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Fractional Integral Inequalities for Strongly $\mathfrak{h}$-Preinvex Functions for a kth Order Differentiable Functions

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Received: 16 October 2019; Accepted: 19 November 2019; Published: 25 November 2019

Abstract: The objective of this paper is to derive Hermite-Hadamard type inequalities for several higher order strongly $\mathfrak{h}$-preinvex functions via Riemann-Liouville fractional integrals. These results are the generalizations of the several known classes of preinvex functions. An identity associated with k-times differentiable function has been established involving Riemann-Liouville fractional integral operator. A number of new results can be deduced as consequences for the suitable choices of the parameters $\mathfrak{h}$ and σ. Our outcomes with these new generalizations have the abilities to be implemented for the evaluation of many mathematical problems related to real world applications.

Keywords: Estimates of upper bound; Riemann-Liouville integral operator; strongly $\mathfrak{h}$-preinvex functions; kth-order differentiablity

1. Introduction

The modeling of a few global problems requires using a fractional calculus which incorporates both derivatives and integrals. This is natural to investigate whether it is possible to present a framework that allows us to include both fractional calculus simultaneously so that you probably can acquire some perception and a better knowledge of the subtle variations among derivatives and integrals domains. To answer this, a principle was formulated by [1,2]. The precept goal of fractional operators are that they construct bridges among continuous and discrete cases. Afterward, this concept was developed through many researchers [3,4]. Fractional calculus has potential applications in pure and applied mathematics. In pure mathematics, fractional calculus has been implemented in mathematical inequalities to unify derivative and integral versions of inequalities. A few decades ago, a diffusion of labor has been done to unify and amplify integral inequalities on fractional calculus [5]. These integral inequalities are utilized in numerous areas for the boundedness, uniqueness of the solutions integro differential equations, force closure properties of robotic grasping, optimization problems, nonlinear programming, dynamic equations, signal and image processing algorithms, etc. (see, [4,6–9] and the references therein). Integral inequalities on convex functions, both derivative and integration, have also been a topic of discussion for quite some time. These inequalities had been advanced via numerous researchers [10–21]. Sarikaya et al. [22] employed the ideas of fractional calculus for establishing a number of integral inequalities that basically rely on Hermite–Hadamard inequality. This approach has opened a new path for research. Additionally, the authors of the manuscript [23,24] illustrated some important inequalities installed by Set et al. Since then numerous scientists have broadly used the ideas of fractional calculus and received several new and novel refinements of inequalities via convex functions and their generalizations. Various papers with new indication, different speculations, and augmentations have appeared in the literature. Inequalities that comprise integrals of functions are of notable importance in mathematics with applications in the concept of differential equations, approximations, convex optimizations, polynomial-time algorithms, automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, and structural optimization, where the approximation concept has proven to be efficient in probability theory, for instance, see [4,6,8,9,25,26] and the references therein. With recent advancements in computing and optimization algorithms, convex programming is nearly as straightforward as linear programming [7].

Recall the concept of convex set and convex functions which are firmly concerned to this paper.

A subset K of $\mathbb{R}$ is said to be convex if

$$(1-\varrho)u_1 + \varrho u_2 \in K, \ \ \forall u_1, \ u_2 \in K, \ \varrho \in [0,1].$$

A function $\Phi : K \to \mathbb{R}$ is said to be convex if

$$\Phi((1-\varrho)u_1 + \varrho u_2) \leq (1-\varrho)\Phi(u_1) + \varrho\Phi(u_2)$$

for all $u_1, \ u_2 \in K$, $\varrho \in [0,1]$.

It is well known that Hermite established the following Hermite–Hadamard integral inequality:

$$\Phi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \Phi(x)dx \leq \frac{\Phi(u_1)+\Phi(u_2)}{2}, \tag{1}$$

where $\Phi : [u_1, u_2] \subset \mathbb{R} \to \mathbb{R}$ is a convex function. This inequality provides a lower and an upper estimate for the integral average of any convex function defined on a compact interval. For generalizations of the classical Hermite–Hadamard inequality, see [27–47] and the references therein.

Craven [48] addressed the "term" for calling this class of functions due to their feature defined as "invariance by convexity". Hanson [49] furnished the idea of differentiable invex functions in reference to their precise global optimum behavior and investigated the significant generalization of convex functions is that of invex functions. Weir and Mond [50] proposed the idea of preinvex functions and implemented it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. In [18], Noor established the celebrated Hermite–Hadamard inequality for preinvex functions. Mohan and Neogy [51] introduced well-known condition C.

Mititelu [52] defined the notion of invex sets as follows: Let $\Omega \subset \mathbb{R}$ be a set and $\delta(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous bifunction. Throughout this paper, the set Ω is an invex set, unless otherwise it is specified.

Definition 1 ([52])**.** *A set Ω is said to be invex if*

$$u_1 + \varrho\delta(u_2, u_1) \in \Omega, \quad \forall u_1, u_2 \in \Omega, \ \varrho \in [0,1].$$

The invex set Ω is also known as the δ-connected set. Note that, if $\delta(u_1, u_2) = u_2 - u_1$, this means that every convex set is an invex set, but the converse is not true.

The concept of preinvex functions was introduced by Weir and Mond [50] as follows:

Definition 2 ([50])**.** *A function $\Phi : \Omega \to \mathbb{R}$ is said to be preinvex if*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq (1-\varrho)\Phi(u_1) + \varrho\Phi(u_2)$$

for all $u_1, u_2 \in \Omega,\ \varrho \in [0,1]$.

For current research on preinvex functions, concerned readers are referred to [11,48,53–58].

The notion of strongly convex functions was introduced by Karamardian [59] and Polyak [21]. Convexity is the only weakening property of strong convexity.

Definition 3. *A function* $\Phi : K \subset \mathbb{R}^n \to \mathbb{R}$ *is said to be a strongly convex function for modulus* $\mu > 0$ *if*

$$\Phi\left((1-\varrho)u_1 + \varrho u_2\right) \leq (1-\varrho)\Phi(u_1) + \varrho\Phi(y) - \mu\varrho(1-\varrho)\|u_2 - u_1\|^2$$

for all $u_1, u_2 \in K,\ \varrho \in [0,1]$.

Karamardian [59] noticed that every strongly monotone has a gradient map if and only if all differentiable function is strongly convex. Higher order strongly convex functions introduced by Lin et al. [60] to abridge the research of linear programming with equilibrium constraints. Bynum [61] and Chen et al. [62] have investigated the basic features and utilizations of the parallelogram laws for the Banach spaces. Xu [63] obtained new attributes of p-uniform convexity and q-uniform smoothness of a Banach space using $\|.\|^p$ and $\|.\|^q$, respectively. These outcomes can be acquired from the ideas of higher order strongly convex (concave) functions, which can be seen as a novel application. For certain examinations on strongly convex functions, see [1,10,12,16,17,19,21].

Definition 4. *A function* $\Phi : K \subset \mathbb{R}^n \to \mathbb{R}$ *is said to be a strongly convex function for modulus* $\mu > 0$ *with order* $\sigma > 0$ *if*

$$\Phi\left((1-\varrho)u_1 + \varrho u_2\right) \leq (1-\varrho)\Phi(u_1) + \varrho\Phi(u_2) - \mu\varrho(1-\varrho)\|u_2 - u_1\|^\sigma$$

for all $u_1, u_2 \in K,\ \varrho \in [0,1]$.

It follows that when $\sigma = 2$ in the above definition, we attain the definition of strongly convex in the classical sense. The gradient map of the function in the higher order strong monotonicity is equivalent to the strong convexity of a function in the higher order sense is investigated by Lin et al. [60].

The class of convex functions involving arbitrary nonnegative auxiliary functions is addressed by Varosanec [64]. The concept on the time of advent has come to be to unify a few generalizations of classical convexity, including Breckner type convex functions [65], P-functions [15], Godunova–Levin type convex and Q-functions [66]. We apprehend that this class defines some other classes of classical convex functions. For information, see [13,20]. $\mathfrak{h}$-convex functions have gained significant consideration from numerous analysts and, therefore, a bulk of articles had been particularly committed to the investigation of this class. Noor et al. [20] prolonged this idea utilizing the invexity property of sets and described the perception of $\mathfrak{h}$-preinvex functions. They have perceived that it comprises of a few new and known classes of convexity.

The principal intention of this research is to introduced the idea of higher order strongly $\mathfrak{h}$-preinvex functions. Furthermore, we observe that the class of higher order strongly $\mathfrak{h}$-preinvex functions unifies several other classes of strong preinvexity. We acquire an identity related to the kth order differentiability. Then, utilizing this identity, we derive our main consequences for some upper bounds for kth order differentiable function via higher order strongly $\mathfrak{h}$-preinvex functions.

2. Preliminaries

We now consider a class of higher order strongly preinvex function with respect to an arbitrary function $\mathfrak{h}$. Moreover, we present some preliminaries related to the fractional calculus and special functions.

Definition 5. *Let $\mathfrak{h} : (0,1) \subseteq J \to \mathbb{R}$ be a nonnegative function. We say that $\Phi : M_\delta \subset \mathbb{R}^n \to \mathbb{R}$ is a higher order strongly $\mathfrak{h}$-preinvex function of order $\sigma > 0$ with modulus $\mu > 0$ if*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq \mathfrak{h}(1-\varrho)\Phi(u_1) + \mathfrak{h}(\varrho)\Phi(u_2) - \mu\varrho(1-\varrho)\|\delta(u_2, u_1)\|^\sigma$$

for all $u_1, u_2 \in M_\delta$, $\varrho \in [0,1]$.

We now discuss several special cases of definition 5.

(I). If $\mathfrak{h}(\varrho) = \varrho$ in Definition 5, then we attain the class of higher order strongly preinvex functions.

Definition 6. *A higher order strongly preinvex function $\Phi : M_\delta \subset \mathbb{R}^n \to \mathbb{R}$ of order $\sigma > 0$ with modulus $\mu > 0$ is defined as*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq (1-\varrho)\Phi(u_1) + (\varrho)\Phi(u_2) - \mu\varrho(1-\varrho)\|\delta(u_2, u_1)\|^\sigma$$

for all $u_1, u_2 \in M_\delta$, $\varrho \in [0,1]$.

(II). If $\mathfrak{h}(\varrho) = \varrho^s$ in Definition 5, then we attain the class of higher order strongly s-preinvex functions, which is called Breakner type of higher order strongly s-preinvex functions.

Definition 7. *For a real number $s \in [0,1]$. We say that $\Phi : M_\delta \subset \mathbb{R}^n \to \mathbb{R}$ is a higher order strongly s-preinvex function of order $\sigma > 0$ with modulus $\mu > 0$ if*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq (1-\varrho)^s\Phi(u_1) + (\varrho)^s\Phi(u_2) - \mu\varrho(1-\varrho)\|\delta(u_2, u_1)\|^\sigma$$

for all $u_1, u_2 \in M_\delta$, $\varrho \in [0,1]$.

(III). If $\mathfrak{h}(\varrho) = \varrho^{-s}$ in Definition 5, then we attain the class of higher order strongly s-preinvex functions, which is called Godunova–Levin type of higher order strongly s-preinvex functions.

Definition 8. *For a real number $s \in [0,1]$. We say that $\Phi : M_\delta \subset \mathbb{R}^n \to \mathbb{R}$ is a higher order strongly s-preinvex function of order $\sigma > 0$ with modulus $\mu > 0$ if*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq (1-\varrho)^{-s}\Phi(u_1) + (\varrho)^{-s}\Phi(u_2) - \mu\varrho(1-\varrho)\|\delta(u_2, u_1)\|^\sigma$$

for all $u_1, u_2 \in M_\delta$, $\varrho \in [0,1]$.

(IV). If $\mathfrak{h}(\varrho) = \varrho^{-1}$ in Definition 5, then we attain the class of higher order strongly Q-preinvex functions.

Definition 9. *A function $\Phi : M_\delta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be higher order strongly Q-preinvex function of order $\sigma > 0$ with modulus $\mu > 0$ if*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq (1-\varrho)^{-1}\Phi(u_1) + (\varrho)^{-1}\Phi(u_2) - \mu\varrho(1-\varrho)\|\delta(u_2, u_1)\|^\sigma$$

for all $u_1, u_2 \in M_\delta$, $\varrho \in [0,1]$.

(V). If $\mathfrak{h}(\varrho) = 1$ in Definition 5, then we attain the class of higher order strongly P-preinvex functions.

Definition 10. *A function $\Phi : M_\delta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be higher order strongly P-preinvex function of order $\sigma > 0$ with modulus $\mu > 0$ if defined as*

$$\Phi\left(u_1 + \varrho\delta(u_2, u_1)\right) \leq (1-\varrho)^{-1}\Phi(u_1) + (\varrho)^{-1}\Phi(u_2) - \mu\varrho(1-\varrho)\|\delta(u_2, u_1)\|^\sigma$$

for all $u_1, u_2 \in M_\delta$, $\varrho \in [0,1]$.

Remark 1. *Observe that if we take* $\mathfrak{h}(\varrho) = \varrho,\ \varrho^s,\ \varrho^{-s},\ \varrho^{-1}$, *and* $\mathfrak{h}(\varrho) = 1$ *in Definition 5, then we acquire Definitions 6–10 respectively. See that, if we substitute* $\delta(u_2, u_1) = u_2 - u_1$ *in Definition 5, then we attain the class of higher order strongly* $\mathfrak{h}$*-convex functions. To the exceptional of our expertise, this class is a new addition in convexity theory. Similarly, for exceptional appropriate selections of* $\mathfrak{h}(.)$*, one can obtain higher order strong convexity. It is worth mentioning that the class of higher order strongly* $\mathfrak{h}$*-preinvex functions is quite general and unifying one.*

Definition 11 ([3])**.** *Let* $\Phi \in L_1([a,b])$ *. The left and right-sided Riemann–Liouville integrals* $J^{\rho}_{a_+}\Phi$ *and* $J^{\rho}_{b_-}\Phi$*, of order* $\rho > 0$*, are defined by*

$$J^{\rho}_{a_+}\Phi(x) = \frac{1}{\Gamma(\rho)}\int_a^x (x-\varrho)^{\rho-1}\Phi(\varrho)d\varrho \quad x > a$$

and

$$J^{\rho}_{b_-}\Phi(x) = \frac{1}{\Gamma(\rho)}\int_x^b (\varrho-x)^{\rho-1}\Phi(\varrho)d\varrho \quad x < b,$$

respectively. Here, $\Gamma(\rho) = \int_0^\infty e^{-\varrho}\varrho^{\rho-1}d\varrho$ *is the gamma function.*

The incomplete Beta function is defined as follows:

$$\mathbb{B}_x(a,b) = \int_0^x \varrho^{a-1}(1-\varrho)^{b-1}d\varrho,\ \ a,\ b > 0,\ 0 < x < 1.$$

3. Auxiliary Result

The subsequent identity plays a key role in inaugurating the main results of this paper. The identification is stated as follows.

Lemma 1. *For* $\rho > 0$*,* $n, k \in \mathbb{N}$*, there is a kth order differentiable function* $\Phi : M_\delta \to \mathbb{R}$ *such that* $c_1,\ c_1 + \delta(c_2, c_1) \in M_\delta$ *with* $\delta(c_2, c_1) > 0$ *and* $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$*. Then*

$$\Psi(k,n,\rho;c_1,c_2)(\Phi) = n^{\rho+\kappa}\Gamma(\rho+\kappa)\sum_{i=1}^{\kappa}\frac{[(-1)^{i-1}-1]}{\Gamma(\rho+\kappa-i+1)}\left(\frac{2}{\delta(c_2,c_1)}\right)^i \Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+\Gamma(\rho+\kappa)\left(\frac{2}{\delta(c_2,c_1)}\right)^{\kappa+\rho}\left[J^{\rho}_{(c_1+\frac{1}{2}\delta(c_2,c_1))^+}\Phi(c_1+\delta(c_2,c_1)) + (-1)^{\kappa}J^{\rho}_{(c_1+\frac{1}{2}\delta(c_2,c_1))^-}\Phi(c_1)\right],$$

where

$$\Psi(k,n,\rho;c_1,c_2)(\Phi)$$
$$= \int_0^n (n-\varrho)^{\rho+\kappa-1}\left[\Phi^{(\kappa)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right) + \Phi^{(\kappa)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)\right]d\varrho.$$

Proof. Let

$$\begin{aligned}
&\int_0^n (n-\varrho)^{\rho+\kappa-1}\left[\Phi^{(\kappa)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)+\Phi^{(\kappa)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)\right]d\varrho\\
&=\int_0^n (n-\varrho)^{\rho+\kappa-1}\Phi^{(\kappa)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho\\
&+\int_0^n (n-\varrho)^{\rho+k-1}\Phi^{(\kappa)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho\\
&=I_1+I_2.
\end{aligned}$$

Now

$$\begin{aligned}
I_1&=\int_0^n (n-\varrho)^{\rho+\kappa-1}\Phi^{(\kappa)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho\\
&=-\frac{2n}{\delta(c_2,c_1)}(n-\varrho)^{\rho+\kappa-1}\Phi^{(\kappa-1)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)\Big|_0^n\\
&-\frac{2n(\rho+\kappa-1)}{\delta(c_2,c_1)}\int_0^n (n-\varrho)^{\rho+k-2}\Phi^{(\kappa-1)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho\\
&=\frac{2n^{\rho+\kappa}}{\delta(c_2,c_1)}\Phi^{(\kappa-1)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)\\
&-\frac{2n(\rho+\kappa-1)}{\delta(c_2,c_1)}\int_0^n (n-\varrho)^{\rho+k-2}\Phi^{(\kappa-1)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho.
\end{aligned}$$

Again, integration by parts, we have

$$\begin{aligned}
I_1=&\frac{2n^{\rho+\kappa}}{\delta(c_2,c_1)}\Phi^{(\kappa-1)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)-\frac{2^2n^{\rho+\kappa}(\rho+\kappa-1)}{(\delta(c_2,c_1))^2}\Phi^{(k-2)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)\\
&+\frac{2^2n^2(\rho+\kappa-1)(\rho+\kappa-2)}{(\delta(c_2,c_1))^2}\int_0^n (n-\varrho)^{\rho+\kappa-3}\Phi^{(k-3)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho.
\end{aligned}$$

Continuing the integration by parts upto k-integration, one obtains

$$\begin{aligned}
I_1=&\,n^{\rho+\kappa}\sum_{i=1}^{\kappa}\frac{(-1)^{i-1}}{\rho+\kappa}\left(\frac{2}{\delta(c_2,c_1)}\right)^i\prod_{p=0}^{i-1}(\rho+\kappa-p)\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)\\
&+\frac{(-1)^{\kappa}}{\rho+\kappa}\left(\frac{2n}{\delta(c_2,c_1)}\right)^{\kappa}\prod_{p=0}^{\kappa}(\rho+\kappa-p)\int_0^n (n-\varrho)^{\rho-1}\Phi\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho\\
=&\,n^{\rho+\kappa}\sum_{i=1}^{\kappa}\frac{(-1)^{i-1}}{\rho+\kappa}\left(\frac{2}{\delta(c_2,c_1)}\right)^i\prod_{p=0}^{i-1}(\rho+\kappa-p)\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)\\
&+\frac{(-1)^{\kappa}\Gamma(\rho+\kappa)}{\Gamma(\rho)}\left(\frac{2n}{\delta(c_2,c_1)}\right)^{\kappa}\int_0^n (n-\varrho)^{\rho-1}\Phi\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho
\end{aligned}$$

$$= n^{\rho+\kappa}\sum_{i=1}^{\kappa}\frac{(-1)^{i-1}\Gamma(\rho+\kappa)}{\Gamma(\rho+\kappa-i+1)}\left(\frac{2}{\delta(c_2,c_1)}\right)^{i}\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+(-1)^{\kappa}\Gamma(\rho+\kappa)\left(\frac{2n}{\delta(c_2,c_1)}\right)^{\kappa+\rho} J^{\rho}_{\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)^-}\Phi(c_1).$$

Analogously, we have

$$I_2=\int_0^n (n-\varrho)^{\rho+\kappa-1}\Phi^{(\kappa)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho$$
$$=\frac{2n}{\delta(c_2,c_1)}(n-\varrho)^{\rho+\kappa-1}\Phi^{(\kappa-1)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)\Big|_0^n$$
$$+\frac{2n(\rho+\kappa-1)}{\delta(c_2,c_1)}\int_0^n (n-\varrho)^{\rho+k-2}\Phi^{(\kappa-1)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho$$
$$=-\frac{2n^{\rho+\kappa}}{\delta(c_2,c_1)}\Phi^{(\kappa-1)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)-\frac{2^2n^{\rho+\kappa}(\rho+\kappa-1)}{(\delta(c_2,c_1))^2}\Phi^{(k-2)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+\frac{2^2n^2(\rho+\kappa-1)(\rho+\kappa-2)}{(\delta(c_2,c_1))^2}\int_0^n (n-\varrho)^{\rho+\kappa-3}\Phi^{(k-3)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho$$
$$\vdots$$
$$=-n^{\rho+\kappa}\sum_{i=1}^{\kappa}\left(\frac{2}{\delta(c_2,c_1)}\right)^{i}\prod_{p=0}^{i-1}(\rho+\kappa-p)\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+\left(\frac{2n}{\delta(c_2,c_1)}\right)^{\kappa}\prod_{p=0}^{\kappa}(\rho+\kappa-p)\int_0^n (n-\varrho)^{\rho-1}\Phi\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho$$
$$=-n^{\rho+\kappa}\sum_{i=1}^{\kappa}\frac{\Gamma(\rho+\kappa)}{\Gamma(\rho+\kappa-i+1)}\left(\frac{2}{\delta(c_2,c_1)}\right)^{i}\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+\frac{\Gamma(\rho+\kappa)}{\Gamma(\rho)}\left(\frac{2}{\delta(c_2,c_1)}\right)^{\kappa}\int_0^n (n-\varrho)^{\rho-1}\Phi\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)d\varrho$$

$$=-n^{\rho+\kappa}\sum_{i=1}^{\kappa}\frac{\Gamma(\rho+\kappa)}{\Gamma(\rho+\kappa-i+1)}\left(\frac{2}{\delta(c_2,c_1)}\right)^{i}\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+\Gamma(\rho+\kappa)\left(\frac{2}{\delta(c_2,c_1)}\right)^{\rho+k} J^{\rho}_{(c_1+\frac{1}{2}\delta(c_2,c_1))^+}\Phi(c_1+\delta(c_2,c_1)).$$

Summing up I_1 and I_2, we have

$$I_1+I_2=n^{\rho+\kappa}\Gamma(\rho+\kappa)\sum_{i=1}^{\kappa}\frac{\left[(-1)^{i-1}-1\right]}{\Gamma(\rho+\kappa-i+1)}\left(\frac{2}{\delta(c_2,c_1)}\right)^{i}\Phi^{(\kappa-i)}\left(c_1+\frac{1}{2}\delta(c_2,c_1)\right)$$
$$+\Gamma(\rho+\kappa)\left(\frac{2}{\delta(c_2,c_1)}\right)^{\kappa+\rho}\left[J^{\rho}_{(c_1+\frac{1}{2}\delta(c_2,c_1))^+}\Phi(c_1+\delta(c_2,c_1))+(-1)^{\kappa}J^{\rho}_{(c_1+\frac{1}{2}\delta(c_2,c_1))^-}\Phi(c_1)\right].$$

□

4. Some New Bounds for Strongly $\mathfrak{h}$-Preinvex Functions for *k*th Order Differentiable Functions

In order to prove our main results for some new upper bounds for the function $\Psi(k,n,\rho;c_1,c_2)(\Phi)$, we need several formulas and lemmas, which we present in this section.

Theorem 1. *For $\rho>0$, $n,\ k\in\mathbb{N}$, and there is a differentiable function $\Phi: M_\delta\to\mathbb{R}$ such that $c_1, c_1+\delta(c_2,c_1)\in M_\delta$ with $\delta(c_2,c_1)>0$. If $\Phi^{(\kappa)}\in L_1([c_1,c_1+\delta(c_2,c_1)])$ and $|\Phi^{(\kappa)}|$ is a higher order strongly $\mathfrak{h}$-preinvex on M_δ, then*

$$|\Psi(k,n,\rho;c_1,c_2)(\Phi)| \le \Upsilon(\rho,\kappa,n,\varrho)\left[|\Phi^{(\kappa)}(c_1)|+|\Phi^{(\kappa)}(c_2)|\right] - \frac{\mu(\delta(c_2,c_1))^{\sigma}n^{\rho+\kappa}}{2}\left[\frac{\rho+\kappa+3}{(\rho+\kappa+1)(\rho+\kappa+2)}\right], \quad (2)$$

where

$$\Upsilon(\rho,\kappa,n,\varrho) := \int_0^n (n-\varrho)^{\rho+\kappa-1}\left[\mathfrak{h}\left(\frac{n+\varrho}{2n}\right)+\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)\right]d\varrho.$$

Proof. Applying Lemma 1, the modulus property, and by given hypothesis, we have

$$\begin{aligned}
&|\Psi(k,n,\rho;c_1,c_2)(\Phi)| \\
&\le \int_0^n (n-\varrho)^{\rho+\kappa-1}\left|\Phi^{(\kappa)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)\right|d\varrho \\
&\quad + \int_0^n (n-\varrho)^{\rho+\kappa-1}\left|\Phi^{(\kappa)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)\right|d\varrho \\
&\le \int_0^n (n-\varrho)^{\rho+\kappa-1}\left[\mathfrak{h}\left(\frac{n+\varrho}{2n}\right)|\Phi^{(\kappa)}(c_1)|+\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)|\Phi^{(\kappa)}(c_2)|\right. \\
&\qquad \left. -\frac{\mu}{(2n)^2}(n+\varrho)(n-\varrho)(\delta(c_2,c_1))^{\sigma}\right]d\varrho \\
&\quad + \int_0^n (n-\varrho)^{\rho+\kappa-1}\left[\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)|\Phi^{(\kappa)}(c_1)|+\mathfrak{h}\left(\frac{n+\varrho}{2n}\right)|\Phi^{(\kappa)}(c_2)|\right. \\
&\qquad \left. -\frac{\mu}{(2n)^2}(n+\varrho)(n-\varrho)(\delta(c_2,c_1))^{\sigma}\right]d\varrho \\
&= \Upsilon(\rho,\kappa,n,\varrho)\left[|\Phi^{(\kappa)}(c_1)|+|\Phi^{(\kappa)}(c_2)|\right] \\
&\qquad - \frac{2\mu(\delta(c_2,c_1))^{\sigma}}{(2n)^2}\int_0^n (n-\varrho)^{\rho+k}(n+\varrho)d\varrho, \quad (3)
\end{aligned}$$

where

$$\int_0^n (n-\varrho)^{\rho+\kappa}(n+\varrho)d\varrho = \frac{n^{\rho+\kappa+2}\left[(\rho+k+3)\right]}{(\rho+\kappa+1)(\rho+\kappa+2)}. \quad (4)$$

Combining (3) and (4), we immediately get the desired inequality (5). □

Some special cases which can be derived immediately from Theorem 1.

(I). Letting $\mathfrak{h}(\varrho)=\varrho$, then we acquire higher order strongly preinvex functions.

Corollary 1. *For $\rho > 0$ and $n, k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$, $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|$ is a higher order strongly preinvex on M_δ, then*

$$|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \le \frac{n^{\rho+k}}{\rho+k}\left[|\Phi^{(\kappa)}(c_1)| + |\Phi^{(\kappa)}(c_2)|\right] - \frac{\mu(\delta(c_2, c_1))^\sigma n^{\rho+\kappa}}{2}\left[\frac{\rho+\kappa+3}{(\rho+\kappa+1)(\rho+\kappa+2)}\right], \quad (5)$$

where

$$\Upsilon_1(\rho, \kappa, n, \varrho) := \int_0^n (n-\varrho)^{\rho+\kappa-1} d\varrho = \frac{n^{\rho+\kappa}}{\rho+\kappa}.$$

(II). Letting $\mathfrak{h}(\varrho) = 1$, then we acquire higher order strongly P-preinvex functions.

Corollary 2. *For $\rho > 0$ and $n, k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$, with $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|$ is a higher order strongly P-preinvex on M_δ, then*

$$|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \le \frac{2n^{\rho+\kappa}}{\rho+\kappa}\left[|\Phi^{(\kappa)}(c_1)| + |\Phi^{(\kappa)}(c_2)|\right] - \frac{\mu(\delta(c_2, c_1))^\sigma n^{\rho+\kappa}}{2}\left[\frac{\rho+\kappa+3}{(\rho+\kappa+1)(\rho+\kappa+2)}\right]. \quad (6)$$

(III). Letting $\mathfrak{h}(\varrho) = \varrho^s$, then we acquire Breckner type of higher order strongly s-preinvex functions.

Corollary 3. *For $\rho > 0$ and $n, k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$ with $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|$ Breckner type of a higher order strongly s-preinvex on M_δ, then*

$$|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \le \Upsilon_2(\rho, \kappa, n, \varrho)\left[|\Phi^{(\kappa)}(c_1)| + |\Phi^{(\kappa)}(c_2)|\right] - \frac{\mu\left(\delta(c_2, c_1)\right)^\sigma n^{\rho+\kappa}}{2}\left[\frac{\rho+\kappa+3}{(\rho+\kappa+1)(\rho+\kappa+2)}\right], \quad (7)$$

where

$$\begin{aligned}\Upsilon_2(\rho, \kappa, n, \varrho) &:= \frac{1}{(2n)^s}\int_0^n (n-\varrho)^{\rho+\kappa-1}\left[(n+\varrho)^s + (n-\varrho)^s\right] d\varrho \\ &= \frac{n^{\rho+\kappa+s}}{(2n)^s(\rho+\kappa+s)} + (2n)^{\rho+\kappa}\,\mathbb{B}_{\frac{1}{2}}(\rho+\kappa, s+1).\end{aligned}$$

(IV). Letting $\mathfrak{h}(\varrho) = \varrho^{-s}$, then we acquire Godunova–Levin type of higher order strongly s-preinvex functions.

Corollary 4. *For $\rho > 0$ and $n, k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$, with $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1[c_1, c_1 + \delta(c_2, c_1)]$ and $|\Phi^{(\kappa)}|$ Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \le \Upsilon_3(\rho, \kappa, n, \varrho) \left[|\Phi^{(\kappa)}(c_1)| + |\Phi^{(\kappa)}(c_2)| \right] - \frac{\mu \left(\delta(c_2, c_1)\right)^\sigma n^{\rho+\kappa}}{2} \left[\frac{\rho + \kappa + 3}{(\rho + \kappa + 1)(\rho + \kappa + 2)} \right], \quad (8)$$

where

$$\Upsilon_2(\rho, \kappa, n, \varrho) := \frac{1}{(2n)^{-s}} \int_0^n (n - \varrho)^{\rho+\kappa-1} \left[(n + \varrho)^{-s} + (n - \varrho)^{-s} \right] d\varrho = \frac{n^{\rho+\kappa-s}}{(2n)^{-s} (\rho + \kappa - s)} + (2n)^{\rho+\kappa} \mathbb{B}_{\frac{1}{2}} (\rho + \kappa, -s + 1).$$

Theorem 2. *For $\rho > 0$ and $n, k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$, and $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1 \left([c_1, c_1 + \delta(c_2, c_1)] \right)$ and $|\Phi^{(\kappa)}|^\alpha$ is a higher order strongly $\mathfrak{h}$-preinvex function, then*

$$|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \le \left(\frac{n^{\alpha(\rho+\kappa-1)+1}}{\alpha(\rho + \kappa - 1) + 1} \right)^{\frac{1}{\alpha}} \times \left[\left\{ \int_0^n \left(\mathfrak{h}\left(\frac{n + \varrho}{2n} \right) |\Phi^{(\kappa)}(c_1)|^\beta + \mathfrak{h}\left(\frac{n - \varrho}{2n} \right) |\Phi^{(\kappa)}(c_2)|^\beta - \frac{\mu n}{6} (\delta(c_2, c_1))^\sigma \right) d\varrho \right\}^{\frac{1}{\beta}} + \left\{ \int_0^n \left(\mathfrak{h}\left(\frac{n - \varrho}{2n} \right) |\Phi^{(\kappa)}(c_1)|^\beta + \mathfrak{h}\left(\frac{n + \varrho}{2n} \right) |\Phi^{(\kappa)}(c_2)|^\beta - \frac{\mu n}{6} (\delta(c_2, c_1))^\sigma \right) d\varrho \right\}^{\frac{1}{\beta}} \right], \quad (9)$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\beta > 1$.

Proof. Applying Lemma 1, the well-known Hölder's inequality, and by given hypothesis, we have

$$\begin{aligned}
&|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \\
&\le \left(\int_0^n (n - \varrho)^{\alpha(\rho+\kappa-1)} d\varrho \right)^{\frac{1}{\alpha}} \left(\int_0^n \left| \Phi^{(\kappa)}\left(c_1 + \left(\frac{n - \varrho}{2n} \right) \delta(c_2, c_1) \right) \right|^\beta d\varrho \right)^{\frac{1}{\beta}} \\
&\quad + \left(\int_0^n (n - \varrho)^{\alpha(\rho+\kappa-1)} d\varrho \right)^{\frac{1}{\alpha}} \left(\int_0^n \left| \Phi^{(\kappa)}\left(c_1 + \left(\frac{n + \varrho}{2n} \right) \delta(c_2, c_1) \right) \right|^\beta d\varrho \right)^{\frac{1}{\beta}} \\
&\le \left(\frac{n^{\alpha(\rho+\kappa-1)+1}}{\alpha(\rho + \kappa - 1) + 1} \right)^{\frac{1}{\alpha}} \left[\left\{ \int_0^n \left(\mathfrak{h}\left(\frac{n + \varrho}{2n} \right) |\Phi^{(\kappa)}(c_1)|^\beta \right. \right. \right.
\end{aligned}$$

$$+\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)|\Phi^{(\kappa)}(c_2)|^{\beta}-\frac{\mu n}{6}(\delta(c_2,c_1))^{\sigma}\bigg)d\varrho\bigg\}^{\frac{1}{\beta}}$$

$$+\left\{\int_0^n\left(\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)|\Phi^{(\kappa)}(c_1)|^{\beta}+\mathfrak{h}\left(\frac{n+\varrho}{2n}\right)|\Phi^{(\kappa)}(c_2)|^{\beta}\right.\right.$$

$$\left.\left.-\frac{\mu n}{6}(\delta(c_2,c_1))^{\sigma}\right)d\varrho\right\}^{\frac{1}{\beta}}\Bigg],$$

which is the required result. □

Some special cases of Theorem 2 are stated as follows.
(I.) Letting $\mathfrak{h}(\varrho)=\varrho$, then we acquire the higher order strongly preinvex functions.

Corollary 5. *For $\rho>0$, $n,\ k\in\mathbb{N}$, and there is a differentiable function $\Phi: M_{\delta}\to\mathbb{R}$ such that $c_1, c_1+\delta(c_2,c_1)\in M_{\delta}$, with $\delta(c_2,c_1)>0$. If $\Phi^{(\kappa)}\in L_1([c_1,c_1+\delta(c_2,c_1)])$ and $|\Phi^{(\kappa)}|^{\alpha}$ is a higher order strongly preinvex function, then*

$$|\Psi(k,n,\rho;c_1,c_2)(\Phi)|\leq\left(\frac{n^{\alpha(\rho+\kappa-1)+1}}{\alpha(\rho+\kappa-1)+1}\right)^{\frac{1}{\alpha}}$$
$$\times\left[\left\{\frac{3n}{4}|\Phi^{(\kappa)}(a)|^{\beta}+\frac{n}{4}|\Phi^{(\kappa)}(b)|^{\beta}-\frac{\mu n}{6}\|\ \delta(c_2,c_1)\|^{\sigma}\right\}^{\frac{1}{\beta}}\right.$$
$$\left.+\left\{\frac{n}{4}|\Phi^{(\kappa)}(a)|^{\beta}+\frac{3n}{4}|\Phi^{(\kappa)}(b)|^{\beta}-\frac{\mu n}{6}(\delta(c_2,c_1))^{\sigma}\right\}\right]^{\frac{1}{\beta}},\quad(10)$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\beta>1$.

(II.) Letting $\mathfrak{h}(\varrho)=\varrho^{s}$, then we acquire the Breckner type of higher order strongly s-preinvex functions.

Corollary 6. *For $\rho>0$, $n,\ k\in\mathbb{N}$, and there is a differentiable function $\Phi: M_{\delta}\to\mathbb{R}$ such that $c_1, c_1+\delta(c_2,c_1)\in M_{\delta}$, with $\delta(c_2,c_1)>0$. If $\Phi^{(\kappa)}\in L_1([c_1,c_1+\delta(c_2,c_1)])$ and $|\Phi^{(\kappa)}|^{\alpha}$ is Breckner type of a higher order strongly s-preinvex function, then*

$$|\Psi(k,n,\rho;c_1,c_2)(\Phi)|\leq\left(\frac{n^{\alpha(\rho+\kappa-1)+1}}{\alpha(\rho+\kappa-1)+1}\right)^{\frac{1}{\alpha}}$$
$$\times\left[\left\{\frac{n[2^{s+1}-1]}{2^{s}(s+1)}|\Phi^{(\kappa)}(c_1)|^{\beta}+\frac{n}{2^{s}(s+1)}|\Phi^{(\kappa)}(c_2)|^{\beta}-\frac{\mu n}{6}\ (\delta(c_2,c_1))^{\sigma}\right\}^{\frac{1}{\beta}}\right.$$
$$\left.+\left\{\frac{n}{2^{s}(s+1)}|\Phi^{(\kappa)}(c_1)|^{\beta}+\frac{n[2^{s+1}-1]}{2^{s}(s+1)}|\Phi^{(\kappa)}(c_2)|^{\beta}-\frac{\mu n}{6}\ (\delta(c_2,c_1))^{\sigma}\right\}^{\frac{1}{\beta}}\right],\quad(11)$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\beta>1$.

(III). Letting $\mathfrak{h}(\varrho)=\varrho^{-s}$, then we acquire the Godunova–Levin type of higher order strongly s-preinvex functions.

Corollary 7. *For $\rho > 0$, $n,\ k \in \mathbb{N}$, and there is a function differentiable funtion $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$ with $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|^\alpha$ is Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$|\Psi(k,n,\rho;a,b)(\Phi)| \le \left(\frac{n^{\alpha(\rho+\kappa-1)+1}}{\alpha(\rho+\kappa-1)+1}\right)^{\frac{1}{\alpha}} \times \left[\left\{\frac{(2n-2^s)}{1-s}|\Phi^{(\kappa)}(a)|^\beta + \frac{2^s n}{1-s}|\Phi^{(\kappa)}(b)|^\beta - \frac{\mu n}{6}\left(\delta(c_2,c_1)\right)^\sigma\right\}^{\frac{1}{\beta}} + \left\{\frac{2^s n}{1-s}|\Phi^{(\kappa)}(a)|^\beta + \frac{(2n-2^s)}{1-s}|\Phi^{(\kappa)}(b)|^\beta - \frac{\mu n}{6}\left(\delta(c_2,c_1)\right)^\sigma\right\}\right]^{\frac{1}{\beta}}, \quad (12)$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\beta > 1$.

(IV). Letting $\mathfrak{h}(\varrho) = 1$, then we acquire the higher order strongly P-preinvex functions.

Corollary 8. *For $\rho > 0$, $n,\ k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$, with $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|^\alpha$ is a higher order strongly P-preinvex function, then*

$$|\Psi(k,n,\rho;a,b)(\Phi)| \le 2\left(\frac{n^{\alpha(\rho+\kappa-1)+1}}{\alpha(\rho+\kappa-1)+1}\right)^{\frac{1}{\alpha}} \times \left\{|\Phi^{(\kappa)}(c_1)|^\beta + |\Phi^{(\kappa)}(c_2)|^\beta - \frac{\mu n}{6}\left(\delta(c_2,c_1)\right)^\sigma\right\}^{\frac{1}{\beta}}, \quad (13)$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\beta > 1$.

Theorem 3. *For $\rho > 0$, $n,\ k \in \mathbb{N}$, and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$, and $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|^\alpha$ is a higher order strongly $\mathfrak{h}$-preinvex function, then*

$$\begin{aligned}|\Psi(k,n,\rho;c_1,c_2)(\Phi)| &\le \Big[\mathcal{H}_1(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^\beta + \mathcal{H}_2(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^\beta \\ &\quad + \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^\sigma}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\Big]^{\frac{1}{\beta}} \\ &\quad + \Big[\mathcal{H}_2(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^\beta + \mathcal{H}_1(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^\beta \\ &\quad + \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^\sigma}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\Big]^{\frac{1}{\beta}}, \quad (14)\end{aligned}$$

where

$$\mathcal{H}_1(\rho,n,k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)}\mathfrak{h}\left(\frac{n+\varrho}{2n}\right)d\varrho$$

and

$$\mathcal{H}_2(\rho,n,k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)}\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)d\varrho.$$

Proof. Applying Lemma 1, the well-known Hölder's inequality, and by given hypothesis, we have

$$
\begin{aligned}
&|\Psi(k,n,\rho;c_1,c_2)(\Phi)| \\
&\leq \left(\int_0^n \frac{1}{n}d\varrho\right)^{\frac{1}{\alpha}} \left(\int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left|\Phi^{(\kappa)}\left(c_1+\left(\frac{n-\varrho}{2n}\right)\delta(c_2,c_1)\right)\right|^{\beta} d\varrho\right)^{\frac{1}{\beta}} \\
&+ \left(\int_0^n \frac{1}{n}d\varrho\right)^{\frac{1}{\alpha}} \left(\int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left|\Phi^{(\kappa)}\left(c_1+\left(\frac{n+\varrho}{2n}\right)\delta(c_2,c_1)\right)\right|^{\beta} d\varrho\right)^{\frac{1}{\beta}} \\
&\leq \left[\int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\mathfrak{h}\left(\frac{n+\varrho}{2n}\right)|\Phi^{(\kappa)}(c_1)|^{\beta} + \mathfrak{h}\left(\frac{n-\varrho}{2n}\right)|\Phi^{(\kappa)}(c_2)|^{\beta}\right.\right. \\
&\qquad \left.\left. - \frac{\mu}{(2n)^2}(n+\varrho)(n-\varrho)(\delta(c_2,c_1))^{\sigma}\right] d\varrho\right]^{\frac{1}{\beta}} \\
&+ \left[\int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\mathfrak{h}\left(\frac{n-\varrho}{2n}\right)|\Phi^{(\kappa)}(c_1)|^{\beta} + \mathfrak{h}\left(\frac{n+\varrho}{2n}\right)|\Phi^{(\kappa)}(c_2)|^{\beta}\right.\right. \\
&\qquad \left.\left. - \frac{\mu}{(2n)^2}(n+\varrho)(n-\varrho)(\delta(c_2,c_1))^{\sigma} d\varrho\right)\right]^{\frac{1}{\beta}} \\
&= \left[\mathcal{H}_1(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^{\beta} + \mathcal{H}_2(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^{\beta}\right. \\
&\qquad \left. - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^{\sigma}}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\right]^{\frac{1}{\beta}} \\
&+ \left[\mathcal{H}_2(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^{\beta} + \mathcal{H}_1(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^{\beta}\right. \\
&\qquad \left. - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^{\sigma}}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\right]^{\frac{1}{\beta}}, \qquad (15)
\end{aligned}
$$

which is the required result. □

Some special cases of Theorem 3 are stated as follows.
(I). Letting $\mathfrak{h}(\varrho) = \varrho$, then we acquire the higher order strongly preinvex functions.

Corollary 9. *For $\rho > 0$, n, $k \in \mathbb{N}$ and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2,c_1) \in M_\delta$ with $\delta(c_2,c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1+\delta(c_2,c_1)])$ and $|\Phi^{(\kappa)}|^{\alpha}$ is a higher order strongly $\mathfrak{h}$-preinvex function, then*

$$
\begin{aligned}
&|\Psi(k,n,\rho;c_1,c_2)(\Phi)| \\
&\leq \left[\mathcal{H}_3(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^{\beta} + \mathcal{H}_4(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^{\beta}\right. \\
&\qquad \left. - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^{\sigma}}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\right]^{\frac{1}{\beta}} \\
&+ \left[\mathcal{H}_4(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^{\beta} + \mathcal{H}_3(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^{\beta}\right. \\
&\qquad \left. - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^{\sigma}}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\right]^{\frac{1}{\beta}}, \qquad (16)
\end{aligned}
$$

where

$$\mathcal{H}_3(\rho,n,k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\frac{n+\varrho}{2n}\right) d\varrho = \frac{n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+3]}{2(\beta(\rho+\kappa-1)+1)(\beta(\rho+\kappa-1)+2)},$$

and

$$\mathcal{H}_4(\rho,n,k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\frac{n-\varrho}{2n}\right) d\varrho = \frac{n^{\beta(\rho+\kappa-1)+1}}{2(\beta(\rho+\kappa-1)+2)}.$$

(II). Letting $\mathfrak{h}(\varrho) = \varrho^s$, then we acquire the Breckner type of higher order strongly preinvex functions.

Corollary 10. *For $\rho > 0$, n, $k \in \mathbb{N}$ and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$ and $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|^\alpha$ is Breckner type of a higher order strongly s-preinvex function, then*

$$\begin{aligned}
&|\Psi(k,n,\rho;c_1,c_2)(\Phi)| \\
&\leq \Big[\mathcal{H}_5(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^\beta + \mathcal{H}_6(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^\beta \\
&\quad - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^\sigma}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\Big]^{\frac{1}{\beta}} \\
&\quad + \Big[\mathcal{H}_6(\rho,n,k)|\Phi^{(\kappa)}(c_1)|^\beta + \mathcal{H}_5(\rho,n,k)|\Phi^{(\kappa)}(c_2)|^\beta \\
&\quad - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2,c_1))^\sigma}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\Big]^{\frac{1}{\beta}}, \qquad (17)
\end{aligned}$$

where

$$\mathcal{H}_5(\rho,n,k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\frac{n+\varrho}{2n}\right)^s d\varrho = (2n)^{\beta(\rho+\kappa-1)+1}\mathbb{B}_{\frac{1}{2}}\left(\beta(\rho+\kappa-1)+1, s+1\right)$$

and

$$\mathcal{H}_6(\rho,n,k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\frac{n-\varrho}{2n}\right)^s d\varrho = \frac{n^{\beta(\rho+\kappa-1)+1}}{2^s(\beta(\rho+\kappa-1)+s+1)}.$$

(III). Letting $\mathfrak{h}(\varrho) = \varrho^{-s}$, then we acquire the Godunova–Levin type of higher order strongly preinvex functions.

Corollary 11. *For $\rho > 0, n,\ k \in \mathbb{N}$ and there is a differentiable $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$ and $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|^\alpha$ is Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$
\begin{aligned}
&|\Psi(k, n, \rho; c_1, c_2)(\Phi)| \\
&\leq \Big[\mathcal{H}_7(\rho, n, k)|\Phi^{(\kappa)}(c_1)|^\beta + \mathcal{H}_8(\rho, n, k)|\Phi^{(\kappa)}(c_2)|^\beta \\
&\quad - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4](\delta(c_2, c_1))^\sigma}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\Big]^{\frac{1}{\beta}} \\
&\quad + \Big[\mathcal{H}_8(\rho, n, k)|\Phi^{(\kappa)}(c_1)|^\beta + \mathcal{H}_7(\rho, n, k)|\Phi^{(\kappa)}(c_2)|^\beta \\
&\quad - \frac{\mu n^{\beta(\rho+\kappa-1)+1}[\beta(\rho+\kappa-1)+4(\delta(c_2, c_1))^\sigma]}{4(\beta(\rho+\kappa-1)+2)(\beta(\rho+\kappa-1)+3)}\Big]^{\frac{1}{\beta}}, \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_7(\rho, n, k) &:= \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\frac{n+\varrho}{2n}\right)^{-s} d\varrho \\
&= (2n)^{\beta(\rho+\kappa-1)+1} \mathbb{B}_{\frac{1}{2}}\left(\beta(\rho+\kappa-1)+1, -s+1\right)
\end{aligned}
$$

and

$$
\mathcal{H}_8(\rho, n, k) := \int_0^n (n-\varrho)^{\beta(\rho+\kappa-1)} \left(\frac{n-\varrho}{2n}\right)^{-s} d\varrho = \frac{2^s n^{\beta(\rho+\kappa-1)+1}}{(\beta(\rho+\kappa-1)-s+1)}.
$$

(IV). Letting $\mathfrak{h}(\varrho) = 1$, then we acquire the higher order strongly P-preinvex functions.

Corollary 12. *For $\rho > 0,\ n,\ k \in \mathbb{N}$ and there is a differentiable function $\Phi : M_\delta \to \mathbb{R}$ such that $c_1, c_1 + \delta(c_2, c_1) \in M_\delta$ with $\delta(c_2, c_1) > 0$. If $\Phi^{(\kappa)} \in L_1([c_1, c_1 + \delta(c_2, c_1)])$ and $|\Phi^{(\kappa)}|^\alpha$ is a higher order strongly P-preinvex function, then*

$$
\begin{aligned}
|\Psi(k, n, \rho; c_1, c_2)(\Phi)| &\leq 2\left(\frac{n^{\beta(\rho+\kappa-1)+1}}{\beta(\rho+\kappa-1)+1}\right)^{\frac{1}{\beta}} \\
&\times \left[|\Phi^{(\kappa)}(c_1)|^\beta + |\Phi^{(\kappa)}(c_2)|^\beta - \frac{\mu\left(\beta(\rho+\kappa-1)+1\right)\left(\beta(\rho+\kappa-1)+4\right)}{4\left(\beta(\rho+\kappa-1)+2\right)\left(\beta(\rho+\kappa-1)+3\right)}(\delta(c_2, c_1))^\sigma\right]^{\frac{1}{\beta}}. \qquad (19)
\end{aligned}
$$

5. Conclusions

We presented the concept of higher order strongly $\mathfrak{h}$-preinvex functions with different kind of preinvexities. We built up the higher order strongly preinvex functions for Breckner type, Godunova–Levin type and P- preinvex functions. We establish an identity associated with differentiable functions of kth order using Riemann–Liouville fractional integral operator. We derived several new upper bounds for the Hermite–Hadamard type by using the strongly $\mathfrak{h}$ preinvexity of the kth order derivative concerning to Riemann–Liouville fractional integral. Further, we have taken some particular cases of these results, choosing specific values of the mapping $\mathfrak{h}$. We can discover a number of inequalities by selecting the values applicable to the restrictions of σ and $\mathfrak{h}$. Here, we emphasis that all the derived outcomes in the present paper endured preserving for strongly preinvex functions, certainly, which can be seen by the unique values of $\sigma = 2$ and $\mathfrak{h}(\varrho) = \varrho$. Moreover, our consequence have potential applications in signal and image processing algorithms based on sparsity convex programming for inverse problems [6] and antimatroids (to model the ordering of events in discrete event simulation systems and mathematical psychology) [8]. We hope that the novel strategies of this paper will inspire the researchers working in the field of analysis, numerical analysis and mathematical inequalities. This is an interesting direction for future research.

Author Contributions: All authors contributed substantially to the research. Conceptualization: S.R. and M.A.L.; methodology, S.R.; software, Y.-M.C. and S.R.; validation, S.R. and M.A.L.; formal analysis, S.R. and Z.H.; investigation, Y.-M.C., Z.H., and M.A.L.; resources, Z.H. and S.R.; data curation, S.R.; writing, original draft preparation, S.R.; writing, review and editing, S.R. and Y.-M.C.; supervision, M.A.L.; project administration, S.R. and Z.H.; funding acquisition, Y.-M.C.

Funding: The work is supported by the Natural Science Foundation of China (Grant Numbers 61673169, 11301127, 11701176, 11626101, 11601485) and The Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

Conflicts of Interest: The authors declare that there are no conflict of interest regarding the publication of this paper.

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