constraints. For other connected ideas to this subject, the reader is directed to Treanţă [27], where several optimization problems governed by simple, multiple, or curvilinear integral functionals (involving second-order Lagrangians) subject to ODE, PDE, or isoperimetric constraints are considered. Also, Treanţă [28] introduced a modified multidimensional optimal control problem and established the associated saddle-point optimality criteria.

In the following, we consider the control variable $\hat{u}(t) = (\hat{u}^\alpha(t))$, $\alpha = \overline{1,k}$, with $\hat{u} \in \mathbb{U}$, as an optimal control for the considered optimization problem (*CP*). Select $\eta > 0$ and define the control variation $u(t, \eta) = \hat{u}(t) + \eta h(t)$, where $h(\cdot)$ is some given function (called *acceptable variation*) selected so that $u(\cdot, \eta)$ is an admissible control for all sufficiently small $\eta > 0$. We shall use this η in our variational arguments.

The state variable $x(t, \eta)$, associated with the control function $u(t, \eta)$, satisfies

$$\dot{y}^a(t, \eta) = X^a\left(t, x(t, \eta), \dot{x}(t, \eta), u(t, \eta)\right), \quad a = \overline{1, r}, \ t \in [0, t_0]$$

and $y(0, \eta) = 0$, $x(0, \eta) = x_0$. For all sufficiently small $\eta > 0$, introduce the following function

$$I(\eta) = \int_0^{t_0} X\left(t, x(t, \eta), u(t, \eta)\right) dt.$$

By hypothesis, $\hat{u}(t)$ is an optimal control in (*CP*) and, consequently, it results $I(0) \geq I(\eta)$, for all sufficiently small $\eta > 0$. Also, the following equality

$$\int_0^{t_0} p_a(t) \left[X^a\left(t, x(t, \eta), \dot{x}(t, \eta), u(t, \eta)\right) - \dot{y}^a(t, \eta)\right] dt = 0$$

is fulfilled for any continuous function $p = (p_a(t))$, $a = \overline{1, r}$, $t \in [0, t_0]$. The variations involve

$$\mathcal{L}\left(t, x(t, \eta), \dot{x}(t, \eta), y(t, \eta), \dot{y}(t, \eta), u(t, \eta), p(t)\right) = X\left(t, x(t, \eta), u(t, \eta)\right)$$

$$+ p_a(t) \left[X^a\left(t, x(t, \eta), \dot{x}(t, \eta), u(t, \eta)\right) - \dot{y}^a(t, \eta)\right],$$

and the corresponding function

$$I(\eta) = \int_0^{t_0} \mathcal{L}\left(t, x(t, \eta), \dot{x}(t, \eta), y(t, \eta), \dot{y}(t, \eta), u(t, \eta), p(t)\right) dt.$$

For the following computations, the co-state variable $p(t) = (p_a(t))$ is considered of C^1-class. Considering the control Hamiltonian involving variations

$$\mathcal{H}\left(t, x(t, \eta), \dot{x}(t, \eta), u(t, \eta), p(t)\right) = X\left(t, x(t, \eta), u(t, \eta)\right) + p_a(t) X^a\left(t, x(t, \eta), \dot{x}(t, \eta), u(t, \eta)\right),$$

it follows

$$I(\eta) = \int_0^{t_0} \left[\mathcal{H}\left(t, x(t, \eta), \dot{x}(t, \eta), u(t, \eta), p(t)\right) - p_a(t) \dot{y}^a(t, \eta)\right] dt.$$

By derivation with respect to η, for $\eta = 0$, and using the adjoint differential equation, we have

$$I'(0) = \int_0^{t_0} \left[\mathcal{H}_{x^i}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t)\right) - p_a(t) X^a_{x^i}\left(t, x(t), \dot{x}(t), \hat{u}(t)\right)\right] x^i_\eta(t, 0) dt$$

$$+ \int_0^{t_0} \left[\mathcal{H}_{\dot{x}^i} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) - p_a(t) X_{\dot{x}^i}^a \left(t, x(t), \dot{x}(t), \hat{u}(t) \right) \right] \dot{x}_\eta^i(t, 0) dt$$

$$+ \int_0^{t_0} \mathcal{H}_{u^\alpha} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) h^\alpha(t) dt$$

$$= \int_0^{t_0} \mathcal{H}_{x^i} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) x_\eta^i(t, 0) dt + \int_0^{t_0} \dot{p}_a(t) v^a(t) dt$$

$$+ \int_0^{t_0} \mathcal{H}_{\dot{x}^i} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) \dot{x}_\eta^i(t, 0) dt$$

$$+ \int_0^{t_0} \mathcal{H}_{u^\alpha} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) h^\alpha(t) dt,$$

where $x(t)$ is the state variable corresponding to the optimal control variable $\hat{u}(t)$. By hypothesis, it follows $I'(0) = 0$, for any acceptable variation $h(t) = (h^\alpha(t))$. Also, the function $y_\eta^a(t, 0)$ solves the following Cauchy problem,

$$\nabla_t y_\eta^a(t, 0) = X_x^a \left(t, x(t), \dot{x}(t), \hat{u}(t) \right) x_\eta(t, 0) + X_{\dot{x}}^a \left(t, x(t), \dot{x}(t), \hat{u}(t) \right) \dot{x}_\eta(t, 0)$$

$$+ X_u^a \left(t, x(t), \dot{x}(t), \hat{u}(t) \right) h(t),$$

$$t \in [0, t_0], \quad y_\eta(0, 0) = 0.$$

Now, taking into account the previous mathematical context, the results are

$$\frac{\partial \mathcal{H}}{\partial u^\alpha} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) = 0, \quad t \in [0, t_0], \ \alpha = \overline{1, k} \tag{5}$$

and, also,

$$\frac{\partial \mathcal{H}}{\partial x^i} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) = 0, \quad t \in [0, t_0], \ i = \overline{1, n}, \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial x^i} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) - \frac{d}{dt} \left(\frac{\partial \mathcal{H}}{\partial \dot{x}^i} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right) \right) = 0, \tag{7}$$

$$t \in [0, t_0], \quad i = \overline{1, n}.$$

Define $\mathcal{P}$ as the set of solutions for

$$\dot{p}_a(t) = 0, \quad a = \overline{1, r}, \tag{8}$$

that is, the Lagrange multiplier $p = (p_a)$ is a constant ($p = (p_a(t))$ is a constant function). Moreover,

$$\dot{y}^a(t) = \frac{\partial \mathcal{H}}{\partial p_a} \left(t, x(t), \dot{x}(t), \hat{u}(t), p(t) \right), \quad t \in [0, t_0], \ a = \overline{1, r}, \ r \leq n, \tag{9}$$

$$y(0) = 0, \quad y(t_0) = l.$$

Finally, taking into account all the previous computations, we are able to formulate a simplified maximum principle. The following theorem represents the main result of the present paper. More precisely, necessary optimality conditions are established for the considered optimal control problem (CP) involving isoperimetric-type constraints.

Theorem 1. *Let $(x, \hat{u}) \in \mathbb{X} \times \mathbb{U}$ be an optimal solution in the considered optimal control problem (CP). Then, there exists the constant co-state variable $p = (p_a)$ such that the relations (5–9) are fulfilled, except at discontinuities.*

Remark 1.

(i) *The algebraic systems in (5) and (6) describe the critical points associated with $\mathcal{H}$ with respect to the control variable $u = (u^\alpha)$ and the state variable $x = (x^i)$.*

(ii) *The differential Equations (8) and (9) and the conditions formulated in (5), (6), and (7) represent the Euler–Lagrange ODEs, respectively,*

$$\frac{\partial \mathcal{L}}{\partial y^a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}^a} = 0, \quad \frac{\partial \mathcal{L}}{\partial p_a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}_a} = 0, \quad a = \overline{1,r},$$

$$\frac{\partial \mathcal{L}}{\partial u^\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} = 0, \quad \alpha = \overline{1,k},$$

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = 0, \quad i = \overline{1,n},$$

corresponding to the new Lagrangian $\mathcal{L}$.

Further, by using the new Lagrangian $\mathcal{L}$ and the aforementioned remark, we establish the following result.

Corollary 1. *Consider $(x, \hat{u}) \in \mathbb{X} \times \mathbb{U}$ is an optimal solution associated with the optimal control problem (CP). Then, there exists the constant co-state variable $p = (p_a)$ such that*

$$\dot{y}^a(t) = X^a\left(t, x(t), \dot{x}(t), u(t)\right), \quad y^a(t_0) = l^a, \quad a = \overline{1,r}, \ r \le n$$

and the following Euler–Lagrange ODEs associated with the Lagrangian $\mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial y^a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}^a} = 0, \quad \frac{\partial \mathcal{L}}{\partial p_a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}_a} = 0, \quad a = \overline{1,r},$$

$$\frac{\partial \mathcal{L}}{\partial u^\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} = 0, \quad \alpha = \overline{1,k},$$

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = 0, \quad i = \overline{1,n},$$

are satisfied, except at discontinuities.

4. Conclusions

In this paper, by using corresponding variational differential systems, adjoint equations and a modified Legendrian duality, we have derived necessary optimality conditions for a feasible point in the considered isoperimetric-type constrained optimal control problem. The main result of this work is original and complements previously known results (see, for instance, Treanță [7,8,14,29]). The current research paper can be extended for other classes of optimal control problems such as variational control problems under uncertainty. Also, the study of sufficient optimality conditions for such optimal control problems is another topic open to research.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Mititelu, Ş.; Treanţă, S. Efficiency conditions in vector control problems governed by multiple integrals. *J. Appl. Math. Comput.* **2018**, *57*, 647–665. [CrossRef]
2. Treanţă, S. On a new class of vector variational control problems. *Numer. Funct. Anal. Optim.* **2018**, *39*, 1594–1603. [CrossRef]
3. Treanţă, S. KT-geodesic pseudoinvex control problems governed by multiple integrals. *J. Nonlinear Convex Anal.* **2019**, *20*, 73–84.
4. Treanţă, S.; Arana-Jiménez, M. KT-pseudoinvex multidimensional control problem. *Optim. Control Appl. Meth.* **2018**, *39*, 1291–1300. [CrossRef]
5. Treanţă, S.; Arana-Jiménez, M. On generalized KT-pseudoinvex control problems involving multiple integral functionals. *Eur. J. Control* **2018**, *43*, 39–45. [CrossRef]
6. Pontriaguine, L.; Boltianski, V.; Gamkrelidze, R.; Michtchenko, E. *Théorie Mathématique des Processus Optimaux*; Edition MIR: Moscou, Russia, 1974.
7. Treanţă, S. PDEs of Hamilton-Pfaff type via multi-time optimization problems. *U.P.B. Sci. Bull. Ser.* **2014**, *76*, 163–168.
8. Treanţă, S. Higher-order Hamilton dynamics and Hamilton-Jacobi divergence PDE. *Comput. Math. Appl.* **2018**, *75*, 547–560. [CrossRef]
9. Treanţă, S. Noether-type first integrals associated with autonomous second-order Lagrangians. *Symmetry* **2019**, *1088*, 11. [CrossRef]
10. Hestenes, M.R. *Calculus of Variations and Optimal Control Theory*; John Wiley & Sons: New York, NY, USA, 1966.
11. Maurer, H.; Pickenhain, S. Second order sufficient conditions for control problems with mixed control-state constraints. *J. Optim. Theory Appl.* **1995**, *86*, 649–667. [CrossRef]
12. Rosenblueth, J.F.; Licea, G.S. A Direct Sufficiency Proof for a Weak Minimum in Optimal Control. *Appl. Math. Sci.* **2010**, *4*, 253–269.
13. Rosenblueth, J.F.; Licea, G.S. Sufficient Variational Conditions for Isoperimetric Control Problems. *Int. Math. Forum* **2011**, *6*, 303–324.
14. Treanţă, S. Sufficient efficiency conditions associated with a multidimensional multiobjective fractional variational problem. *J. Multid. Model. Optim.* **2018**, *1*, 1–13.
15. Agrachev, A.A.; Stefani, G.; Zezza, P. Strong optimality for a bang-bang trajectory. *SIAM J. Control Optim.* **2002**, *41*, 991–1014. [CrossRef]
16. Caputo, M.R. Economic characterization of reciprocal isoperimetric control problems. *J. Optim. Theory Appl.* **1998**, *98*, 325–350. [CrossRef]
17. Caputo, M.R. Economic characterization of reciprocal isoperimetric control problems revisited. *J. Optim. Theory Appl.* **1999**, *101*, 723–730. [CrossRef]
18. Aronsson, G.; Barron, E.N. L^∞ variational problems with running costs and constraints. *Appl. Math. Optim.* **2012**, *65*, 53–90. [CrossRef]
19. Evans, L.C. *An Introduction to Mathematical Optimal Control Theory*; Lecture Notes; Department of Mathematics, University of California: Berkeley, CA, USA, 2008.
20. Kalaba, R.; Spingarn, K. *Control, Identification, and Input Optimization*; Plenum: New York, NY, USA, 1982.
21. Kalaba, R.; Spingarn, K. Automatic Solution of Optimal Control Problems III: Differential and Integral Constraints. *Control Syst. Mag.* **1984**, *4*, 3–8. [CrossRef]
22. Lee, E.B.; Markus, L. *Foundations of Optimal Control Theory*; Wiley: New York, NY, USA, 1967.
23. Barbu, V.; Lasiecka, I.; Tiba, D.; Vârsan, C. Analysis and Optimization of Differential Systems. In Proceedings of the International Working Conference on Analysis and Optimization of Differential Systems, IFIP TC7/WG7.2, Constanta, Romania, 10–14 September 2002; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2003.
24. van Brunt, B. *The Calculus of Variations*; Springer: Berlin/Heidelberg, Germany, 2004.
25. Treanţă, S. PDE-constrained vector variational problems governed by curvilinear integral functionals. *Appl. Anal. Optim.* **2019**, *3*, 83–101.

26. Treanţă, S. On Controlled Variational Inequalities Involving Convex Functionals. In *Optimization of Complex Systems: Theory, Models, Algorithms and Applications. WCGO 2019, Advances in Intelligent Systems and Computing*; Le Thi, H., Le, H., Pham Dinh, T., Eds.; Springer: Cham, Switzerland, 2020; Volume 991, pp. 164–174.

27. Treanţă, S. Constrained variational problems governed by second-order Lagrangians. *Appl. Anal.* **2018**. [CrossRef]

28. Treanţă, S. On a modified optimal control problem with first-order PDE constraints and the associated saddle-point optimality criterion. *Eur. J. Control* **2019**. [CrossRef]

29. Treanţă, S. Multiobjective fractional variational problem on higher-order jet bundles. *Commun. Math. Stat.* **2016**, *4*, 323–340. [CrossRef]