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A Study on Hypergraph Representations of Complex Fuzzy Information

Anam Luqman [1], Muhammad Akram [1,*], Ahmad N. Al-Kenani [2] and José Carlos R. Alcantud [3]

1 Department of Mathematics, University of the Punjab, New Campus, Lahore 54590, Pakistan; anamluqman7@yahoo.com
2 Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80219, Jeddah 21589, Saudi Arabia; aalkenani10@hotmail.com
3 BORDA Research Unit and IME, University of Salamanca, 37007 Salamanca, Spain; jcr@usal.es
* Correspondence: m.akram@pucit.edu.pk

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Abstract: The paradigm shift prompted by Zadeh's fuzzy sets in 1965 did not end with the fuzzy model and logic. Extensions in various lines have produced e.g., intuitionistic fuzzy sets in 1983, complex fuzzy sets in 2002, or hesitant fuzzy sets in 2010. The researcher can avail himself of graphs of various types in order to represent concepts like networks with imprecise information, whether it is fuzzy, intuitionistic, or has more general characteristics. When the relationships in the network are symmetrical, and each member can be linked with groups of members, the natural concept for a representation is a hypergraph. In this paper we develop novel generalized hypergraphs in a wide fuzzy context, namely, complex intuitionistic fuzzy hypergraphs, complex Pythagorean fuzzy hypergraphs, and complex q-rung orthopair fuzzy hypergraphs. Further, we consider the transversals and minimal transversals of complex q-rung orthopair fuzzy hypergraphs. We present some algorithms to construct the minimal transversals and certain related concepts. As an application, we describe a collaboration network model through a complex q-rung orthopair fuzzy hypergraph. We use it to find the author having the most outstanding collaboration skills using score and choice values.

Keywords: complex q-rung orthopair fuzzy set; complex q-rung orthopair fuzzy graphs; complex q-rung orthopair fuzzy hypergraphs; transversals

1. Introduction

In 1965, fuzzy sets (FSs) were originally defined by Zadeh [1] as a novel approach to represent uncertainty arising in various fields. The idea of "partial membership" was questioned by many researchers at that time. The extension of crisp sets to FSs, i.e., the extension of membership function $\mu(x)$ from $\{0,1\}$ to $[0,1]$, bears comparison to the generalization of $\mathbb{Q}$ to $\mathbb{R}$. Just like $\mathbb{R}$ was extended to $\mathbb{C}$ with the incorporation of imaginary quantities, FSs have been extended to complex fuzzy sets (CFSs) by Ramot et al. [2]. A CFS is characterized by a membership function $\mu(x)$ whose range is not limited to [0,1] but extends to the unit circle in the complex plane. Hence, $\mu(x)$ is a complex-valued function that assigns a grade of membership of the form $v(x)e^{i\alpha(x)}$, $i=\sqrt{-1}$ to any element x in the universe of discourse. The membership function $\mu(x)$ of CFS consists of two terms, namely, an amplitude term $v(x)$ which lies in the unit interval $[0,1]$ and a phase term (periodic term) $\alpha(x)$ which lies in the interval $[0,2\pi]$. During the last few years, many researchers have paid special attention to CFSs. Yazdanbakhsh and Dick [3] gives an updated review of the development of CFSs.

Atanassov [4] had proposed a different extension of FSs by intuitionistic fuzzy sets (IFSs). Fuzzy sets give the degree of membership of an element in a given set (the non-membership of degree equals one minus the degree of membership), while IFSs give both a degree of membership and a degree of non-membership, which are to some extent independent from each other. The truth (T) and falsity (F) membership functions are used to characterize an IFS in such a way that the sum of truth and falsity degrees should not be greater than one at any point. These figures allow for some indeterminacy in the expression of memberships. Progress on the investigation of IFSs and related extensions of the FS concept continues to be made. Liu et al. [5] introduced different types of centroid transformations of IF values. Feng et al. [6] defined various new operations for generalized IF soft sets. Recently, Shumaiza et al. [7] have proposed group decision-making based on the VIKOR method with trapezoidal bipolar fuzzy information. Akram and Arshad [8] proposed a novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making. Alcantud et al [9] proposed a novel modelization of the party formation process, in which citizens' private opinions are described by means of continuous fuzzy profiles. A novel hesitant fuzzy model for group decision-making was proposed by Alcantud and Giarlotta [10].

Of particular importance are two extensions of IFSs proposed by Yager [11–13]. In these papers he introduced Pythagorean fuzzy sets (PFSs) and q-rung orthopair fuzzy sets (q-ROFSs). A q-ROFS is characterized by means of truth and falsity degrees satisfying the constraint that the sum of the qth powers of both degrees should be less than one. PFSs consist of the case where $q = 2$. Thus, q-ROFSs generalize both the notions of IFSs and PFSs so that the uncertain information can be dealt with in a more widened range. After that, Liu and Wang [14] applied certain simple weighted operators to aggregate q-ROFSs in decision-making. Intertemporal choice problems have been investigated with the help of fuzzy soft sets [15]. These problems appear in the analysis of environmental issues and sustainable development with an infinitely long horizon, project evaluations, or health care [16,17]. In multi-attribute decision making, q-ROF Heronian mean operators were defined by Wei et al. [18]. For further applications of q-ROFSs, we refer the readers to the work presented in [19,20]. Complex intuitionistic fuzzy sets (CIFSs) were introduced by Alkouri and Salleh [21] in order to generalize IFSs in the spirit of [2] by adding non-membership degree $\nu(x) = s(x)e^{i\beta(x)}$ to the CFSs subjected to the constraint $0 \leq r + s \leq 1$. The CIFSs are used to handle information about uncertainty and periodicity simultaneously. As an extension of both PFSs and CIFSs, Ullah et al. [22] proposed complex Pythagorean fuzzy sets (CPFSs) and discussed some applications.

The vagueness in the representation of various objects and the uncertain interactions between them originated the necessity of fuzzy graphs (FGs), that were first defined by Rosenfeld [23] (see also the remarks made by Bhattacharya [24]). The notion of FGs was extended to complex fuzzy graphs (CFGs) by Thirunavukarasu et al. [25]. Intuitionistic fuzzy graphs (IFGs) were defined by Parvathi and Karunambigai [26]. The energy of Pythagorean fuzzy graphs (PFGs) was discussed by Akram and Naz [27]. Akram and Habib [28] defined q-ROF competition graphs and discussed their applications. Akram et al. [29] proposed a novel description on edge-regular q-ROFGs. Yaqoob et al. [30] defined complex intuitionistic fuzzy graphs (CIFGs) and discussed an application of CIFGs in cellular networks to test the proposed model. Later on, complex neutrosophic graphs were studied by Yaqoob and Akram [31]. Recently, complex Pythagorean fuzzy graphs (CPFGs) and their applications in decision making have been put forward by Akram and Naz [32].

A hypergraph, as an extension of a crisp graph, is a powerful tool to model different practical problems in various fields, including biological sciences, computer science, sustainable development and social networks [33–35]. Co-authorship networks, an important type of social network, have been studied extensively from various angles such as degree distribution analysis, social community extraction and social entity ranking. Most of the previous studies consider the co-authorship relation between two or more authors as a collaboration using crisp hypergraphs. Han et al. [36] proposed a hypergraph analysis approach to understand the importance of collaborations in co-authorship networks. Zhang and Liu [37] proposed a hypergraph model of social tagging networks. Ouvrard et al. [38] studied the hypergraph modeling and visualization of collaboration networks.

In order to allow for uncertainty in crisp hypergraphs, fuzzy hypergraphs (FHGs) were defined by Kaufmann [39] as an extension of FGs. Lee-Kwang and Lee [40] discussed fuzzy partitions using

FHGs. A valuable contribution to FGs and FHGs has been proposed by Mordeson and Nair [41]. Fuzzy transversals of FHGs were studied by Goetschel et al. [42]. Intuitionistic fuzzy hypergraphs (IFHGs) were defined by Parvathi et al. [43]. Further discussion on IFHGs can be seen in [44,45]. Akram and Luqman [46] defined bipolar neutrosophic hypergraphs with applications. Transversals and minimal transversals of m-polar FHGs were studied by Akram and Sarwar [47]. Luqman et al. [48] presented q-ROFHGs and their applications. Further, Luqman et al. [49,50] have proposed m-polar and q-rung picture fuzzy hypergraph models of granular computing.

The proposed research generalizes the concepts of CIFGs and CPFGs. These existing models can only depict the uncertainty having periodic nature occurring in pairwise relationships. The existence of various complex network models in which the relationships are more generalized rather than the pairwise relationships motivates the extension of CIFGs and CPFGs to complex intuitionistic fuzzy and complex Pythagorean fuzzy hypergraphs. Let us consider the modeling of research collaborations through CIFGs and CPFGs. The uncertainty and periodicity of the given data are dealt with with the help of phase terms and amplitude terms, respectively. Two research articles are connected through an edge if both have the same author but if more than two articles are written by the same author then CIFGs and CPFGs fail to model this situation. Thus the main objective of this study is to generalize the concepts of CIFGS and CPFGs to complex q-rung orthopair fuzzy hypergraphs. As argued above, complex q-ROF models provide more flexibility than IFSs and FSs. Therefore a complex q-rung orthopair fuzzy hypergraph model proves to be a very general framework to deal with vagueness in complex hypernetworks when the symmetrical relationships go beyond pairwise interactions. The generality of the proposed model can be observed from the reduction of complex q-rung orthopair fuzzy models to CIF and CPF models for $q = 1$ and $q = 2$, respectively. Moreover, most of the previous studies consider the co-authorship relation between two or more authors as a collaboration using crisp hypergraphs. Here we consider a complex q-rung orthopair fuzzy hypergraph model of co-authorship network to represent the collaboration relations between authors having uncertainty and vagueness of periodic nature simultaneously.

The contents of this paper are as follows. In Sections 2 and 3, we define complex intuitionistic fuzzy hypergraphs and complex Pythagorean fuzzy hypergraphs, respectively. In Section 4, complex q-ROF hypergraphs are discussed. In Section 5, we define the q-ROF transversals and minimal transversals of q-ROF hypergraphs. Section 6 illustrates an application of q-ROF hypergraphs in research collaboration networks. We also present an algorithm to select an author with powerful collaboration characteristics using the score and choice values of q-rung orthopair fuzzy hypergraphs and give a brief comparison of our proposed model with CIF and CPF models. The final Section 7 contains conclusions and future research directions.

2. Complex Intuitionistic Fuzzy Hypergraphs

In this section, we define the notion of complex intuitionistic fuzzy hypergraphs. A complex intuitionistic fuzzy hypergraph extends the concept of CIFGs. The proposed hypergraph model is used to handle the uncertain and periodic real-life situations when the relationships are analyzed between more than two objects. The main model that we use in our research design is given in the next definition:

Definition 1. *[21] A complex intuitionistic fuzzy set (CIFS) I on the universal set Y is defined as,*

$$I = \{(u, T_I(u)e^{i\phi_I(u)}, F_I(u)e^{i\psi_I(u)})|u \in Y\},$$

where $i = \sqrt{-1}$, $T_I(u), F_I(u) \in [0,1]$, $\phi_I(u), \psi_I(u) \in [0, 2\pi]$, *and for every* $u \in Y$, $0 \leq T_I(u) + F_I(u) \leq 1$.

For every $u \in Y$, $T_I(u)$ and $F_I(u)$ are the amplitude terms for membership and non-membership of u, and $\phi_I(u)$ and $\psi_I(u)$ are the phase terms for membership and non-membership of u. CIFSs where the phase terms equal zero (for all u) reduce to ordinary IFSs. When in addition, the amplitude terms for non-membership of all elements equal zero, we obtain a FS.

The application of this concept to graphs was produced in [30]. We represent definition of complex intuitionistic fuzzy graphs as follows:

Definition 2. *A complex intuitionistic fuzzy graph (CIFG) on Y is an ordered pair $G = (A, B)$, where A is a complex intuitionistic fuzzy set on Y and B is complex intuitionistic fuzzy relation on Y such that,*

$$T_B(ab) \leq \min\{T_A(a), T_A(b)\},\ F_B(ab) \leq \max\{F_A(a), F_A(b)\}, \text{(for amplitude terms)}$$
$$\phi_B(ab) \leq \min\{\phi_A(a), \phi_A(b)\},\ \psi_B(ab) \leq \max\{\psi_A(a), \psi_A(b)\}, \text{(for phase terms)}$$

$0 \leq T_B(ab) + F_B(ab) \leq 1$, *and* $\phi, \psi \in [0, 2\pi]$, *for all* $a, b \in Y$.

When we apply Definition 1 to hypergraphs we obtain the following structure that generalizes Definition 2:

Definition 3. *Let Y be a non-trivial set of universe. A complex intuitionistic fuzzy hypergraph (CIFHG) is defined as an ordered pair $H = (\mathcal{C}, \mathcal{D})$, where $\mathcal{C} = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is a finite family of complex intuitionistic fuzzy sets on Y and $\mathcal{D}$ is a complex intuitionistic fuzzy relation on complex intuitionistic fuzzy sets α_j's such that the following conditions hold:*

(i)

$$\begin{aligned}
T_{\mathcal{D}}(\{r_1, r_2, \cdots, r_l\}) &\leq \min\{T_{\alpha_j}(r_1), T_{\alpha_j}(r_2), \cdots, T_{\alpha_j}(r_l)\},\\
F_{\mathcal{D}}(\{r_1, r_2, \cdots, r_l\}) &\leq \max\{F_{\alpha_j}(r_1), F_{\alpha_j}(r_2), \cdots, F_{\alpha_j}(r_l)\}, \text{(for amplitude terms)}\\
\phi_{\mathcal{D}}(\{r_1, r_2, \cdots, r_l\}) &\leq \min\{\phi_{\alpha_j}(r_1), \phi_{\alpha_j}(r_2), \cdots, \phi_{\alpha_j}(r_l)\},\\
\psi_{\mathcal{D}}(\{r_1, r_2, \cdots, r_l\}) &\leq \max\{\psi_{\alpha_j}(r_1), \psi_{\alpha_j}(r_2), \cdots, \psi_{\alpha_j}(r_l)\}, \text{(for phase terms)}
\end{aligned}$$

$0 \leq T_{\mathcal{D}} + F_{\mathcal{D}} \leq 1$, *and* $\phi_{\mathcal{D}}, \psi_{\mathcal{D}} \in [0, 2\pi]$, *for all* $r_1, r_2, \cdots, r_l \in Y$.

(ii) $\bigcup\limits_j supp(\alpha_j) = Y$, *for all* $\alpha_j \in \mathcal{C}$.

Notice that $E_k = \{r_1, r_2, \cdots, r_l\}$ *is the crisp hyperedge of* $H = (\mathcal{C}, \mathcal{D})$.

Note that the above formula reduces to Definition 2 if we consider only two vertices in an hyperedge.

We illustrate the previous definition with a graphical example.

Example 1. *Consider a CIFHG $H = (\mathcal{C}, \mathcal{D})$ on $Y = \{v_1, v_2, v_3, v_4\}$. The CIFR is defined as, $\mathcal{D}(\{v_1, v_2, v_3, v_4\}) = (0.2e^{i0.4\pi}, 0.6e^{i0.3\pi})$, $\mathcal{D}(\{v_1, v_2\}) = (0.3e^{i0.6\pi}, 0.6e^{i0.3\pi})$, and $\mathcal{D}(\{v_3, v_4\}) = (0.2e^{i0.4\pi}, 0.5e^{i0.3\pi})$. The corresponding CIFHG is shown in Figure 1.*

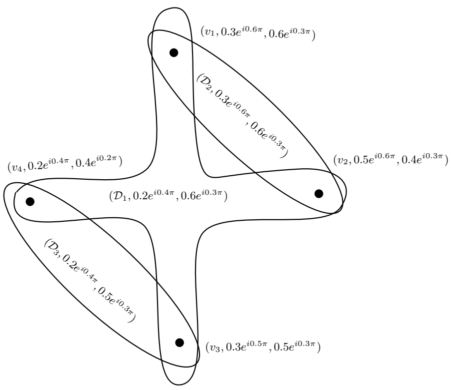

Figure 1. Complex intuitionistic fuzzy hypergraph.

Simple CIFHGs are the following special types of CIFHGs:

Definition 4. *A CIFHG $H = (\mathcal{C}, \mathcal{D})$ is simple if whenever $\mathcal{D}_j, \mathcal{D}_k \in \mathcal{D}$ and $\mathcal{D}_j \subseteq \mathcal{D}_k$, then $\mathcal{D}_j = \mathcal{D}_k$.*
A CIFHG $H = (\mathcal{C}, \mathcal{D})$ is support simple if whenever $\mathcal{D}_j, \mathcal{D}_k \in \mathcal{D}$, $\mathcal{D}_j \subseteq \mathcal{D}_k$, and $supp(\mathcal{D}_j) = supp(\mathcal{D}_k)$, then $\mathcal{D}_j = \mathcal{D}_k$.

Our next notion produces a link between CIFHGs and crisp hypergraphs. The subsequent example illustrates this construction.

Definition 5. *Let $H = (\mathcal{C}, \mathcal{D})$ be a CIFHG. Suppose that $\alpha, \beta \in [0, 1]$ and $\theta, \varphi \in [0, 2\pi]$ such that $0 \leq \alpha + \beta \leq 1$. The $(\alpha e^{i\theta}, \beta e^{i\varphi})-$level hypergraph of H is defined as an ordered pair $H^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = (\mathcal{C}^{(\alpha e^{i\theta}, \beta e^{i\varphi})}, \mathcal{D}^{(\alpha e^{i\theta}, \beta e^{i\varphi})})$, where*

- **(i)** $\mathcal{D}^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = \{D_j^{(\alpha e^{i\theta}, \beta e^{i\varphi})} : D_j \in \mathcal{D}\}$ *and* $D_j^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = \{u \in Y : T_{D_j}(u) \geq \alpha, \phi_{D_j}(u) \geq \theta$, and $F_{D_j}(u) \leq \beta, \psi_{D_j}(u) \leq \varphi\}$,
- **(ii)** $\mathcal{C}^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = \bigcup\limits_{D_j \in \mathcal{D}} D_j^{(\alpha e^{i\theta}, \beta e^{i\varphi})}$.

Note that the $(\alpha e^{i\theta}, \beta e^{i\varphi})-$level hypergraph of H is a crisp hypergraph.

Example 2. *Consider a CIFHG $H = (\mathcal{C}, \mathcal{D})$ as shown in Figure 1. Let $\alpha = 0.2$, $\beta = 0.5$, $\theta = 0.5\pi$, and $\varphi = 0.2\pi$. The $(\alpha e^{i\theta}, \beta e^{i\varphi})-$level hypergraph of H is shown in Figure 2.*

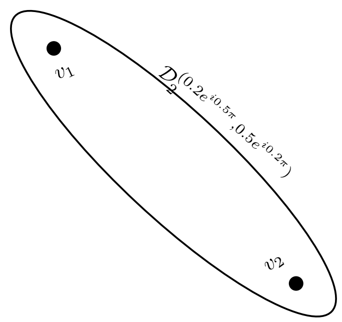

Figure 2. $(0.2e^{i(0.5)\pi}, 0.5e^{i(0.2)\pi})-$level hypergraph of H.

Definition 6. *Let $H = (\mathcal{C}, \mathcal{D})$ be a CIFHG. The complex intuitionistic fuzzy line graph of H is defined as an ordered pair $l(H) = (\mathcal{C}_l, \mathcal{D}_l)$, where $\mathcal{C}_l = \mathcal{D}$ and there exists an edge between two vertices in $l(H)$ if $|supp(D_j) \cap supp(D_k)| \geq 1$. The membership degrees of $l(H)$ are given as,*

- **(i)** $\mathcal{C}_l(E_k) = \mathcal{D}(E_k)$,
- **(ii)** $\mathcal{D}_l(E_jE_k) = (\min\{T_\mathcal{D}(E_j), T_\mathcal{D}(E_k)\}e^{i\min\{\phi_\mathcal{D}(E_j), \phi_\mathcal{D}(E_k)\}}, \max\{F_\mathcal{D}(E_j), F_\mathcal{D}(E_k)\}e^{i\max\{\psi_\mathcal{D}(E_j), \psi_\mathcal{D}(E_k)\}})$.

Definition 7. *A CIFHG $H = (\mathcal{C}, \mathcal{D})$ is said to be linear if for every $D_j, D_k \in \mathcal{D}$,*

- **(i)** $supp(D_j) \subseteq supp(D_k) \Rightarrow j = k$,
- **(ii)** $|supp(D_j) \cap supp(D_k)| \leq 1$.

Example 3. *Consider a CIFHG $H = (\mathcal{C}, \mathcal{D})$ as shown in Figure 1. By direct calculations, we have*

$$supp(\mathcal{D}_1) = \{v_1, v_2, v_3, v_4\},\ supp(\mathcal{D}_2) = \{v_1, v_2\},\ supp(\mathcal{D}_3) = \{v_3, v_4\}.$$

Note that, $supp(D_j) \subseteq supp(D_k) \Rightarrow j \neq k$ and $|supp(D_j) \cap supp(D_k)| \nleq 1$. Hence, CIFHG $H = (\mathcal{C}, \mathcal{D})$ is not linear. The corresponding CIFHG $H = (\mathcal{C}, \mathcal{D})$ and its line graph is shown in Figure 3.

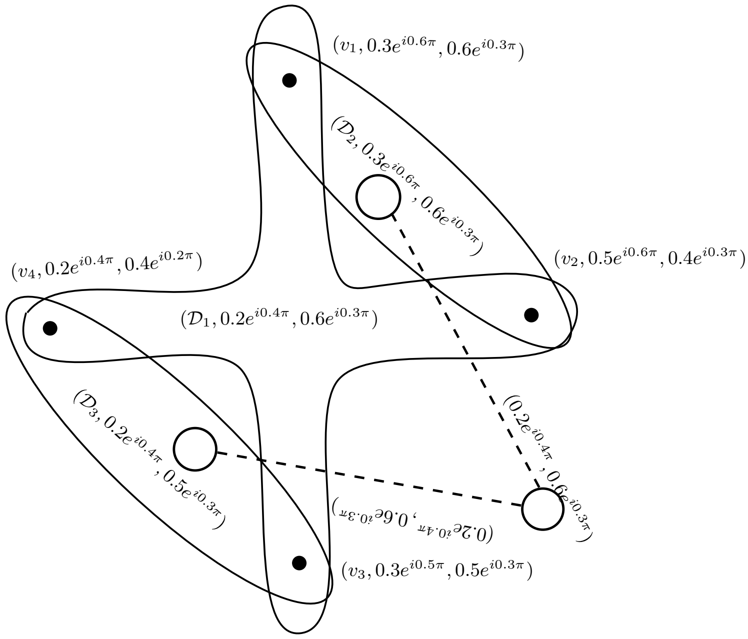

Figure 3. Complex intuitionistic fuzzy line graph of H.

Theorem 1. *A simple strong CIFG is the complex intuitionistic line graph of a linear CIFHG.*

Definition 8. *The* 2*-section* $H_2 = (\mathcal{C}_2, \mathcal{D}_2)$ *of a CIFHG* $H = (\mathcal{C}, \mathcal{D})$ *is a CIFG having same set of vertices as that of* H*,* $\mathcal{D}_2$ *is a CIFS on* $\{e = u_j u_k | u_j, u_k \in E_l,\ l = 1, 2, 3, \cdots\}$*, and* $\mathcal{D}_2(u_j u_k) = (\min\{\min T_{\alpha_l}(u_j), \min T_{\alpha_l}(u_k)\} e^{i \min\{\min \phi_{\alpha_l}(u_j), \min \phi_{\alpha_l}(u_k)\}}, \max\{\max F_{\alpha_l}(u_j), \max F_{\alpha_l}(u_k)\} e^{i \max\{\max \psi_{\alpha_l}(u_j), \max \psi_{\alpha_l}(u_k)\}})$ *such that* $0 \leq T_{\mathcal{D}_2}(u_j u_k) + F_{\mathcal{D}_2}(u_j u_k) \leq 1$, $\phi_{\mathcal{D}_2}, \psi_{\mathcal{D}_2} \in [0, 2\pi]$.

Example 4. *An example of a CIFHG is given in Figure 4. The* 2*-section of* H *is presented with dashed lines.*

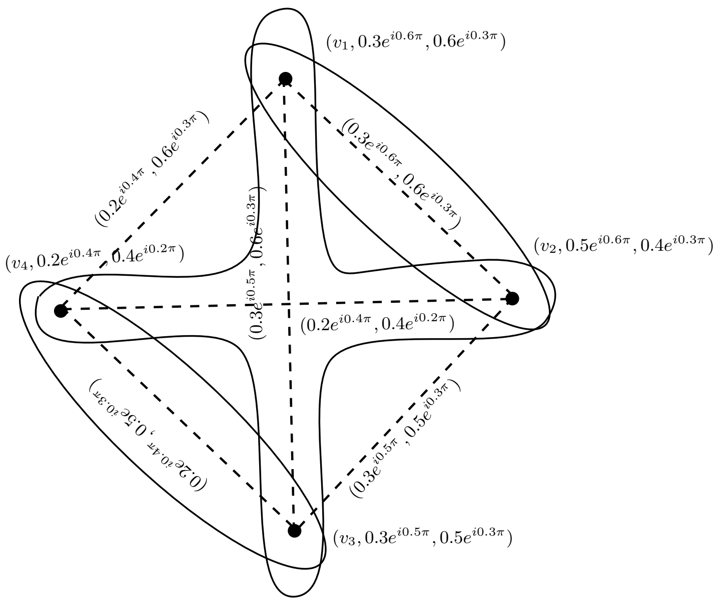

Figure 4. Two-section of complex intuitionistic fuzzy hypergraph.

Definition 9. *Let $H = (\mathcal{C}, \mathcal{D})$ be a CIFHG. A complex intuitionistic fuzzy transversal (CIFT) τ is a CIFs of Y satisfying the condition $\rho^{h(\rho)} \cap \tau^{h(\rho)} \neq \emptyset$, for all $\rho \in \mathcal{D}$, where $h(\rho)$ is the height of ρ.*

A minimal complex intuitionistic fuzzy transversal t is the CIFT of H having the property that if $\tau \subset t$, then τ is not a CIFT of H.

3. Complex Pythagorean Fuzzy Hypergraphs

We now turn our attention to the next class of hypergraphs called complex Pythagorean fuzzy hypergraphs. A complex Pythagorean fuzzy hypergraph is the generalization of CPFGs and CIFHGs. The occurrence of truth and falsity degrees whose sum is not less than one but the sum of squares does not exceed one in complex hypernetworks motivates the necessity of this proposed model.

Definition 10. *[32] A complex Pythagorean fuzzy graph (CPFG) on Y is an ordered pair $G^* = (C, D)$, where C is a CPFS on Y and D is CPFR on Y such that,*

$$T_D(ab) \leq \min\{T_C(a), T_C(b)\},\ F_D(ab) \leq \max\{F_C(a), F_C(b)\},\ \text{for amplitude terms}$$
$$\phi_D(ab) \leq \min\{\phi_C(a), \phi_C(b)\},\ \psi_D(ab) \leq \max\{\psi_C(a), \psi_C(b)\},\ \text{for phase terms}$$

$0 \leq T_D^2(ab) + F_D^2(ab) \leq 1$, and $\phi_D, \psi_D \in [0, 2\pi]$, for all $a, b \in Y$.

Definition 11. *A complex Pythagorean fuzzy hypergraph (CPFHG) on Y is defined as an ordered pair $H^* = (\mathcal{C}^*, \mathcal{D}^*)$, where $\mathcal{C}^* = \{\beta_1, \beta_2, \cdots, \beta_k\}$ is a finite family of CPFSs on Y and $\mathcal{D}^*$ is a CPFR on CPFSs β_j's such that,*

(i)

$$\begin{aligned} T_{\mathcal{D}^*}(\{s_1, s_2, \cdots, s_l\}) &\leq \min\{T_{\beta_j}(s_1), T_{\beta_j}(s_2), \cdots, T_{\beta_j}(s_l)\}, \\ F_{\mathcal{D}^*}(\{s_1, s_2, \cdots, s_l\}) &\leq \max\{F_{\beta_j}(s_1), F_{\beta_j}(s_2), \cdots, F_{\beta_j}(s_l)\},\ \text{for amplitude terms} \\ \phi_{\mathcal{D}^*}(\{s_1, s_2, \cdots, s_l\}) &\leq \min\{\phi_{\beta_j}(s_1), \phi_{\beta_j}(s_2), \cdots, \phi_{\beta_j}(s_l)\}, \\ \psi_{\mathcal{D}^*}(\{s_1, s_2, \cdots, s_l\}) &\leq \max\{\psi_{\beta_j}(s_1), \psi_{\beta_j}(s_2), \cdots, \psi_{\beta_j}(s_l)\},\ \text{for phase terms} \end{aligned}$$

$0 \leq T_{\mathcal{D}^*}^2 + F_{\mathcal{D}^*}^2 \leq 1$, and $\phi_{\mathcal{D}^*}, \psi_{\mathcal{D}^*} \in [0, 2\pi]$, for all $s_1, s_2, \cdots, s_l \in Y$.

(ii) *$\bigcup\limits_j supp(\beta_j) = Y$, for all $\beta_j \in \mathcal{C}^*$.*

Note that, $E_k = \{s_1, s_2, \cdots, s_l\}$ is the crisp hyperedge of $H^* = (\mathcal{C}^*, \mathcal{D}^*)$.

Example 5. *Consider a CPFHG $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ on $Y = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The CPFR is defined as, $\mathcal{D}^*(s_1, s_2, s_3) = ((0.6e^{i(0.2)\pi}, 0.5e^{i(0.9)\pi}))$, $\mathcal{D}^*(s_4, s_5, s_6) = (0.6e^{i(0.4)\pi}, 0.4e^{i(0.6)\pi})$, $\mathcal{D}^*(s_3, s_6) = (0.6e^{i(0.6)\pi}, 0.5e^{i(0.6)\pi})$, $\mathcal{D}^*(s_2, s_5) = (0.6e^{i(0.4)\pi}, 0.5e^{i(0.6)\pi})$, and $\mathcal{D}^*(s_1, s_4) = (0.6e^{i(0.2)\pi}, 0.9e^{i(0.9)\pi})$. The corresponding CPFHG is shown in Figure 5.*

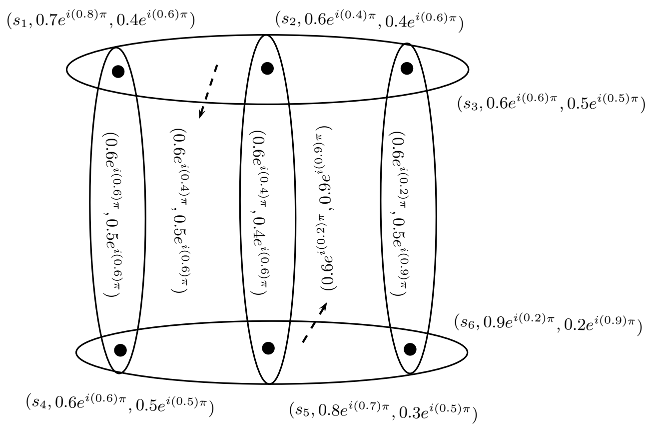

Figure 5. Complex Pythagorean fuzzy hypergraph.

Definition 12. *A CPFHG* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *is simple if whenever* $\mathcal{D}_j^*, \mathcal{D}_k^* \in \mathcal{D}^*$ *and* $\mathcal{D}_j^* \subseteq \mathcal{D}_k^*$*, then* $\mathcal{D}_j^* = \mathcal{D}_k^*$*. A CPFHG* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *is support simple if whenever* $\mathcal{D}_j^*, \mathcal{D}_k^* \in \mathcal{D}^*$, $\mathcal{D}_j^* \subseteq \mathcal{D}_k^*$*, and* $supp(\mathcal{D}_j^*) = supp(\mathcal{D}_k^*)$*, then* $\mathcal{D}_j^* = \mathcal{D}_k^*$*.*

Definition 13. *Let* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *be a CPFHG. Suppose that* $\alpha_1, \beta_1 \in [0,1]$ *and* $\theta, \varphi \in [0, 2\pi]$ *such that* $0 \leq \alpha_1^2 + \beta_1^2 \leq 1$*. The* $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})-$*level hypergraph of* H^* *is defined as an ordered pair* $H^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = (\mathcal{C}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})}, \mathcal{D}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})})$*, where*

(i) $\mathcal{D}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = \{D_j^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} : D_j^* \in \mathcal{D}^*\}$ *and* $D_j^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = \{y \in Y : T_{D_j^*}(y) \geq \alpha_1, \phi_{D_j^*}(y) \geq \theta$, and $F_{D_j^*}(y) \leq \beta_1, \psi_{D_j^*}(y) \leq \varphi\}$,

(ii) $\mathcal{C}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = \bigcup_{D_j^* \in \mathcal{D}^*} D_j^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})}$.

Note that, $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})-$*level hypergraph of* H^* *is a crisp hypergraph.*

Example 6. *Consider a CPFHG* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *as shown in Figure 5. Let* $\alpha_1 = 0.5$, $\beta_1 = 0.6$, $\theta = 0.3\pi$*, and* $\varphi = 0.7\pi$*. Then,* $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})-$*level hypergraph of* H^* *is shown in Figure 6.*

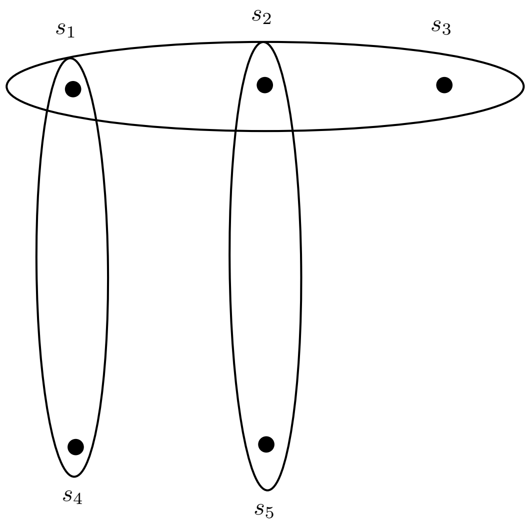

Figure 6. $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})-$level hypergraph of H^*.

Definition 14. *Let* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *be a CPFHG. The complex Pythagorean fuzzy line graph of* H^* *is defined as an ordered pair* $l(H^*) = (\mathcal{C}_l^*, \mathcal{D}_l^*)$*, where* $\mathcal{C}_l^* = \mathcal{D}^*$ *and there exists an edge between two vertices in* $l(H^*)$ *if* $|supp(D_j) \cap supp(D_k)| \geq 1$*, for all* $D_j, D_k \in \mathcal{D}^*$*. The membership degrees of* $l(H^*)$ *are given as,*

(i) $\mathcal{C}_l^*(E_k) = \mathcal{D}^*(E_k)$,

(ii) $\mathcal{D}_l^*(E_j E_k) = (\min\{T_{\mathcal{D}^*}(E_j), T_{\mathcal{D}^*}(E_k)\} e^{i \min\{\phi_{\mathcal{D}^*}(E_j), \phi_{\mathcal{D}^*}(E_k)\}}, \max\{F_{\mathcal{D}^*}(E_j), F_{\mathcal{D}^*}(E_k)\}$
$e^{i \max\{\psi_{\mathcal{D}^*}(E_j), \psi_{\mathcal{D}^*}(E_k)\}})$.

Definition 15. *A CPFHG* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *is said to be linear if for every* $D_j, D_k \in \mathcal{D}^*$,

(i) $supp(D_j) \subseteq supp(D_k) \Rightarrow j = k$,

(ii) $|supp(D_j) \cap supp(D_k)| \leq 1$.

Example 7. *Consider a CPFHG* $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ *as shown in Figure 5. By direct calculations, we have*

$$\begin{aligned} supp(\mathcal{D}_1) &= \{s_1, s_2, s_3\},\ supp(\mathcal{D}_2) = \{s_4, s_5, s_6\},\ supp(\mathcal{D}_3) = \{s_1, s_4\}, \\ supp(\mathcal{D}_4) &= \{s_2, s_5\},\ supp(\mathcal{D}_5) = \{s_3, s_6\}. \end{aligned}$$

Note that, $supp(D_j) \subseteq supp(D_k) \Rightarrow j = k$ and $|supp(D_j) \cap supp(D_k)| \leq 1$. Hence, CPFHG $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ is linear. The corresponding CPFHG $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ and its line graph is shown in Figure 7.

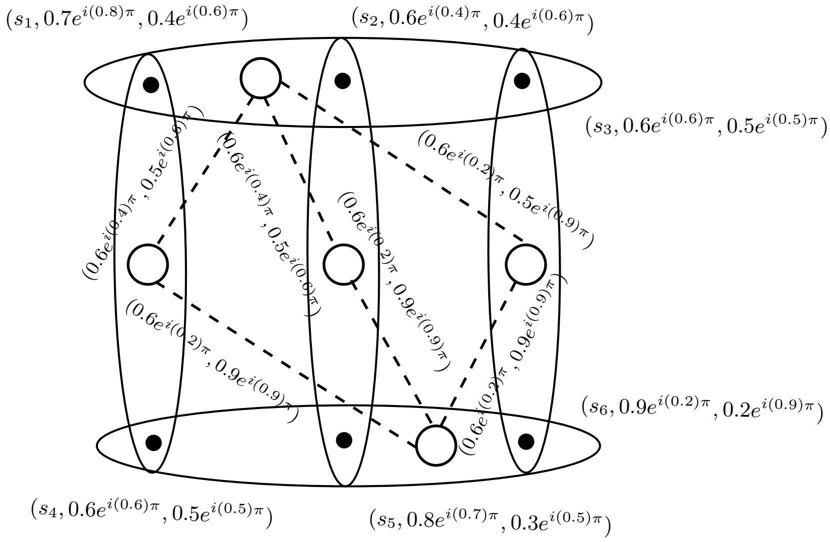

Figure 7. Line graph of complex Pythagorean fuzzy hypergraph H^*.

Theorem 2. *A simple strong CPFG is the complex Pythagorean fuzzy line graph of a linear CPFHG.*

Definition 16. *The 2-section $H_2^* = (\mathcal{C}_2^*, \mathcal{D}_2^*)$ of a CPFHG $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ is a CPFG having same set of vertices as that of H^*, $\mathcal{D}_2^*$ is a CPFS on $\{e = u_j u_k | u_j, u_k \in E_l,\ l = 1, 2, 3, \cdots\}$, and $\mathcal{D}_2^*(u_j u_k) = (\min\{\min T_{\beta_l}(u_j), \min T_{\beta_l}(u_k)\} e^{i \min\{\min \phi_{\beta_l}(u_j), \min \phi_{\beta_l}(u_k)\}}, \max\{\max F_{\beta_l}(u_j), \max F_{\beta_l}(u_k)\}$ $e^{i \max\{\max \psi_{\beta_l}(u_j), \max \psi_{\beta_l}(u_k)\}})$ such that $0 \leq T^2_{\mathcal{D}_2^*}(u_j u_k) + F^2_{\mathcal{D}_2^*}(u_j u_k) \leq 1$, $\phi_{\mathcal{D}_2^*}, \psi_{\mathcal{D}_2^*} \in [0, 2\pi]$.*

Example 8. *An example of a CPFHG is given in Figure 8. The 2-section of H^* is presented with dashed lines.*

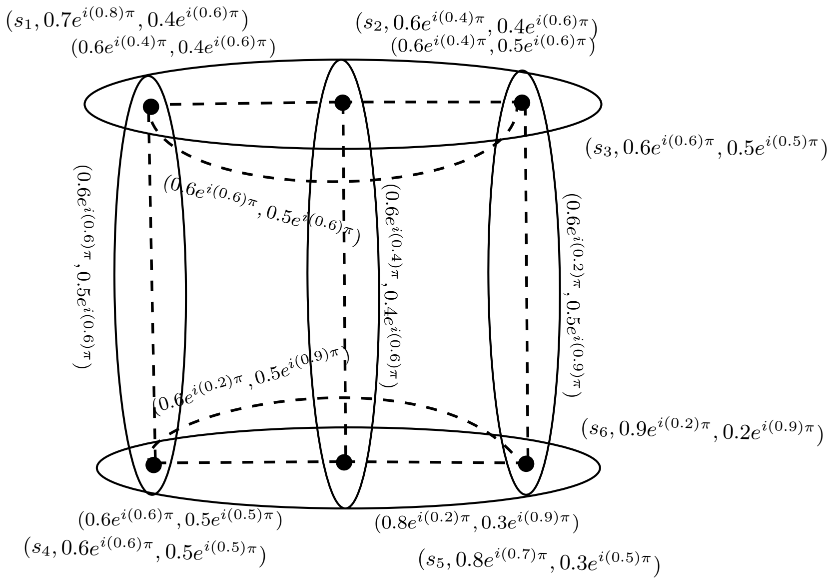

Figure 8. Two-section of complex Pythagorean fuzzy hypergraph H^*.

Definition 17. *Let $H^* = (\mathcal{C}^*, \mathcal{D}^*)$ be a CPFHG. A complex Pythagorean fuzzy transversal (CPFT) τ is a CPFS of Y satisfying the condition $\rho^{h(\rho)} \cap \tau^{h(\rho)} \neq \emptyset$, for all $\rho \in \mathcal{D}^*$, where $h(\rho)$ is the height of ρ.*

A minimal complex Pythagorean fuzzy transversal t is the CPFT of H^* having the property that if $\tau \subset t$, then τ is not a CPFT of H^*.

4. Complex q-Rung Orthopair Fuzzy Hypergraphs

This section explores the class of complex q-rung orthopair fuzzy graphs and complex q-rung orthopair fuzzy hypergraphs. Complex q-rung orthopair fuzzy hypergraphs generalize the notions of CIFHGs and CPFHGs. The class of Cq-ROFSs extends the classes of CIFSs and CPFSs. The space of Cq-ROFSs increases as the value of parameter q increases. Based on these advantages of Cq-ROFSs, we combine the theories of Cq-ROFSs and graphs to define complex q-rung orthopair fuzzy graphs and complex q-rung orthopair fuzzy hypergraphs.

Definition 18. *[13] A q-rung orthopair fuzzy set (q-ROFS) Q in the universal set Y is defined as, $Q = \{(u, T_Q(u), F_Q(u))|u \in Y\}$, where the function $T_Q : Y \to [0,1]$ defines the truth-membership and $F_Q : Y \to [0,1]$ defines the falsity-membership of the element $u \in Y$ and for every $u \in Y$, $0 \leq T_Q^q(u) + F_Q^q(u) \leq 1$, $q \geq 1$. Furthermore, $\pi_Q(u) = \sqrt[q]{1 - T_Q^q(u) - F_Q^q(u)}$ is called the indeterminacy degree or q-ROF index of u to the set Q.*

Definition 19. *A complex q-rung orthopair fuzzy set (Cq-ROFS) S in the universal set Y is given as,*

$$S = \{(u, T_S(u)e^{i\phi_S(u)}, F_S(u)e^{i\psi_S(u)})|u \in Y\},$$

where $i = \sqrt{-1}$, $T_S(u), F_S(u) \in [0,1]$, $\phi_S(u), \psi_S(u) \in [0, 2\pi]$, and for every $u \in Y$, $0 \leq T_S^q(u) + F_S^q(u) \leq 1$, $q \geq 1$.

Remark 1.

- *When $q = 1$, C1-ROFS is called a CIFS.*
- *When $q = 2$, C2-ROFS is called a CPFS.*

Definition 20. *Let $S_1 = \{(u, T_{S_1}(u)e^{i\phi_{S_1}(u)}, F_{S_1}(u)e^{i\psi_{S_1}(u)})|u \in Y\}$ and $S_2 = \{(u, T_{S_2}(u)e^{i\phi_{S_2}}(u), F_{S_2}(u)e^{i\psi_{S_2}}(u))|u \in Y\}$ be two Cq-ROFSs in Y, then*

(i) *$S_1 \subseteq S_2 \Leftrightarrow T_{S_1} \leq T_{S_2}(u)$, $F_{S_1}(u) \geq F_{S_2}(u)$, and $\phi_{S_1}(u) \leq \phi_{S_2}(u)$, $\psi_{S_1}(u) \geq \psi_{S_2}(u)$ for amplitudes and phase terms, respectively, for all $u \in Y$.*
(ii) *$S_1 = S_2 \Leftrightarrow T_{S_1} = T_{S_2}(u)$, $F_{S_1}(u) = F_{S_2}(u)$, and $\phi_{S_1}(u) = \phi_{S_2}(u)$, $\psi_{S_1}(u) = \psi_{S_2}(u)$ for amplitudes and phase terms, respectively, for all $u \in Y$.*

Definition 21. *Let $S_1 = \{(u, T_{S_1}(u)e^{i\phi_{S_1}(u)}, F_{S_1}(u)e^{i\psi_{S_1}(u)})|u \in Y\}$ and $S_2 = \{(u, T_{S_2}(u)e^{i\phi_{S_2}}(u), F_{S_2}(u)e^{i\psi_{S_2}}(u))|u \in Y\}$ be two Cq-ROFSs in Y, then*

(i) $S_1 \cup S_2 = \{(u, \max\{T_{S_1}(u), T_{S_2}(u)\}e^{i\max\{\phi_{S_1}(u), \phi_{S_2}(u)\}}, \min\{F_{S_1}(u), F_{S_2}(u)\}e^{i\min\{\psi_{S_1}(u), \psi_{S_2}(u)\}}) |u \in Y\}$.
(ii) $S_1 \cap S_2 = \{(u, \min\{T_{S_1}(u), T_{S_2}(u)\}e^{i\min\{\phi_{S_1}(u), \phi_{S_2}(u)\}}, \max\{F_{S_1}(u), F_{S_2}(u)\}e^{i\max\{\psi_{S_1}(u), \psi_{S_2}(u)\}}) |u \in Y\}$.

Definition 22. *A complex q-rung orthopair fuzzy relation (Cq-ROFR) is a Cq-ROFS in $Y \times Y$ given as,*

$$R = \{(rs, T_R(rs)e^{i\phi_R(rs)}, F_R(rs)e^{i\psi_R(rs)})|rs \in Y \times Y\},$$

where $i = \sqrt{-1}$, $T_R : Y \times Y \to [0,1]$, $F_R : Y \times Y \to [0,1]$ characterize the truth and falsity degrees of R, and $\phi_R(rs), \psi_R(rs) \in [0, 2\pi]$ such that for all $rs \in Y \times Y$, $0 \leq T_R^q(rs) + F_R^q(rs) \leq 1$, $q \geq 1$.

Example 9. *Let $Y = \{b_1, b_2, b_3\}$ be the universal set and $\{b_1b_2, b_2b_3, b_1b_3\}$ be the subset of $Y \times Y$. Then, the C5-ROFR R is given as,*

$$R = \{(b_1b_2, 0.9e^{i(0.7)\pi}, 0.7e^{i(0.9)\pi}), (b_2b_3, 0.6e^{i(0.7)\pi}, 0.8e^{i(0.9)\pi}), (b_1b_3, 0.7e^{i(0.8)\pi}, 0.5e^{i(0.6)\pi})\}.$$

Note that, $0 \leq T_R^5(xy) + F_R^5(xy) \leq 1$, for all $xy \in Y \times Y$. Hence, R is a C5-ROFR on Y.

Definition 23. *A complex q-rung orthopair fuzzy graph(Cq-ROFG) on Y is an ordered pair $\mathcal{G} = (\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ is a complex q-rung orthopair fuzzy set on Y and $\mathcal{B}$ is complex q-rung orthopair fuzzy relation on Y such that,*

$$\begin{aligned}
&T_{\mathcal{B}}(ab) \leq \min\{T_{\mathcal{A}}(a), T_{\mathcal{A}}(b)\},\\
&F_{\mathcal{B}}(ab) \leq \max\{F_{\mathcal{A}}(a), F_{\mathcal{A}}(b)\}, \text{ (for amplitude terms)}\\
&\phi_{\mathcal{B}}(ab) \leq \min\{\phi_{\mathcal{A}}(a), \phi_{\mathcal{A}}(b)\},\\
&\psi_{\mathcal{B}}(ab) \leq \max\{\psi_{\mathcal{A}}(a), \psi_{\mathcal{A}}(b)\}, \text{ (for phase terms)}
\end{aligned}$$

$0 \leq T_{\mathcal{B}}^q(ab) + F_{\mathcal{B}}^q(ab) \leq 1$, $q \geq 1$, *for all $a, b \in Y$.*

Remark 2. *Note that,*

- *When $q = 1$, C1-ROFG is called a CIFG.*
- *When $q = 2$, C2-ROFG is called a CPFG.*

Example 10. *Let $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ be a C6-ROFG on $Y = \{s_1, s_2, s_3, s_4\}$, where $\mathcal{A} = \{(s_1, 0.7e^{i(0.9)\pi}, 0.9e^{i(0.7)\pi}), (s_2, 0.5e^{i(0.6)\pi}, 0.6e^{i(0.5)\pi}), (s_3, 0.7e^{i(0.4)\pi}, 0.4e^{i(0.7)\pi}), (s_4, 0.8e^{i(0.5)\pi}, 0.5e^{i(0.8)\pi})\}$ and $\mathcal{B} = \{(s_1s_4, 0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi}), (s_2s_4, 0.5e^{i(0.5)\pi}, 0.6e^{i(0.8)\pi}), (s_3s_4, 0.7e^{i(0.4)\pi}, 0.5e^{i(0.8)\pi})\}$ are C6-ROFS and C6-ROFR on Y, respectively. The corresponding C6-ROFG $\mathcal{G}$ is shown in Figure 9.*

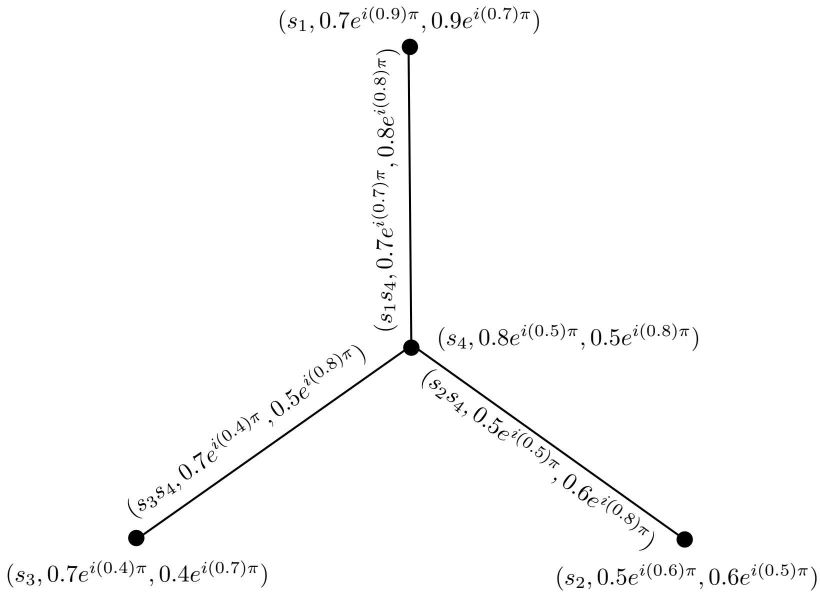

Figure 9. Complex six-rung orthopair fuzzy graph.

We now define the more extended concept of complex q-ROF hypergraphs.

Definition 24. *The support of a Cq-ROFS* $S = \{(u, T_S(u)e^{i\phi_S(u)}, F_S(u)e^{i\psi_S(u)})|u \in Y\}$ *is defined as* $supp(S) = \{u|T_S(u) \neq 0, F_S(u) \neq 1, 0 < \phi_S(u), \psi_S(u) < 2\pi\}$.. *The height of a Cq-ROFS* $S = \{(u, T_S(u)e^{i\phi_S(u)}, F_S(u)e^{i\psi_S(u)})|u \in Y\}$ *is defined as*

$$h(S) = \{\max_{u\in Y} T_S(u)e^{i\max\limits_{u\in Y}\phi_S(u)}, \min_{u\in Y} F_S(u)e^{i\min\limits_{u\in Y}\psi_S(u)}\}.$$

If $h(S) = (1e^{i2\pi}, 0e^{i0})$, *then S is called normal.*

Definition 25. *Let Y be a non-trivial set of universe. A complex q-rung orthopair fuzzy hypergraph (Cq-ROFHG) is defined as an ordered pair* $\mathcal{H} = (\mathcal{Q}, \eta)$, *where* $\mathcal{Q} = \{Q_1, Q_2, \cdots, Q_k\}$ *is a finite family of complex q-rung orthopair fuzzy sets on Y and* η *is a complex q-rung orthopair fuzzy relation on complex q-rung orthopair fuzzy sets* Q_j*'s such that,*

(i)

$$\begin{aligned}
T_\eta(\{a_1, a_2, \cdots, a_l\}) &\leq \min\{T_{Q_j}(a_1), T_{Q_j}(a_2), \cdots, T_{Q_j}(a_l)\},\\
F_\eta(\{a_1, a_2, \cdots, a_l\}) &\leq \max\{F_{Q_j}(a_1), F_{Q_j}(a_2), \cdots, F_{Q_j}(a_l)\}, \text{ (for amplitude terms)}\\
\phi_\eta(\{a_1, a_2, \cdots, a_l\}) &\leq \min\{\phi_{Q_j}(a_1), \phi_{Q_j}(a_2), \cdots, \phi_{Q_j}(a_l)\},\\
\psi_\eta(\{a_1, a_2, \cdots, a_l\}) &\leq \max\{\psi_{Q_j}(a_1), \psi_{Q_j}(a_2), \cdots, \psi_{Q_j}(a_l)\}, \text{ (for phase terms)}
\end{aligned}$$

$0 \leq T_\eta^q + F_\eta^q \leq 1$, $q \geq 1$, *for all* $a_1, a_2, \cdots, a_l \in Y$.

(ii) $\bigcup\limits_j supp(Q_j) = X$, *for all* $Q_j \in \mathcal{Q}$.

Note that, $E_k = \{a_1, a_2, \cdots, a_l\}$ *is the crisp hyperedge of* $\mathcal{H} = (\mathcal{Q}, \eta)$.

Remark 3. *Note that,*

- *When* $q = 1$, *C1-ROFHG is a CIFHG.*
- *When* $q = 2$, *C2-ROFHG is a CPFHG.*

Definition 26. *Let* $\mathcal{H} = (\mathcal{Q}, \eta)$ *be a Cq-ROFHG. The height of* $\mathcal{H}$, *given as* $h(\mathcal{H})$, *is defined as* $h(\mathcal{H}) = (\max \eta_l e^{i\max\phi}, \min \eta_m e^{i\min\psi})$, *where* $\eta_l = \max T_{\rho_j}(x_k)$, $\phi = \max \phi_{\rho_j}(x_k)$, $\eta_m = \min F_{\rho_j}(x_k)$, $\psi = \min \psi_{\rho_j}(x_k)$. *Here,* $T_{\rho_j}(x_k)$ *and* $F_{\rho_j}(x_k)$ *denote the truth and falsity degrees of vertex* x_k *to hyperedge* ρ_j, *respectively.*

Definition 27. *Let* $\mathcal{H} = (\mathcal{Q}, \eta)$ *be a Cq-ROFHG. Suppose that* $\mu, \nu \in [0, 1]$ *and* $\theta, \varphi \in [0, 2\pi]$ *such that* $0 \leq \mu^q + \nu^q \leq 1$. *The* $(\mu e^{i\theta}, \nu e^{i\varphi})-$*level hypergraph of* $\mathcal{H}$ *is defined as an ordered pair* $\mathcal{H}^{(\mu e^{i\theta}, \nu e^{i\varphi})} = (\mathcal{Q}^{(\mu e^{i\theta}, \nu e^{i\varphi})}, \eta^{(\mu e^{i\theta}, \nu e^{i\varphi})})$, *where*

(i) $\eta^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \{\rho_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} : \rho_j \in \eta\}$ *and* $\rho_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \{u \in Y : T_{\rho_j}(u) \geq \mu, \phi_{\rho_j}(u) \geq \theta, \text{ and } F_{\rho_j}(u) \leq \nu, \psi_{\rho_j}(u) \leq \varphi\}$,

(ii) $\mathcal{Q}^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \bigcup\limits_{\rho_j \in \eta} \rho_j^{(\mu e^{i\theta}, \nu e^{i\varphi})}$.

Note that, $(\mu e^{i\theta}, \nu e^{i\varphi})-$*level hypergraph of* $\mathcal{H}$ *is a crisp hypergraph.*

Example 11. *Consider a C6-ROFHG* $\mathcal{H} = (\mathcal{Q}, \eta)$ *on* $Y = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. *The C6-ROFR* η *is given as,* $\eta(u_1, u_2, u_3) = (0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$, $\eta(u_3, u_4, u_5) = (0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$, $\eta(u_1, u_6) = (0.8e^{i(0.8)\pi}, 0.8e^{i(0.8)\pi})$ *and* $\eta(u_4, u_6) = (0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$. *The incidence matrix of* $\mathcal{H}$ *is given in Table 1.*

Table 1. Incidence matrix of C6-ROFHG $\mathcal{H}$.

$u \in Y$	η_1	η_2	η_3	η_4
u_1	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$	$(0,0)$	$(0,0)$	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$
u_2	$(0.7e^{i(0.7)\pi}, 0.6e^{i(0.6)\pi})$	$(0,0)$	$(0,0)$	$(0,0)$
u_3	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	$(0,0)$	$(0,0)$
u_4	$(0,0)$	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	$(0,0)$
u_5	$(0,0)$	$(0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$	$(0,0)$	$(0,0)$
u_6	$(0,0)$	$(0,0)$	$(0.9e^{i(0.9)\pi}, 0.8e^{i(0.8)\pi})$	$(0.9e^{i(0.9)\pi}, 0.8e^{i(0.8)\pi})$

The corresponding C6-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ is shown in Figure 10.

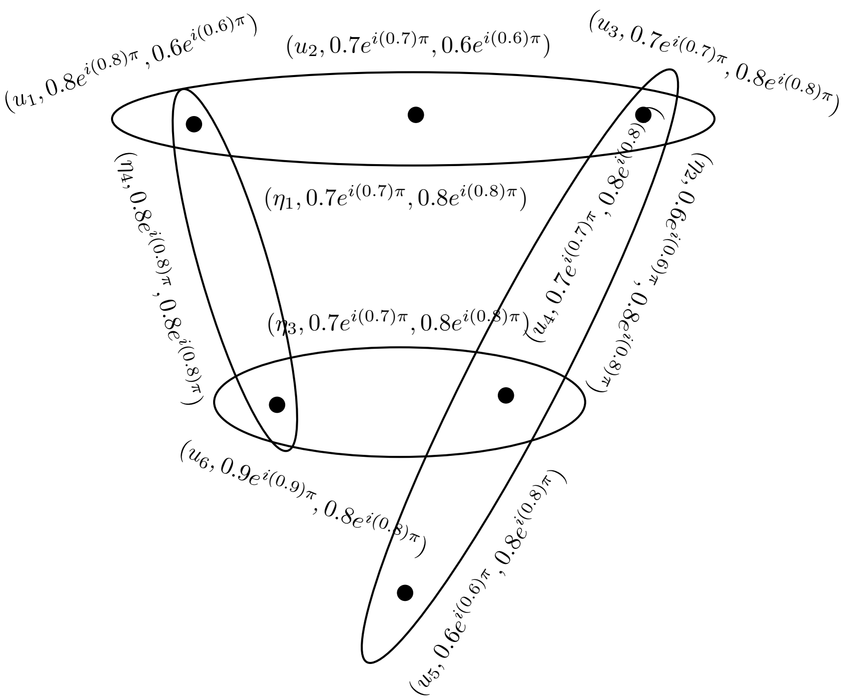

Figure 10. Complex six-rung orthopair fuzzy hypergraph.

Let $\mu = 0.7$, $\nu = 0.6$, $\theta = 0.7\pi$, and $\varphi = 0.6\pi$, then $(0.7e^{i(0.7)\pi}, 0.6e^{i(0.6)\pi})-$level hypergraph of $\mathcal{H}$ is shown in Figure 11.

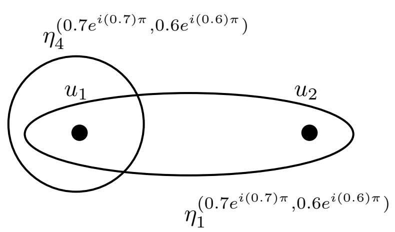

Figure 11. The $(0.7e^{i(0.7)\pi}, 0.6e^{i(0.6)\pi})-$level hypergraph of $\mathcal{H}$.

Note that,

$$\begin{aligned}\eta_1^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} &= \{u_1,u_2\},\ \eta_2^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} = \{\emptyset\},\\ \eta_3^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} &= \{\emptyset\},\ \eta_4^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} = \{u_1\}.\end{aligned}$$

5. Transversals of Complex q-Rung Orthopair Fuzzy Hypergraphs

In this section we study transversality. Prior to the main definition we need the following auxiliary concept:

Definition 28. *Let $\mathcal{H} = (\mathcal{Q}, \eta)$ be a Cq-ROFHG and for $0 < \mu \leq T(h(\mathcal{H}))$, $\nu \geq F(h(\mathcal{H})) > 0$, $0 < \theta \leq \phi(h(\mathcal{H}))$, and $\varphi \geq \psi(h(\mathcal{H})) > 0$ let $\mathcal{H}^{(\mu e^{i\theta},\nu e^{i\varphi})} = (\mathcal{Q}^{(\mu e^{i\theta},\nu e^{i\varphi})}, \eta^{(\mu e^{i\theta},\nu e^{i\varphi})})$ be the level hypergraph of $\mathcal{H}$. The sequence of complex numbers $\{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ such that $0 < \mu_1 < \mu_2 < \cdots < \mu_n = T(h(\mathcal{H}))$, $\nu_1 > \nu_2 > \cdots > \nu_n = F(h(\mathcal{H})) > 0$, $0 < \theta_1 < \theta_2 < \cdots < \theta_n = \phi(h(\mathcal{H}))$, and $\varphi_1 > \varphi_2 > \cdots > \varphi_n = \psi(h(\mathcal{H})) > 0$ satisfying the conditions,*

- **(i)** *if $\mu_{k+1} < \alpha \leq \mu_k$, $\nu_{k+1} > \beta \geq \nu_k$, $\theta_{k+1} < \phi \leq \theta_k$, $\varphi_{k+1} > \psi \geq \varphi_k$, then $\eta^{(\alpha e^{i\phi},\beta e^{i\psi})} = \eta^{(\mu_k e^{i\theta_k},\nu_k e^{i\varphi_k})}$, and*
- **(ii)** $\eta^{(\mu_k e^{i\theta_k},\nu_k e^{i\varphi_k})} \subset \eta^{(\mu_{k+1} e^{i\theta_{k+1}},\nu_{k+1} e^{i\varphi_{k+1}})}$,

is called the fundamental sequence of $\mathcal{H} = (\mathcal{Q}, \eta)$, denoted by $\mathcal{F}_s(\mathcal{H})$. The set of $(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})-$level hypergraphs $\{\mathcal{H}^{(\mu_1 e^{i\theta_1},\nu_1 e^{i\varphi_1})}, \mathcal{H}^{(\mu_2 e^{i\theta_2},\nu_2 e^{i\varphi_2})}, \cdots, \mathcal{H}^{(\mu_n e^{i\theta_n},\nu_n e^{i\varphi_n})}\}$ is called the set of core hypergraphs or the core set of $\mathcal{H}$, denoted by $cor(\mathcal{H})$.

Now we are ready to define:

Definition 29. *Let $\mathcal{H} = (\mathcal{Q}, \eta)$ be a Cq-ROFHG. A complex q-rung orthopair fuzzy transversal (Cq-ROFT) τ is a Cq-ROFs of Y satisfying the condition $\rho^{h(\rho)} \cap \tau^{h(\rho)} \neq \emptyset$, for all $\rho \in \eta$, where $h(\rho)$ is the height of ρ.*

A minimal complex q-rung orthopair fuzzy transversal t is the Cq-ROFT of $\mathcal{H}$ having the property that if $\tau \subset t$, then τ is not a Cq-ROFT of $\mathcal{H}$.

Let us denote the family of minimal Cq-ROFTs of $\mathcal{H}$ by $t_r(\mathcal{H})$.

Example 12. *Consider a C5-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ on $Y = \{a_1, a_2, a_3, a_4, a_5\}$. The C5-ROFR η is given as, $\eta(\{a_1 a_3, a_4\}) = (0.6e^{i(0.6)\pi}, 0.9e^{i(0.9)\pi})$, $\eta(\{a_2, a_3, a_5\}) = (0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi})$, and $\eta(\{a_1, a_2, a_4\}) = (0.6e^{i(0.6)\pi}, 0.9e^{i(0.9)\pi})$. The incidence matrix of $\mathcal{H}$ is given in Table 2.*

Table 2. Incidence matrix of C5-ROFHG $\mathcal{H}$.

$a \in Y$	η_1	η_2	η_3
a_1	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$	$(0,0)$
a_2	$(0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi})$	$(0,0)$	$(0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi})$
a_3	$(0,0)$	$(0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$	$(0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$
a_4	$(0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$	$(0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$	$(0,0)$
a_5	$(0,0)$	$(0,0)$	$(0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})$

The corresponding C5-ROFHG is shown in Figure 12.

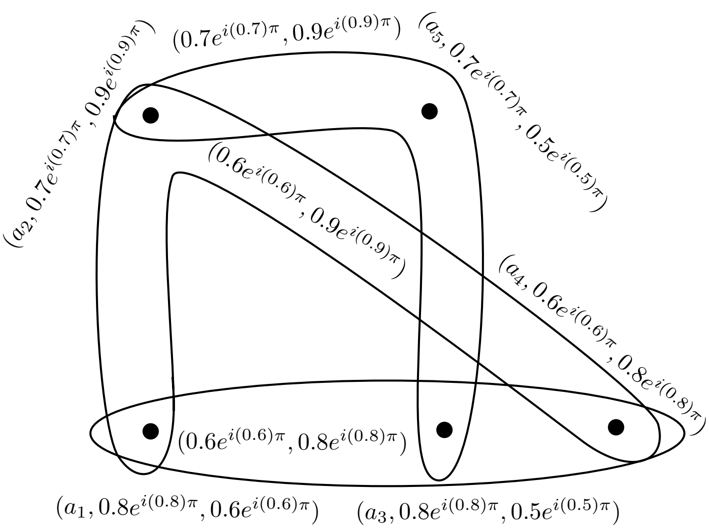

Figure 12. Complex five-rung orthopair fuzzy hypergraph.

By routine calculations, we have $h(\eta_1) = (0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$, $h(\eta_2) = (0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$, *and* $h(\eta_3) = (0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$. *Consider a C5-ROFS* τ_1 *of* Y *such that* $\tau_1 = \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_2, 0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})\}$. *Note that,*

$$\begin{aligned}
\eta_1^{(0.8e^{i(0.8)\pi},0.6e^{i(0.6)\pi})} &= \{a_1\}, \ \eta_2^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\}, \ \eta_3^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\}, \\
\tau_1^{(0.8e^{i(0.8)\pi},0.6e^{i(0.6)\pi})} &= \{a_1, a_3\}, \ \tau_1^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\}, \ \tau_1^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\}.
\end{aligned}$$

Thus, we have $\eta_j^{h(\eta_j)} \cap \tau_1^{h(\eta_j)} \neq \emptyset$, *for all* $\eta_j \in \eta$. *Hence,* τ_1 *is a C5-ROFT of* $\mathcal{H}$. *Similarly,*

$$\begin{aligned}
\tau_2 &= \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})\}, \\
\tau_3 &= \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi}), (a_4, 0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})\}, \\
\tau_4 &= \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi}), (a_5, 0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})\},
\end{aligned}$$

are C5-ROFTs of $\mathcal{H}$.

Definition 30. *A Cq-ROFHG* $\mathcal{H}_1 = (\mathcal{Q}_1, \eta_1)$ *is a partial Cq-ROFHG of* $\mathcal{H}_2 = (\mathcal{Q}_2, \eta_2)$ *if* $\eta_1 \subseteq \eta_2$, *denoted by* $\mathcal{H}_1 \subseteq \mathcal{H}_2$. *A Cq-ROFHG* $\mathcal{H}_1 = (\mathcal{Q}_1, \eta_1)$ *is ordered if the core set* $cor(\mathcal{H}) = \{\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \dots, \mathcal{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$ *is ordered, i.e.,* $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})} \subseteq \mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})} \subseteq \dots \subseteq \mathcal{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}$. $\mathcal{H}$ *is simply ordered if* $\mathcal{H}$ *is ordered and* $\eta' \subset \eta^{(\mu_{l+1} e^{i\theta_{l+1}}, \nu_{l+1} e^{i\varphi_{l+1}})} \setminus \eta^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})} \Rightarrow \eta' \nsubseteq \mathcal{Q}^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})}$.

Definition 31. *A Cq-ROFS* S *on* Y *is elementary if* S *is single-valued on* $supp(S)$. *A Cq-ROFHG* $\mathcal{H} = (\mathcal{Q}, \eta)$ *is elementary if every* $Q_j \in \mathcal{Q}$ *and* η *are elementary.*

Proposition 1. *If* τ *is a Cq-ROFT of* $\mathcal{H} = (\mathcal{Q}, \eta)$, *then* $h(\tau) \geq h(\rho)$, *for all* $\rho \in \eta$. *Furthermore, if* τ *is minimal Cq-ROFT of* $\mathcal{H} = (\mathcal{Q}, \eta)$, *then* $h(\tau) = \max\{h(\rho) | \rho \in \eta\} = h(\mathcal{H})$.

Lemma 1. *Let* $\mathcal{H}_1 = (\mathcal{Q}_1, \eta_1)$ *be a partial Cq-ROFHG of* $\mathcal{H}_2 = (\mathcal{Q}_2, \eta_2)$. *If* τ_2 *is minimal Cq-ROFT of* $\mathcal{H}_2$, *then there is a minimal Cq-ROFT of* $\mathcal{H}_1$ *such that* $\tau_1 \subseteq \tau_2$.

Proof. Let S_1 be a Cq-ROFS on Y, which is defined as $S_1 = \tau_2 \cap (\cup_{Q_{1j \in \mathcal{Q}_1}} Q_{1j})$. Then, S_1 is a Cq-ROFT of $\mathcal{H}_1 = (\mathcal{Q}_1, \eta_1)$. Thus, there exists a minimal Cq-ROFT of $\mathcal{H}_1$ such that $\tau_1 \subseteq S_1 \subseteq \tau_2$. □

Lemma 2. *Let $\mathcal{H} = (\mathcal{Q}, \eta)$ be a Cq-ROFHG then $f_s(t_r(\mathcal{H})) \subseteq f_s(\mathcal{H})$.*

Proof. Let $f_s(\mathcal{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ and $\tau \in t_r(\mathcal{H})$. Suppose that for $u \in supp(\tau)$, $(T_\tau(u), F_\tau(u)) \in (\mu_{j+1}, \mu_j] \times (\nu_{j+1}, \nu_j]$, $\phi_\tau(u) \in (\theta_{j+1}, \theta_j]$, and $\psi_\tau(u) \in (\varphi_{j+1}, \varphi_j]$. Define a function λ by

$$T_\lambda(v)e^{i\phi} = \begin{cases} \mu_j e^{i\theta_j}, & \text{if} \quad u = v, \\ T_\tau(u)e^{i\phi_\tau(u)}, & \text{otherwise.} \end{cases}, \quad F_\lambda(v)e^{i\psi} = \begin{cases} \mu_j e^{i\varphi_j}, & \text{if} \quad u = v, \\ F_\tau(u)e^{i\psi_\tau(u)}, & \text{otherwise.} \end{cases}$$

From definition of λ, we have $\lambda^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} = \tau^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$. Definition 28 implies that for every $t \in (\mu_{j+1} e^{i\theta_{j+1}}, \mu_j e^{\theta_j}] \times (\nu_{j+1} e^{i\varphi_{j+1}}, \nu_j e^{i\varphi_j}]$, $\mathcal{H}^t = \mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$. Thus, $\lambda^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ is a Cq-ROFT of $\mathcal{H}^t$. Since, τ is minimal Cq-ROFT and $\lambda^t = \tau^t$, for all $t \notin (\mu_{j+1} e^{i\theta_{j+1}}, \mu_j e^{\theta_j}] \times (\nu_{j+1} e^{i\varphi_{j+1}}, \nu_j e^{i\varphi_j}]$. This implies that λ is also a Cq-ROFT and $\lambda \leq \tau$ but the minimality of τ implies that $\lambda = \tau$. Hence, $\tau(u) = \lambda(u) = (\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})$, which implies that for every Cq-ROFT $\tau \in t_r(\mathcal{H})$ and for each $u \in Y$, $\tau(u) \in f_s(\mathcal{H})$ and so we have $f_s(t_r(\mathcal{H})) \subseteq f_s(\mathcal{H})$. □

We now illustrate a recursive procedure to find $t_r(\mathcal{H})$ in Algorithm 1.

Algorithm 1: To find the family of minimal Cq-ROFTs $t_r(\mathcal{H})$.

Let $\mathcal{H} = (\mathcal{Q}, \eta)$ be a Cq-ROFHG having the fundamental sequence $f_s(\mathcal{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ and core set $cor(\mathcal{H}) = \{\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \cdots, \mathcal{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$. The minimal transversal of $\mathcal{H} = (\mathcal{Q}, \eta)$ is determined as follows,

1. Determine a crisp minimal transversal t_1 of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$.
2. Determine a crisp minimal transversal t_2 of $\mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ satisfying the condition $t_1 \subseteq t_2$, i.e., obtain an hypergraph H_2 having the hyperedges $\eta^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ and a loop at every vertex $u \in t_1$. Thus, we have $\eta(H_2) = \eta(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}) \cup \{\{u \in t_1\}\}$.
3. Let t_2 be the minimal transversal of H_2.
4. Obtain a sequence of minimal transversals $t_1 \subseteq t_2 \subseteq \cdots \subseteq t_j$ such that t_j is the minimal transversal of $\mathcal{H}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ satisfying the condition $t_{j-1} \subseteq t_j$.
5. Define an elementary Cq-ROFS S_j having the support t_j and $h(S_j) = (\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})$, $1 \leq j \leq n$.
6. Determine a minimal Cq-ROFT of $\mathcal{H}$ as $\tau = \bigcup_{j=1}^{n} \{S_j | 1 \leq j \leq n\}$.

Example 13. *Consider a C5-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ on $Y = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown in Figure 13. Let $(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}) = (0.9e^{i(0.9)\pi}, 0.7e^{i(0.7)\pi})$, $(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}) = (0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$, $(\mu_3 e^{i\theta_3}, \nu_3 e^{i\varphi_3}) = (0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi})$, and $(\mu_4 e^{i\theta_4}, \nu_4 e^{i\varphi_4}) = (0.3e^{i(0.3)\pi}, 0.2e^{i(0.2)\pi})$. Clearly, the sequence $\{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), (\mu_3 e^{i\theta_3}, \nu_3 e^{i\varphi_3}), (\mu_4 e^{i\theta_4}, \nu_4 e^{i\varphi_4})\}$ satisfies all the conditions of Definition 28. Hence, it is the fundamental sequence of $\mathcal{H}$.*

Note that, $t_1 = t_2 = \{v_4\}$ is the minimal transversal of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$ and $\mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$, $t_3 = \{v_1\}$ is the minimal transversal of $\mathcal{H}^{(\mu_3 e^{i\theta_3}, \nu_3 e^{i\varphi_3})}$, and $t_4 = \{v_1, v_4\}$ is the minimal transversal of $\mathcal{H}^{(\mu_4 e^{i\theta_4}, \nu_4 e^{i\varphi_4})}$. Consider

$$\begin{aligned} S_1 &= \{(v_4, 0.9e^{i(0.9)\pi}, 0.7e^{i(0.7)\pi})\} = S_2, \\ S_3 &= \{(v_1, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})\}, \\ S_4 &= \{(v_1, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi}), (v_4, 0.9e^{i(0.9)\pi}, 0.7e^{i(0.7)\pi})\}. \end{aligned}$$

Hence, $\bigcup_{j=1}^{4} = \{(v_1, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi}), (v_4, 0.9e^{i(0.9)\pi}, 0.7e^{i(0.7)\pi})\}$ *is a C5-ROFT of* $\mathcal{H}$.

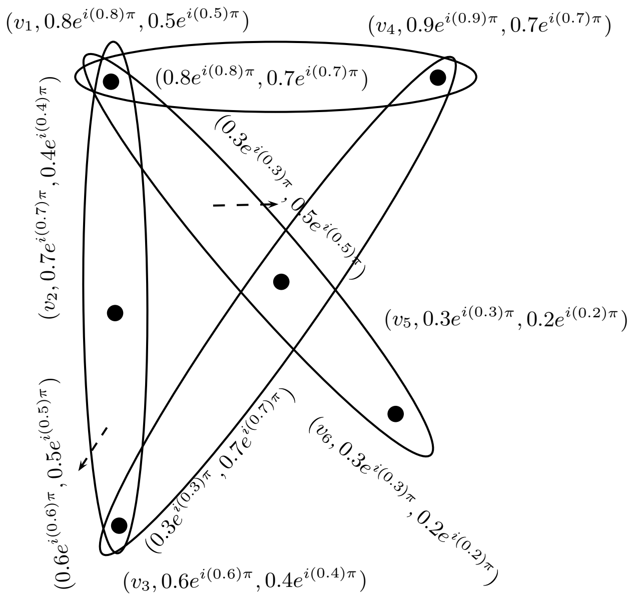

Figure 13. Complex five-rung orthopair fuzzy hypergraph.

Lemma 3. *Let* $\mathcal{H} = (\mathcal{Q}, \eta)$ *be a Cq-ROFHG with* $f_s(\mathcal{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$. *If* τ *is a Cq-ROFT of* $\mathcal{H}$, *then* $h(\tau) \geq h(Q_j)$, *for every* $Q_j \in \mathcal{Q}$. *If* $\tau \in t_r(\mathcal{H})$ *then* $h(\tau) = \max\{h(Q_j) | Q_j \in \mathcal{Q}\} = (\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})$.

Proof. Since τ is a Cq-ROFT of $\mathcal{H}$, implies that $\tau^{h(Q_j)} \cap Q_j^{h(Q_j)} \neq \emptyset$. Let $a \in supp(\tau)$, then $T_\tau(a) \geq T(h(Q_j))$, $F_\tau(a) \leq F(h(Q_j))$, $\phi_\tau(a) \geq \phi(h(Q_j))$, and $\psi_\tau(a) \leq \psi(h(Q_j))$. This shows that $h(\tau) \geq h(Q_j)$. If $\tau \in t_r(\mathcal{H})$, i.e., τ is minimal Cq-ROFT then $h(Q_j) = (\max T_{Q_j}(a) e^{i \max \phi_{Q_j}(a)}, \min F_{Q_j}(a) e^{i \min \psi_{Q_j}(a)}) = (\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})$. Thus, we have $h(\tau) = \max\{h(Q_j) | Q_j \in \mathcal{Q}\} = (\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})$. $\square$

Lemma 4. *Let* β *be a Cq-ROFT of a Cq-ROFHG* $\mathcal{H}$. *Then, there exists* $\gamma \in t_r(\mathcal{H})$ *such that* $\gamma \leq \beta$.

Proof. Let $f_s(\mathcal{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$. Suppose that $\lambda^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ is a transversal of $\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ and $\tau^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} \in t_r(\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$, for $1 \leq k \leq n$ such that $\tau^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} \subseteq \lambda^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$. Let β_k be an elementary Cq-ROFS having support λ_k and γ_k be an elementary Cq-ROFS having support τ_k, for $1 \leq k \leq n$. Then, Algorithm 1 implies that $\beta = \bigcup_{k=1}^{n} \beta_k$ is a Cq-ROFT of $\mathcal{H}$ and $\gamma = \bigcup_{k=1}^{n} \gamma_k$ is minimal Cq-ROFT of $\mathcal{H}$ such that $\gamma \leq \beta$. $\square$

Theorem 3. *Let* $\mathcal{H}_1 = (\mathcal{Q}_1, \eta_1)$ *and* $\mathcal{H}_2 = (\mathcal{Q}_2, \eta_2)$ *be Cq-ROFHGs. Then,* $\mathcal{Q}_2 = t_r(\mathcal{H}_1) \Leftrightarrow \mathcal{H}_2$ *is simple,* $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$, $h(\eta_k) = h(\mathcal{H}_1)$, *for every* $\rho_k \in \eta_2$, *and for every Cq-ROFS* $\xi \in \mathcal{P}(Y)$, *exactly one of the conditions must satisfy,*

(i) $\rho \leq \xi$, *for some* $\rho \in \mathcal{Q}_2$ *or*

(ii) *there is $Q_j \in \mathcal{Q}_1$ and $(\mu e^{i\theta}, \nu e^{i\varphi})$, where $(\mu,\nu) \in [0, T_{h(Q_j)}] \times [0, F_{h(Q_j)}]$, $\theta \in [0, \phi_{h(Q_j)}]$, $\varphi \in [0, \psi_{h(Q_j)}]$ such that $Q_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} \cap \xi^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \emptyset$, i.e., ξ is not a Cq-ROFT of $\mathcal{H}_1$.*

Proof. Let $\mathcal{Q}_2 = t_r(\mathcal{H}_1)$. Since, the family of all minimal Cq-ROFTs form a simple Cq-ROFHG on $Y_1 \subseteq Y_2$. Lemma 3 implies that every edge of $t_r(\mathcal{H}_1)$ has height $(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}) = h(\mathcal{H}_1)$. Let ξ be an arbitrary Cq-ROFS.

Case(i) If ξ is a Cq-ROFT of $\mathcal{H}_1$), then Lemma 4 implies the existence of a minimal Cq-ROFT ρ such that $\rho \leq \xi$. Thus, the condition (i) holds and (ii) violates.

Case(ii) If ξ is not a Cq-ROFT of $\mathcal{H}_1$), then there is an edge $Q_j \in \mathcal{Q}_1$ such that $Q_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} \cap \xi^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \emptyset$. If condition (i) holds, $\rho \leq \xi$ implies that $Q_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} \cap \rho^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \emptyset$, which is the contradiction against the fact that ρ is Cq-ROFT. Hence, condition (i) does not hold and (ii) is satisfied.

Conversely, suppose that $\mathcal{Q}_2$ satisfies all properties as mentioned above and $\rho \in \mathcal{Q}_2$. Let $\rho = \xi$, then we obtain $\rho \leq \rho$ and conditions (ii) is not satisfied, so ρ is Cq-ROFT of $\mathcal{H}_1$. If t is minimal Cq-ROFT of $\mathcal{H}_1$ and $t \leq \rho$, t does not satisfy (ii), this implies the existence of $\rho_2 \in \mathcal{Q}_2$ such that $\rho_2 \leq t$, hence $\mathcal{Q}_2 \subseteq t_r(\mathcal{H}_1)$. Since, t is minimal Cq-ROF which implies that $\rho = t$, ρ and t were chosen arbitrarily therefore, we have $\mathcal{Q}_2 = t_r(\mathcal{H}_1)$. □

The construction of fundamental subsequence and subcore of Cq-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ is discussed in Algorithm 2.

Algorithm 2: Construction of fundamental subsequence and subcore.

Let $\mathcal{H} = (\mathcal{Q}, \eta)$ be a Cq-ROFHG and $\mathcal{H}_1 = (\mathcal{Q}_1, \eta_1)$ be a partial Cq-ROFHG of $\mathcal{H}$. The fundamental subsequence $f_{ss}(\mathcal{H})$ is constructed as follows:
Let $f_s(\mathcal{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ and
$cor(\mathcal{H}) = \{\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \ldots, \mathcal{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$.

1. Construct $\widetilde{\mathcal{H}}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$, a partial hypergraph of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$, by removing all hyperedges of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$, which contain properly any other hyperedge of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$.
2. In the same way, a partial hypergraph $\widetilde{\mathcal{H}}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ of $\mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ is constructed by removing all hyperedges of $\mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$, which contain properly any other hyperedge of $\mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ or any other hyperedge of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$. $\widetilde{\mathcal{H}}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ is non-trivial iff there exists a Cq-ROFT $\tau \in t_r(\mathcal{H})$ and a vertex $u \in \mathcal{Q}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ such that $(T_\tau(u) e^{i\phi_\tau(u)}, F_\tau(u) e^{i\psi_\tau(u)}) = (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})$.
3. Continuing the same procedure, construct $\widetilde{\mathcal{H}}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, a partial hypergraph of $\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, by removing all hyperedges of $\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, which contain properly any other hyperedge of $\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ or contain any other hyperedge of $\mathcal{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathcal{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \cdots, \mathcal{H}^{(\mu_{k-1} e^{i\theta_{k-1}}, \nu_{k-1} e^{i\varphi_{k-1}})}$. $\widetilde{\mathcal{H}}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ is non-trivial iff there exists a Cq-ROFT $\tau \in t_r(\mathcal{H})$ and an element $u \in \mathcal{Q}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ such that $(T_\tau(u) e^{i\phi_\tau(u)}, F_\tau(u) e^{i\psi_\tau(u)}) = (\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})$.
4. Let $\{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1}), (\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2}), \cdots, (\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})\}$ be the set of complex numbers such that the corresponding partial hypergraphs $\widetilde{\mathcal{H}}^{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1})}, \widetilde{\mathcal{H}}^{(\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2})}, \cdots, \widetilde{\mathcal{H}}^{(\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})}$ are non-empty.
5. Then, $f_{ss}(\mathcal{H}) = \{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1}), (\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2}), \cdots, (\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})\}$ and $\widetilde{cor}(\mathcal{H}) = \{\widetilde{\mathcal{H}}^{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1})}, \widetilde{\mathcal{H}}^{(\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2})}, \cdots, \widetilde{\mathcal{H}}^{(\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})}\}$ are subsequence and subcore set of $\mathcal{H}$, respectively.

Definition 32. *Let $\mathcal{H}=(\mathcal{Q},\eta)$ be a Cq-ROFHG having fundamental subsequence $f_{ss}(\mathcal{H})$ and subcore $\widetilde{cor}(\mathcal{H})$ of $\mathcal{H}$. The Cq-ROFT core of $\mathcal{H}$ is defined as an elementary Cq-ROFHG $\widehat{\mathcal{H}}=(\widehat{\mathcal{Q}},\widehat{\eta})$ such that,*

- **(i)** *$f_{ss}(\mathcal{H})=f_{ss}(\widehat{\mathcal{H}})$, i.e., $f_{ss}(\mathcal{H})$ is also a fundamental subsequence of $\widehat{\mathcal{H}}$,*
- **(ii)** *height of every $\widehat{Q}_j\in\widehat{\mathcal{Q}}$ is $(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})\in f_{ss}(\mathcal{H})$ iff $supp(\widehat{Q}_j)$ is an hyperedge of $\widehat{\mathcal{H}}^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}$.*

Theorem 4. *For every Cq-ROFHG, we have $t_r(\mathcal{H})=t_r(\widehat{\mathcal{H}})$.*

Proof. Let $t\in t_r(\mathcal{H})$ and $\widehat{Q}_j\in\widehat{\mathcal{Q}}$. Definition 32 implies that $h(\widehat{Q}_j)=(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})$ and $\widehat{Q}_j^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}$ is an hyperedge of $\widetilde{\mathcal{H}}^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}$. Since $\widetilde{\mathcal{H}}^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}\subseteq\mathcal{H}^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}$ and $\tau^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}$ is a transversal of $\mathcal{H}^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}$ therefore $\widehat{Q}_j^{(\tilde{\mu}_je^{i\tilde{\theta}_j},\tilde{\nu}_je^{i\tilde{\varphi}_j})}\cap\tau^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}\neq\emptyset$. Thus, τ is a Cq-ROFT of $\widehat{\mathcal{H}}$.

Let $\widehat{\tau}\in t_r(\widehat{\mathcal{H}})$ and $Q_j\in\mathcal{Q}$. Definition 28 implies that $Q_j^{h(Q_j)}\in\mathcal{H}^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}$, for $h(Q_j)\leq(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})\in f_s(\mathcal{H})$. Definition of subcore $\widetilde{cor}(\mathcal{H})$ implies the existence of an hyperedge $\widehat{Q}_j^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}$ of $\widetilde{\mathcal{H}}^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}$ such that $\widehat{Q}_j^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}\subseteq Q_j^{h(Q_j)}$ and $(\mu_ke^{i\theta_k},\nu_ke^{i\varphi_k})\geq(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})$ $\geq h(Q_j)$. For $\widehat{\tau}\in t_r(\widehat{\mathcal{H}})$, we have $u\in\widehat{Q}_j^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}\cap\widehat{\tau}^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}\subseteq\widehat{Q}_j^{h(Q_j)}\cap\widehat{\tau}^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}$. Hence, $\widehat{\tau}$ is a Cq-ROFT of $\mathcal{H}$.

Let $\tau\in t_r(\mathcal{H})\Rightarrow\tau$ is a Cq-ROFT of $\widehat{\mathcal{H}}$. This implies that there is $\widehat{\tau}$ such that $\widehat{\tau}\subseteq\tau$. But $\widehat{\tau}$ is a Cq-ROFT of $\mathcal{H}$ and $\tau\in t_r(\mathcal{H})$ implies that $\widehat{\tau}=\tau$. Thus, $t_r(\mathcal{H})\subseteq t_r(\widehat{\mathcal{H}})$. Also $t_r(\widehat{\mathcal{H}})\subseteq t_r(\mathcal{H})$ implies that $t_r(\mathcal{H})=t_r(\widehat{\mathcal{H}})$. □

Although τ can be taken as a minimal transversal of $\mathcal{H}$, it is not necessary for $\tau^{(\mu e^{i\theta},\nu e^{i\varphi})}$ to be the minimal transversal of $\mathcal{H}^{(\mu e^{i\theta},\nu e^{i\varphi})}$, for all $\mu,\nu\in[0,1]$, and $\theta,\varphi\in[0,2\pi]$. Furthermore, it is not necessary for the family of minimal Cq-ROFTs to form a hypergraph on Y. For those Cq-ROFTs that satisfy the above property, we have:

Definition 33. *A Cq-ROFT τ having the property that $\tau^{(\mu e^{i\theta},\nu e^{i\varphi})}\in t_r(\mathcal{H}^{(\mu e^{i\theta},\nu e^{i\varphi})})$, for all $\mu,\nu\in[0,1]$, and $\theta,\varphi\in[0,2\pi]$ is called the locally minimal Cq-ROFT of $\mathcal{H}$. The collection of all locally minimal Cq-ROFTs of $\mathcal{H}$ is represented by $t_r^*(\mathcal{H})$.*

Note that, $t_r^*(\mathcal{H})\subseteq t_r(\mathcal{H})$, but the converse is not generally true.

Example 14. *Consider a C6-ROFHG $\mathcal{H}=(\mathcal{Q},\eta)$ as shown in Figure 14. The C6-ROFS*

$$\{(x_1,0.6e^{i(0.6)\pi},0.4e^{i(0.4)\pi}),(x_5,0.4e^{i(0.4)\pi},0.7e^{i(0.7)\pi}),(x_6,0.4e^{i(0.4)\pi},0.7e^{i(0.7)\pi})\}$$

is a locally minimal C6-ROFT of $\mathcal{H}$.

Theorem 5. *Let $\mathcal{H}=(\mathcal{Q},\eta)$ be an ordered Cq-ROFHG with $f_s(\mathcal{H})=\{(\mu_1e^{i\theta_1},\nu_1e^{i\varphi_1}),(\mu_2e^{i\theta_2},\nu_2e^{i\varphi_2}),\cdots,(\mu_ne^{i\theta_n},\nu_ne^{i\varphi_n})\}$. If λ_k is a minimal transversal of $\mathcal{H}^{(\mu_ke^{i\theta_k},\nu_ke^{i\varphi_k})}$, then there exists $\alpha\in t_r(\mathcal{H})$ such that $\alpha^{(\mu_ke^{i\theta_k},\nu_ke^{i\varphi_k})}=\lambda_k$ and $\alpha^{(\mu_le^{i\theta_l},\nu_le^{i\varphi_l})}$ is a minimal transversal of $\mathcal{H}^{(\mu_le^{i\theta_l},\nu_le^{i\varphi_l})}$, for all $l<k$. In particular, if $\lambda_j\in t_r(\mathcal{H}^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})})$, then there exists a locally minimal Cq-ROFT $\alpha^{(\mu_je^{i\theta_j},\nu_je^{i\varphi_j})}=\lambda_j$ and $t_r^*(\mathcal{H})\neq\emptyset$.*

Proof. Let $\lambda_k\in t_r(\mathcal{H}^{(\mu_ke^{i\theta_k},\nu_ke^{i\varphi_k})})$. Since, $\mathcal{H}=(\mathcal{Q},\eta)$ is an ordered Cq-ROFHG, therefore $\mathcal{H}^{(\mu_{k-1}e^{i\theta_{k-1}},\nu_{k-1}e^{i\varphi_{k-1}})}\subseteq\mathcal{H}^{(\mu_ke^{i\theta_k},\nu_ke^{i\varphi_k})}$. Also, there exists $\lambda_{k-1}\in t_r(\mathcal{H}^{(\mu_{k-1}e^{i\theta_{k-1}},\nu_{k-1}e^{i\varphi_{k-1}})})$ such that $\lambda_{k-1}\subseteq\lambda_k$. Following this iterative procedure, we have a nested sequence $\lambda_1\subseteq\lambda_2\subseteq\cdots\subseteq\lambda_{k-1}\subseteq\lambda_k$ of minimal transversals, where every $\lambda_l\in t_r(\mathcal{H}^{(\mu_le^{i\theta_l},\nu_le^{i\varphi_l})})$. Let α_l be an elementary Cq-ROFS having height $(\mu_le^{i\theta_l},\nu_le^{i\varphi_l})$ and support α_l. Let us define $\alpha(x)$ such that

$\alpha(x) = \{(\max T_{\alpha_l}(x)e^{i \max \phi_{\alpha_l}(x)}, \min F_{\alpha_l}(x)e^{i \min \psi_{\alpha_l}(x)}) | 1 \leq l \leq n\}$, that generates the required minimal Cq-ROFT of $\mathcal{H}$. If $k = n$, α is locally minimal Cq-ROFT of $\mathcal{H}$. Hence, $t_r^*(\mathcal{H}) \neq \emptyset$. □

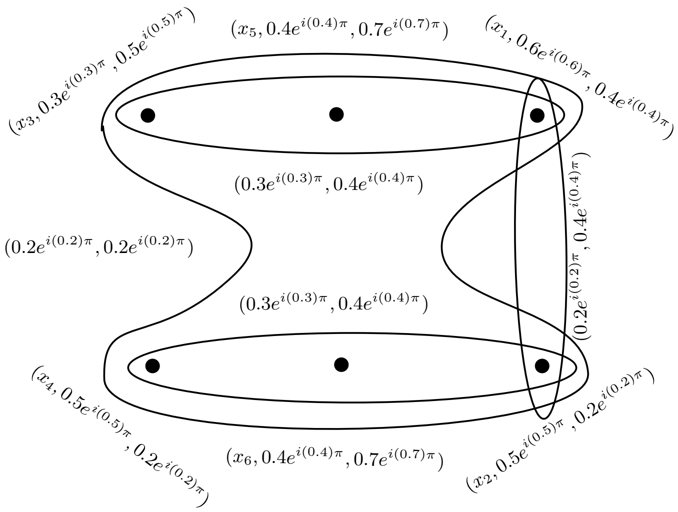

Figure 14. Complex six-rung orthopair fuzzy hypergraph.

Theorem 6. *Let $\mathcal{H} = (\mathcal{Q}, \eta)$ be a simply ordered Cq-ROFHG with $f_s(\mathcal{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \cdots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$. If $\lambda_k \in t_r(\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$, then there exists $\alpha \in t_r^*(\mathcal{H})$ such that $\alpha^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} = \lambda_k$.*

Proof. Let $\lambda_k \in t_r(\mathcal{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$ and $\mathcal{H} = (\mathcal{Q}, \eta)$ is a simply ordered Cq-ROFHG. Theorem 5 implies that a nested sequence $\lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_{k-1} \subseteq \lambda_k$ of minimal transversals can be constructed. Let Let α_l be an elementary Cq-ROFS having height $(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})$ and support α_l such that $\alpha(x) = \{(\max T_{\alpha_l}(x)e^{i \max \phi_{\alpha_l}(x)}, \min F_{\alpha_l}(x)e^{i \min \psi_{\alpha_l}(x)}) | 1 \leq l \leq n\}$ generates the locally minimal Cq-ROFT of $\mathcal{H}$ with $\alpha^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} = \lambda_k$. □

6. Application

Most of the previous studies use crisp hypergraphs to analyze the co-authorship relation between two or more authors as a collaboration. In this section, we consider a Cq-ROFHG model of co-authorship network to represent the collaboration relations between authors having uncertainty and vagueness of periodic nature simultaneously. The next comparison law between Cq-ROFNs will be helpful in our application:

Definition 34. *Let $\mathcal{Q} = (Te^{i\phi}, Fe^{i\psi})$ be a Cq-ROFN. Then, the score function of $\mathcal{Q}$ is defined as,*

$$s(\mathcal{Q}) = (T^q - F^q) + \frac{1}{2^q \pi^q}(\phi^q - \psi^q).$$

The accuracy of $\mathcal{Q}$ is defined as,

$$a(\mathcal{Q}) = (T^q + F^q) + \frac{1}{2^q \pi^q}(\phi^q + \psi^q).$$

For two Cq-ROFNs $\mathcal{Q}_1$ and $\mathcal{Q}_2$,

1. *if $s(\mathcal{Q}_1) > s(\mathcal{Q}_2)$, then $\mathcal{Q}_1 \succ \mathcal{Q}_2$,*

2. *if* $s(\mathcal{Q}_1) = s(\mathcal{Q}_2)$, *then*

- *if* $a(\mathcal{Q}_1) > a(\mathcal{Q}_2)$, *then* $\mathcal{Q}_1 \succ \mathcal{Q}_2$,
- *if* $a(\mathcal{Q}_1) = a(\mathcal{Q}_2)$, *then* $\mathcal{Q}_1 \sim \mathcal{Q}_2$.

6.1. A C6-ROFHG Model of Research Collaboration Network

A collaboration network is a group of independent organizations or people that interact to complete a particular goal for achieving better collective results by means of the joint execution of a task. The entities of a collaborative network may be geographically distributed and heterogeneous in terms of their culture, goals, and operating environment but they collaborate to achieve compatible or common goals. For decades, science academies have been interested in research collaboration. The most common reasons for research collaboration are funding, more experts working on the same project imply the more chances for effectiveness, productivity, and innovativeness. Nowadays, most of the public research is based on the collaboration of different types of expertise from different disciples and different economic sectors. In this section, we study a research collaboration network model through C6-ROFHG. Consider a science academy that wants to select an author among a group of researchers that has the best collaborative skills. For this purpose, the following characteristics can be considered:

- Cooperative spirit
- Mutual respect
- Critical thinking
- Innovations
- Creativity
- Embrace diversity

We construct a C6-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ on $Y = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$. The universe Y represents the group of authors as the vertices of $\mathcal{H}$ and these authors are grouped through hyperedges if they have worked together on some projects. The truth-membership of each author represents the collaboration strength and falsity-membership describes the opposite behavior of the corresponding author. Suppose that a team of experts assigns that the collaboration power of A_1 is 60% and non-collaborative behavior is 50% after carefully observing the different attributes. The corresponding phase terms illustrate the specific period of time in which the collaborative behavior of an author varies. We model this data as $(A_1, 0.6e^{i(0.5)\pi}, 0.5e^{i(0.5)\pi})$. The C6-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ model of collaboration network is shown in Figure 15.

The membership degrees of hyperedges represent the collective degrees of collaboration and non-collaboration of the corresponding authors combined through an hyperedge. The adjacency matrix of this network is given in Tables 3–5.

Table 3. Adjacency matrix of collaboration network.

η	A_1	A_2	A_3	A_4
A_1	$(0,0)$	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$
A_2	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0,0)$	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0,0)$
A_3	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0.6e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0,0)$	$(0,0)$
A_4	$(0.6e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0,0)$	$(0,0)$	$(0,0)$
A_5	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_6	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_7	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_8	$(0,0)$	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$	$(0,0)$
A_9	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_{10}	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

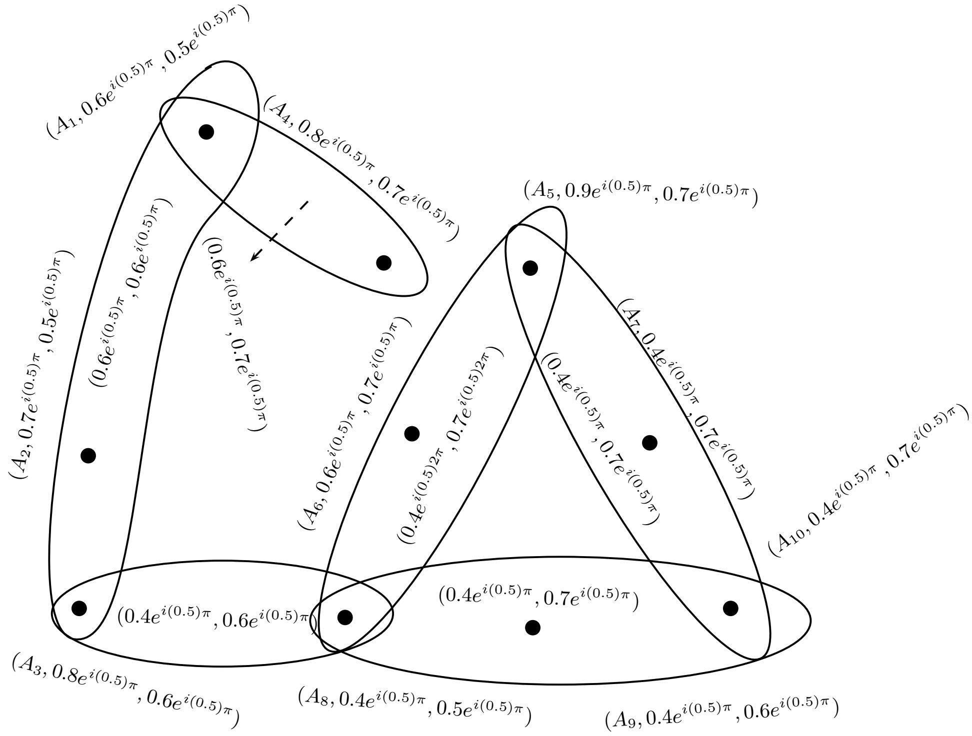

Figure 15. Complex six-rung orthopair fuzzy hypergraph model of collaboration network.

Table 4. Adjacency matrix of collaboration network.

η	A_5	A_6	A_7	A_8
A_1	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_2	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_3	$(0,0)$	$(0,0)$	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.6e^{i(0.5)\pi})$
A_4	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
A_5	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0.6e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_6	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0,0)$	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_7	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0,0)$	$(0,0)$	$(0,0)$
A_8	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0,0)$	$(0,0)$
A_9	$(0,0)$	$(0,0)$	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_{10}	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$

Table 5. Adjacency matrix of collaboration network.

η	A_9	A_{10}
A_1	$(0,0)$	$(0,0)$
A_2	$(0,0)$	$(0,0)$
A_3	$(0,0)$	$(0,0)$
A_4	$(0,0)$	$(0,0)$
A_5	$(0,0)$	$(0.6e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_6	$(0,0)$	$(0,0)$
A_7	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_8	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_9	$(0,0)$	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$
A_{10}	$(0.4e^{i(0.5)\pi}, 0.7e^{i(0.5)\pi})$	$(0,0)$

The score values and choice values of a C6-ROFHG $\mathcal{H} = (\mathcal{Q}, \eta)$ are calculated as follows,

$$s_{jk} = (T_{jk}^q + F_{jk}^q) + \frac{1}{2^q \pi^q}(\phi_{jk}^q + \psi_{jk}^q), \; c_j = \sum_k s_{jk} + (T_j^q + F_j^q) + \frac{1}{2^q \pi^q}(\phi_j^q + \psi_j^q),$$

respectively. These values are given in Table 6.

Table 6. Score and choice values.

s_{jk}	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	c_j
A_1	0	0.1245	0.1245	0.1245	0	0	0	0	0	0	0.88690
A_2	0.1245	0	0.1245	0	0	0	0	0	0	0	0.41377
A_3	0.1245	0.1245	0	0	0	0	0	0.0820	0	0	0.67105
A_4	0.1955	0	0	0	0	0	0	0	0	0	0.60654
A_5	0	0	0	0	0	0.1529	0.1955	0.1529	0	0.1955	1.37714
A_6	0	0	0	0	0.1529	0	0	0.1529	0	0	0.53480
A_7	0	0	0	0	0.1955	0	0	0	0	0.1529	0.50139
A_8	0	0	0.0820	0	0.1529	0.1529	0	0	0.1529	0.1529	0.74457
A_9	0	0	0	0	0	0	0	0.1529	0	0.1529	0.38780
A_{10}	0	0	0	0	0.1529	0	0.1529	0.1529	0.1529	0	0.76459

The choice values of Table 6 show that A_5 is the author having maximum strength of collaboration and good collective skills among all the authors. Similarly, the choice values of all authors represent the strength of their respective collaboration skills in a specific period of time. The method adopted in our model to select the author having best collaboration skills is given in Algorithm 3.

Algorithm 3: Selection of author having maximum collaboration skills.

Input the set of vertices (authors) $A_1, A_2, \cdots, A_j$.
Input the Cq-ROFS Q of vertices such that $Q(A_k) = (T_k e^{i\phi_k}, F_k e^{i\psi_k})$, $1 \leq k \leq j$, $0 \leq T_k^q + F_k^q \leq 1$, $q \geq 1$. Here, $k = 1, 2, \cdots, j$ denotes the number of authors, $q \geq 1$ is the parameter, T and F characterize the truth and falsity membership degrees of corresponding authors.
Input the adjacency matrix $\eta = [(T_{kl} e^{i\phi_{kl}}, F_{kl} e^{i\psi_{kl}})]_{j \times j}$ of vertices.
  do k from $1 \to j$
      $c_k = 0$
    do l from $1 \to j$
      $s_{jk} = (T_{kl}^q + F_{kl}^q) + \frac{1}{2^q \pi^q}(\phi_{kl}^q + \psi_{kl}^q)$
      $c_k = c_k + s_{jk}$
    end do
      $c_k = c_k + (T_k^q + F_k^q) + \frac{1}{2^q \pi^q}(\phi_k^q + \psi_k^q)$
  do
Select a vertex of $\mathcal{H} = (\mathcal{Q}, \eta)$ having maximum choice value as the author possessing strong collaboration powers.

6.2. Comparative Analysis

The proposed Cq-ROF model is more flexible and compatible to the system when the given data ranges over complex subset with unit disk instead of real subset with $[0, 1]$. We illustrate the flexibility of our proposed model by taking an example. Consider an educational institute that wants to establish its minimum branches in a particular city in order to facilitate the maximum number of students according to some parameters such as transportation, suitable place, connectivity with the main branch, and expenditures. Suppose a team of three decision-makers selects the different places.

Let $Y = \{p_1, p_2, p_3\}$ be the set of places where the team is interested to establish the new branches. After carefully observing the different attributes, the first decision-makers assign the membership and non-membership degrees to support the place p_1 as 60% and 40%, respectively. The phase terms represent the period of time for which the place p_1 can attract maximum number of students. This information is modeled using a CIFS as $(p_1, 0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi})$. Note that, $0 \leq 0.6 + 0.4 \leq 1$ and $0 \leq (0.6)\pi + (0.4)\pi \leq \pi$. Similarly, he models the other places as, $(p_2, 0.7e^{i(0.7)\pi}, 0.2e^{i(0.2)\pi})$, $(p_3, 0.5e^{i(0.5)\pi}, 0.2e^{i(0.2)2\pi})$. We denote this CIF model as

$$I = \{(p_1, 0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi}), (p_2, 0.7e^{i(0.7)\pi}, 0.2e^{i(0.2)\pi}), (p_3, 0.5e^{i(0.5)\pi}, 0.2e^{i(0.2)\pi})\}.$$

All CIF grades are CPF as well as Cq-ROF grades. We find the score functions of the above values using the formulas $s(p_j) = (T - F) + \frac{1}{2\pi}(\phi - \psi)$, $s(p_j) = (T^2 - F^2) + \frac{1}{2^2\pi^2}(\phi^2 - \psi^2)$, and $s(p_j) = (T^3 - F^3) + \frac{1}{2^3\pi^3}(\phi^3 - \psi^3)$. The results corresponding to these three approaches are given in Table 7.

Table 7. Comparative analysis of CIF, CPF, and C3-ROF models.

Methods	Score Values	Ranking
CIF model	0.4 1.0 0.6	$p_2 > p_3 > p_1$
CPF model	0.4 0.9 0.42	$p_2 > p_3 > p_1$
C3-ROF model	0.104 0.67 0.234	$p_2 > p_3 > p_1$

Suppose that the second decision-maker assigns the membership values to these places as, $(p_1, 0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi}), (p_2, 0.7e^{i(0.7)\pi}, 0.2e^{i(0.2)\pi}), (p_3, 0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})$. This information can not be modeled using CIFS as $0.7 + 0.5 = 1.2 > 1$. We model this information using a CPFS and the corresponding model is given as,

$$P = \{(p_1, 0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi}), (p_2, 0.7e^{i(0.7)\pi}, 0.2e^{i(0.2)\pi}), (p_3, 0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})\}.$$

All CPF grades are also Cq-ROF grades. We find the score functions of the above values using the formulas $s(p_j) = (T^2 - F^2) + \frac{1}{2^2\pi^2}(\phi^2 - \psi^2)$ and $s(p_j) = (T^3 - F^3) + \frac{1}{2^3\pi^3}(\phi^3 - \psi^3)$. The results corresponding to these two approaches are given in Table 8.

Table 8. Comparative analysis of CPF, and C3-ROF models.

Methods	Score Values	Ranking
CPF model	0.4 0.9 0.48	$p_2 > p_3 > p_1$
C3-ROF model	0.104 0.67 0.436	$p_2 > p_3 > p_1$

We now suppose that the third decision-maker assigns the membership values to these places as, $(p_1, 0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi}), (p_2, 0.8e^{i(0.8)\pi}, 0.7e^{i(0.7)\pi}), (p_3, 0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})$. This information can not be modeled using CIFS and CPFS as $0.7 + 0.8 = 1.5 > 1$, $0.7^2 + 0.8^2 = 1.13 > 1$. We model this information using a C3-ROFS and the corresponding model is given as,

$$Q = \{(p_1, 0.6e^{i(0.6)\pi}, 0.4e^{i(0.4)\pi}), (p_2, 0.8e^{i(0.8)\pi}, 0.7e^{i(0.7)\pi}), (p_3, 0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})\}.$$

We find the score functions of the above values using the formula $s(p_j) = (T^3 - F^3) + \frac{1}{2^3\pi^3}(\phi^3 - \psi^3)$. The score values of C3-ROF information are given as,

$$s(p_1) = 0.304,\ s(p_2) = 0.438,\ s(p_3) = 0.436.$$

Note that p_2 is the best optimal choice to establish a new branch according to the given parameters. We see that every CIF grade is a CPF grade, as well as a Cq-ROF grade, however there are Cq-ROF grades that are not CIF nor CPF grades. This implies the generalization of Cq-ROF values. Thus the proposed Cq-ROF model provides more flexibility due to its most prominent feature that is the adjustment of the range of demonstration of given information by changing the value of parameter q, $q \geq 1$. The generalization of our proposed model can also be observed from the reduction of Cq-ROF model to CIF and CPF models for $q = 1$ and $q = 2$, respectively.

7. Conclusions and Future Directions

Fuzzy sets and intuitionistic fuzzy sets cannot handle imprecise, inconsistent, and incomplete information of periodic nature. They lack the capability to model two-dimensional phenomena. To vercome this difficulty, the concept of complex fuzzy sets was introduced by Ramot et al. [2]. Their phase term is the critical feature of the complex fuzzy set model. The potential of a complex fuzzy set for representing two-dimensional phenomena makes it superior when it comes to handle ambiguous and intuitive information, especially in time-periodic phenomena.

A Cq-ROF model is a generalized form of both the complex intuitionistic fuzzy and complex Pythagorean fuzzy models. Indeed, a Cq-ROF model reduces to a CIF model when $q = 1$, and it becomes a CPF model when $q = 2$. The Cq-ROF model provides a sufficiently wide space of permissible complex orthopairs.

Hypergraphs are mathematical tools for the representation and understanding of problems in a wide variety of scientific fields. In this article, we have applied the most fruitful concept of Cq-ROFSs to hypergraphs. We have defined the novel concepts of Cq-ROFSs, Cq-ROFGs, Cq-ROFHGs, level hypergraphs, and Cq-ROF transversals of Cq-ROFHGs. Further, we have proved that a C1-ROFHG is a CIFHG and a C2-ROFHG is a CPFHG. We have also designed algorithms to construct minimal transversals, fundamental subsequence and subcore of a Cq-ROFHG. Finally, we have illustrated a real-life application of Cq-ROFHGs in collaboration networks that enhances the motivation of this research article.

We aim to broaden our study in the future with the analysis of (1) Complex fuzzy directed hypergraphs, (2) Complex bipolar neutrosophic hypergraphs, (3) Fuzzy rough soft directed hypergraphs and (4) Fuzzy rough neutrosophic hypergraphs.

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