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Some New Observations and Results for Convex Contractions of Istratescu's Type

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Abstract: The purpose is to ensure that a continuous convex contraction mapping of order two in b-metric spaces has a unique fixed point. Moreover, this result is generalized for convex contractions of order n in b-metric spaces and also in almost and quasi b-metric spaces.

Keywords: convex contraction; fixed point; b-metric space; almost b-metric space; coincidence point

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1. Introduction

In [1,2], the notion of a b-metric space was initiated and some usual fixed point results have been provided. Many new results in this space were obtained over the past ten years (see for example [3–6]). Istratescu [7] considered convex contraction mappings in metric spaces and showed that each convex contraction mapping of order two admits a unique fixed point. The Istratescu's result has recently caused the attention and was the object of examination in b-metric spaces (see [8]). Our paper is a generalization of the Istratescu's result for convex contractions of order n in b-metric spaces (and also in almost b-metric spaces and in quasi b-metric spaces).

2. Preliminaries

Definition 1. *Given a nonempty set* Υ *and* $s \geq 1$*. Let* $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ *satisfy:*

(1) $\eta(\omega, \upsilon) = 0$ *if* $\omega = \upsilon$*;*
(2) $\eta(\omega, \upsilon) = \eta(\upsilon, \omega)$*;*
(3) $\eta(\omega, \varsigma) \leq s[\eta(\omega, \upsilon) + \eta(\upsilon, \varsigma)]$*,*

for all $\omega, \upsilon, \varsigma \in \Upsilon$, then d is a b-metric. Here, (Υ, η, s) is called a b-metric space.

Definition 2. *Let $\{\varsigma_n\}$ be a sequence in a b-metric space (Υ, η, s). Take $\varsigma \in \Upsilon$.*

(a) $\{\varsigma_n\}$ is convergent to ς, if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $\eta(\varsigma_n, \varsigma) < \varepsilon$ for all $n \geq n_0$;
(b) $\{\varsigma_n\}$ is Cauchy if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $\eta(\varsigma_n, \varsigma_m) < \varepsilon$ for all $n, m \geq n_0$;
(c) (Υ, η, s) is complete if every Cauchy sequence is convergent.

Miculescu and Mihail [9] (Lemma 2.2) and Suzuki [10] (Lemma 6) gave the following result (see also [11]).

Lemma 1. *Let $\{\varsigma_n\}$ be a sequence in the b-metric space (Υ, η, s) so that there is $\gamma \in [0, 1)$ in order that for every $n \geq 1$, $\eta(\varsigma_{n+1}, \varsigma_n) \leq \gamma\eta(\varsigma_n, \varsigma_{n-1})$. Then $\{\varsigma_n\}$ is Cauchy.*

The above lemma is an important tool to get variant results in b-metric spaces since it facilitates many proofs concerning various contraction conditions. The following is a consequence of the proof of Lemma 1.3 in [9]).

Lemma 2. *Let $\{\varsigma_n\}$ be a sequence in the b-metric space (Υ, η, s) so that there are $\gamma \in [0, 1)$ and $C > 0$ in order that for each $n \geq 0$,*

$$\eta(\varsigma_{n+1}, \varsigma_n) \leq C\gamma^n.$$

Then $\{\varsigma_n\}$ is Cauchy.

Definition 3. *Let (Υ, η, s) be a b-metric space. The mapping $\Omega : \Upsilon \to \Upsilon$ is a convex Reich type contraction if*

$$\begin{aligned} \eta(\Omega^2\varsigma, \Omega^2\omega) \quad \leq \quad & \alpha\eta(\varsigma, \omega) + \beta\eta(\Omega\varsigma, \Omega\omega) + \gamma[\eta(\varsigma, \Omega\varsigma) + \eta(\omega, \Omega\omega)] + \\ & \delta[\eta(\Omega\varsigma, \Omega^2\varsigma) + \eta(\Omega\omega, \Omega^2\omega)], \end{aligned}$$

for all $\varsigma, \omega \in \Upsilon$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2(\gamma + \delta) < 1$.

Definition 4. *Let (Υ, η, s) be a b-metric space. A mapping $\Omega : \Upsilon \to \Upsilon$ is a convex contraction of Reich type of order $n (n \geq 2)$ if*

$$\begin{aligned} \eta(\Omega^n\varsigma, \Omega^n\omega) \quad \leq \quad & a_0\eta(\varsigma, \omega) + a_1\eta(\Omega\varsigma, \Omega\omega) \\ + \quad & \sum_{i=0}^{n-1} a_{i+2}[\eta(\Omega^i\varsigma, \Omega^{i+1}\varsigma) + \eta(\Omega^i\omega, \Omega^{i+1}\omega)], \end{aligned}$$

for all $\varsigma, \omega \in \Upsilon$, where $a_0, a_1, a_2, \ldots, a_{n+1}$ are nonnegative constants with $a_0 + a_1 + 2\sum_{i=0}^{n-1} a_{i+2} < 1$.

A slight modification of the definition of contraction mappings of order n was as follows:

Definition 5. *([7]) Let (Υ, η) be a complete metric space. A mapping $\Omega : \Upsilon \to \Upsilon$ is a convex contraction of order n if there are $a_0, \ldots, a_{n-1}$ in $(0, 1)$ so that for all $\varsigma, \omega \in \Upsilon$,*

$$\eta(\Omega^n(\varsigma), \Omega^n(\omega)) \leq a_0\eta(\varsigma, \omega) + a_1\eta(\Omega(\varsigma), \Omega(\omega)) + \cdots + a_{n-1}\eta(\Omega^{n-1}(\varsigma), \Omega^{n-1}(\omega)), \quad (1)$$

where $\Omega^n(\varsigma) = \Omega(\Omega^{n-1}(\varsigma))$ and $a_0 + a_1 + \cdots + a_{n-1} < 1$.

If we exclude in Definition 1 the symmetry condition, (Υ, η, s) is said to be a quasi b-metric space. Now, we recall the definition of almost b-metric spaces, which relies on some symmetry-type limiting conditions of defined almost b-metrics.

Definition 6. *([12]) Let* Υ *be a nonempty set and* $s \geq 1$. *Let* $\eta_{ab} : \Upsilon \times \Upsilon \to [0, +\infty)$ *and* $\varsigma, \omega, \sigma, \varsigma_n \in \Upsilon$ *so that:*
(bM1) $\eta_{ab}(\varsigma, \omega) = 0$ *if and only if* $\varsigma = \omega$,
(bM2l) $\eta_{ab}(\varsigma_n, \varsigma) \to 0$, $n \to \infty$ *implies* $\eta_{ab}(\varsigma, \varsigma_n) \to 0$, $n \to \infty$,
(bM2r) $\eta_{ab}(\varsigma, \varsigma_n) \to 0$, $n \to \infty$ *implies* $\eta_{ab}(\varsigma_n, \varsigma) \to 0$, $n \to \infty$,
(bM3) $\eta_{ab}(\varsigma, \omega) \leq s(\eta_{ab}(\varsigma, \sigma) + \eta_{ab}(\sigma, \omega))$.

1. *If (bM1), (bM2l) and (bM3) are verified, then* η_{ab} *is said to be an l-almost b-metric on* Υ;
2. *If (bM1), (bM2r) and (bM3) are verified, then* η_{ab} *is said to be an r-almost b-metric on* Υ;
3. *If (bM1), (bM2l), (bM2r) and (bM3) are verified, then* η_{ab} *is said to be an almost b-metric on* Υ.

We refer readers to see more about those spaces in the papers [1,2,7–11,13–33]. For all contractive type conditions, see [34,35]. One of applications of contractive mappings was used for maximum likelihood estimation of the multiple linear regression parameters in the generalized Gauss–Laplace distribution assumption of the measurement's errors [36].

3. Main Results

3.1. Some Lemmas

The first lemma is an auxiliary result. We use it to be ensured that in a convex contraction, the Cauchyness holds. The same result was obtained for metric and *b*-metric spaces.

Lemma 3. *Let* $k \in \mathbb{N}$ *and* $\{\varsigma_n\}$ *be a sequence in a b-metric space* (Υ, η, s) *so that*

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{n+i}, \varsigma_{n+i-1}), \tag{2}$$

for all $n \in \mathbb{N}$, *where* $a_i \geq 0$ *such that* $\sum_{i=0}^{k-1} a_i < 1$. *Then*

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq c\gamma^n, \tag{3}$$

for all $n \in \mathbb{N}$, *where* $c = \max\{\eta(\varsigma_1, \varsigma_0), \ldots, \eta(\varsigma_k, \varsigma_{k-1})\}$ *and* $\gamma = \left(\sum_{i=0}^{k-1} a_i\right)^{1/k}$.

Proof. From (2), we obtain

$$\eta(\varsigma_{k+1}, \varsigma_k) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{i+1}, \varsigma_i) \leq c \sum_{i=0}^{k-1} a_i = c\gamma^k \leq c\gamma.$$

Similarly,

$$\eta(\varsigma_{k+2}, \varsigma_{k+1}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{i+2}, \varsigma_{i+1}) \leq c \sum_{i=0}^{k-2} a_i + c a_{k-1}\gamma \leq c \sum_{i=0}^{k-2} a_i + c a_{k-1} = c\gamma^k \leq c\gamma^2.$$

$$\vdots$$

$$\eta(\varsigma_{2k}, \varsigma_{2k-1}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{i+k}, \varsigma_{i+k-1}) \leq c \sum_{i=0}^{k-1} a_i \gamma^i \leq c \sum_{i=0}^{k-1} a_i = c\gamma^k.$$

So, by induction for $n \geq k$ we have that

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{n+i}, \varsigma_{n+i-1}) \leq c \sum_{i=0}^{k-1} a_i \gamma^{n+i-k} \leq c\gamma^{n-k} \sum_{i=0}^{k-1} a_i = c\gamma^n.$$

□

Lemma 4. *Let $k \in \mathbb{N}$ and $\{\varsigma_n\}$ be a sequence in a b-metric space (Υ, η, s) so that*

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{n+i}, \varsigma_{n+i-1}), \tag{4}$$

for all $n \in \mathbb{N}$, where $a_i \geq 0$ such that $\sum_{i=0}^{k-1} a_i < 1$. Then $\{\varsigma_n\}$ is Cauchy.

Proof. It follows directly from Lemma 2 and Lemma 3. □

Remark 1. *Lemma 4 with $k = 1$ corresponds to Lemma 1, while for $k > 1$, Lemma 4 is more general.*

3.2. On Convex Contractions of Order k in b-Metric Spaces

Our next theorem is a generalization of Istratescu's result about convex contractions. We generalize the result of [7] in two directions by proving it for any $k \in \mathbb{N}$ and by considering the class of b-metric spaces. What distinguishes our obtained result is the fact that it is the same as in usual metric spaces and in b-metric spaces. There are two reasons for our new result: The first is due to Lemma 4 and the other is adding the assumption that the considered mapping is continuous.

Theorem 1. *Let $\Omega : \Upsilon \to \Upsilon$ be a continuous convex contraction of order k, on a complete b-metric space (Υ, η), so that*

$$\eta(\Omega^k \varsigma, \Omega^k \omega) \leq \sum_{i=0}^{k-1} a_i \eta(\Omega^i \varsigma, \Omega^i \omega), \tag{5}$$

for all $\varsigma, \omega \in \Upsilon$, where $a_i \geq 0$ such that $\sum_{i=0}^{k-1} a_i < 1$. Then there is a unique fixed point of Ω.

Proof. Let t_0 be in Υ. Consider $t_n = \Omega^n(t_0)$.

$$\eta(t_{n+k}, t_{n+k-1}) = \eta(\Omega^k(t_n), \Omega^k(t_{n-1})) \leq \sum_{i=0}^{k-1} a_i \eta(\Omega^i(t_n), \Omega^i(t_{n-1}))$$
$$= \sum_{i=0}^{k-1} a_i \eta(t_{n+i}, t_{n+i-1})$$

and directly from Lemma 4, we conclude that $\{t_n\}$ is a Cauchy sequence in (Υ, η) (which is complete). Hence, there is $t \in \Upsilon$ so that $\lim_{n\to\infty} t_n = t$. Since Ω is continuous, we obtain that

$$\Omega(t) = \lim_{n\to\infty} \Omega(t_n) = \lim_{n\to\infty} t_{n+1} = t,$$

i.e., t is a fixed point of Ω. Its uniqueness follows from (5). □

Example 1. *The space* $\Upsilon = l^p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}, p \in (0,1)$, *together with the function* $d : l^p \times l^p \to \mathbb{R}$,

$$d(x,y) = \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = (x_n), y = (y_n) \in l^p$, *is a b-metric space with* $s = 2^{\frac{1}{p}}$, *[1]. Indeed, by an elementary calculation, we obtain*

$$d(x,z) \leq 2^{\frac{1}{p}}[d(x,y) + d(y,z)].$$

Let $\Omega : l^p \to l^p$ *be a mapping defined by*

$$\Omega(x_1, x_2, x_3, x_4, \ldots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{2}, \frac{x_4}{2}, \ldots).$$

Note that Ω *has a unique fixed point, which is* $(0,0,0,\ldots)$ *(see also [37]). We have*

$$\Omega^2(x_1, x_2, x_3, x_4, \ldots) = (0, 0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^2}, \frac{x_4}{2^2}, \ldots),$$

$$\Omega^3(x_1, x_2, x_3, x_4, \ldots) = (0, 0, 0, \frac{x_1}{2^2}, \frac{x_2}{2^3}, \frac{x_3}{2^3}, \frac{x_4}{2^3}, \ldots),$$

$$\vdots$$

$$\Omega^n(x_1, x_2, x_3, x_4, \ldots) = (\underbrace{0, \ldots, 0}_{n}, \frac{x_1}{2^{n-1}}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}, \ldots).$$

Further, for $x, y \in l^p$, *we have*

$$\begin{aligned} d(\Omega^n x, \Omega^n y) &= \left(\frac{|x_1 - y_1|^p}{2^{p(n-1)}} + \frac{|x_2 - y_2|^p}{2^{pn}} + \frac{|x_3 - y_3|^p}{2^{pn}} + \cdots \right)^{\frac{1}{p}} \\ &\leq \left[\frac{1}{2^{p(n-1)}} \left(|x_1 - y_1|^p + |x_2 - y_2|^p + |x_3 - y_3|^p + \cdots \right) \right]^{\frac{1}{p}} \\ &\leq \frac{1}{2^{n-1}} d(x,y). \end{aligned}$$

So, the mapping $\Omega : l^p \to l^p$ *is a convex contraction of order n. On the other hand,* Ω *is not a contraction. Indeed, for* $x = (1,0,0,\ldots)$ *and* $y = (2,0,0,\ldots)$, *we have* $\Omega x = (0,1,0,0,\ldots)$, $\Omega y = (0,2,0,0,\ldots)$, $d(x,y) = 1$ *and* $d(\Omega x, \Omega y) = 1$. *Consequently,*

$$d(\Omega x, \Omega y) > \lambda d(x,y),$$

for each $\lambda \in (0,1)$.

Example 2. *(a) Let* $\Upsilon = [1,4]$ *be endowed with* $\eta(\varsigma, \omega) = |\varsigma - \omega|$. *Consider* $\Omega : \Upsilon \to \Upsilon$ *defined by* $\Omega\varsigma = 2\sqrt{\varsigma}$. *Here,* Ω *is not a contraction on the metric space* (Υ, η). *Also,* Ω *is a convex contraction mapping of order 2. Namely, we have*

$$|\Omega^2\varsigma - \Omega^2\omega| \leq \frac{\sqrt{2}}{2}|\varsigma - \omega|,$$

for all $\varsigma, \omega \in \Omega$ *and* $\varsigma = 4$ *is the unique fixed point of* Ω.
(b) Let $\Upsilon = [1,4]$. *Consider* $\Omega\varsigma = 2\sqrt{\varsigma}$ *and* $\eta(\varsigma, \omega) = (\varsigma - \omega)^2$. *Here,* $(\Upsilon, \eta, s = 2)$ *is a b-metric space. Note that* Ω *is not contraction, but since*

$$\eta(\Omega^2\varsigma, \Omega^2, \omega) = 8(\sqrt[4]{\varsigma} - \sqrt[4]{\omega})^2 \leq \frac{1}{4}\left(\eta(\Omega\varsigma, \Omega\omega) + \eta(\varsigma, \omega)\right)$$

Ω is also a convex contraction of order 2.

Example 3. *Let $\Upsilon = \mathbb{R}$. Take $\Omega(\zeta) = \frac{\zeta+3}{4}$ and $\eta(\omega, v) = |\omega - v|^2$. Then (Υ, η) is a complete $b-$metric space with the coefficient $s = 2$. Given $\Omega : \Upsilon \to \Upsilon$ as $\Omega(\zeta) = \frac{\zeta+3}{4}$ for all $\zeta \in \Upsilon$. We obtain that*

$$d\left(\Omega^2\omega, \Omega^2 v\right) = \left|\frac{\omega+15}{16} - \frac{v+15}{16}\right|^2 = \frac{|\omega - v|^2}{256} \quad \text{and} \quad d(\Omega\omega, \Omega v) = \frac{|\omega - v|^2}{16},$$

that is,

$$d\left(\Omega^2\omega, \Omega^2 v\right) \leq a \cdot d(\Omega\omega, \Omega v) + b \cdot d(\omega, v),$$

i.e.,

$$\frac{|\omega - v|^2}{256} \leq a \cdot \frac{|\omega - v|^2}{16} + b \cdot |\omega - v|^2.$$

Or, equivalently $\frac{1}{256} \leq \frac{a}{16} + b$ where $a, b \geq 0$ and $a + b < 1$. Putting, for example $a = b$ we get that for $a = b \geq \frac{1}{272}$, all the conditions of Theorem 1 are satisfied, i.e., Ω is a convex contraction of order 2 and therefore has a unique fixed point (which is, $\zeta = 2$).

Remark 2. *Theorem 2.1 of [7] and Theorem 4 of [8] follow from Theorem 1. Also Theorem 1 improves the contraction condition (in b-metric spaces) stated for convex contraction mappings of order 2 in [8].*

In the previous lemmas and in Theorem 1, we did not use the symmetry of the b-metric η. Thus, with minor changes in the contraction conditions, the main results are similarly valid in the setting of almost b-metric spaces (and also in quasi b-metric spaces). Notice that previous formulations can also applied in l-almost b-metric spaces. Next, we state results for r-almost b-metric spaces.

Remark 3. *Let $k \in \mathbb{N}$ and $\{\varsigma_n\}$ be a sequence in the r-almost b-metric space (Υ, η, s) so that*

$$\eta(\varsigma_{n+k-1}, \varsigma_{n+k}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{n+i-1}, \varsigma_{n+i}), \tag{6}$$

for all $n \in \mathbb{N}$, where $a_i \geq 0$ such that $\sum\limits_{i=0}^{k-1} a_i < 1$. Then

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq c\gamma^n, \tag{7}$$

for all $n \in \mathbb{N}$, where $c = \max\{\eta(\varsigma_0, \varsigma_1), \ldots, \eta(\varsigma_{k-1}, \varsigma_k)\}$ and $\gamma = \left(\sum\limits_{i=0}^{k-1} a_i\right)^{1/k}$.

Since the result is similar to Lemma 2 and is valid in r-almost b-metric spaces (see [12]), the r-almost version of Lemma 4 is as follows: let $k \in \mathbb{N}$ and $\{\varsigma_n\}$ be a sequence in the r-almost b-metric space (Υ, η, s) so that

$$\eta(\varsigma_{n+k-1}, \varsigma_{n+k}) \leq \sum_{i=0}^{k-1} a_i \eta(\varsigma_{n+i-1}, \varsigma_{n+i}),$$

for all $n \in \mathbb{N}$, where $a_i \geq 0$ such that $\sum\limits_{i=0}^{k-1} a_i < 1$. Then $\{\varsigma_n\}$ is right-Cauchy sequence.

Next, we extend Lemma 3 to quasi b-metric spaces.

Remark 4. *Let* (Υ, η, s) *be a quasi b-metric space and let* $\{\varsigma_n\}$ *be a sequence in* Υ *and* $k \in \mathbb{N}$ *such that (4) is satisfied for all* $n \in \mathbb{N}$*, where* $a_i \geq 0$ *such that* $\sum_{i=0}^{k-1} a_i < 1$*. Then*

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq c\gamma^n, \tag{8}$$

for all $n \in \mathbb{N}$*, where* $c = \max\{\eta(\varsigma_1, \varsigma_0), \eta(\varsigma_0, \varsigma_1) \ldots, \eta(\varsigma_k, \varsigma_{k-1}), \eta(\varsigma_{k-1}, \varsigma_k)\}$ *and* $\gamma = \left(\sum_{i=0}^{k-1} a_i\right)^{1/k}$.

3.3. On Convex Reich Type Contractions of Order k in b-Metric Spaces

Our next result is about convex Reich type contractions of order k ($k \geq 2$), (see [18,35,38,39]).

Theorem 2. *Let* (Υ, η, s) *be a complete b-metric space and* $\Omega : \Upsilon \to \Upsilon$ *be a continuous convex Reich type contraction mapping of order* k $(k \geq 2)$ *so that*

$$\begin{aligned} \eta(\Omega^k\varsigma, \Omega^k\omega) &\leq a_0\eta(\varsigma, \omega) + a_1\eta(\Omega\varsigma, \Omega\omega) \\ &+ \sum_{i=0}^{k-1} a_{i+2}[\eta(\Omega^i\varsigma, \Omega^{i+1}\varsigma) + \eta(\Omega^i\omega, \Omega^{i+1}\omega)], \end{aligned}$$

for all $\varsigma, \omega \in \Upsilon$*, where* $a_0, a_1, a_2, \ldots, a_{k+1} \geq 0$ *with* $a_0 + a_1 + 2\sum_{i=0}^{k-1} a_{i+2} < 1$*. Then there is a unique fixed point of* Υ*.*

Proof. Let ς_0 be Υ. Take $\varsigma_n = \Omega^n(\varsigma_0)$ for all $n \in \mathbb{N}$. Now, since Ω is a convex Reich type contraction of order k we obtain

$$\begin{aligned} \eta(\varsigma_{n+k}, \varsigma_{n+k-1}) &\leq a_0\eta(\varsigma_n, \varsigma_{n-1}) + a_1\eta(\varsigma_{n+1}, \varsigma_n) \\ &+ \sum_{i=0}^{k-1} a_{i+2}[\eta(\varsigma_{n+i}, \varsigma_{n+i+1}) + \eta(\varsigma_{n+i-1}, \varsigma_{n+i})]. \end{aligned}$$

Since

$$\sum_{i=0}^{k-1} a_{i+2}\eta(\varsigma_{n+i-1}, \varsigma_{n+i}) = a_2\eta(\varsigma_{n-1}, \varsigma_n) + \sum_{i=0}^{k-2} a_{i+3}\eta(\varsigma_{n+i}, \varsigma_{n+i+1}),$$

we obtain

$$\begin{aligned} \eta(\varsigma_{n+k}, \varsigma_{n+k-1}) &\leq (a_0 + a_2)\eta(\varsigma_n, \varsigma_{n-1}) + a_1\eta(\varsigma_{n+1}, \varsigma_n) \\ &+ \sum_{i=0}^{k-2}(a_{i+2} + a_{i+3})\eta(\varsigma_{n+i}, \varsigma_{n+i+1}) + a_{k+1}\eta(\varsigma_{n+k-1}, \varsigma_{n+k}). \end{aligned}$$

That is,

$$\begin{aligned} (1 - a_{k+1})\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) &\leq (a_0 + a_2)\eta(\varsigma_n, \varsigma_{n-1}) + a_1\eta(\varsigma_{n+1}, \varsigma_n) \\ &+ \sum_{i=0}^{k-2}(a_{i+2} + a_{i+3})\eta(\varsigma_{n+i}, \varsigma_{n+i+1}). \end{aligned}$$

Thus,

$$\eta(\varsigma_{n+k}, \varsigma_{n+k-1}) \leq \sum_{i=0}^{k-1} b_i\eta(\varsigma_{n+i}, \varsigma_{n+i-1}).$$

for all $n \in \mathbb{N}$, where $b_0 = \frac{a_0+a_2}{1-a_{k+1}}$, $b_1 = \frac{a_1+a_2+a_3}{1-a_{k+1}}$, $b_i = \frac{a_{i+1}+a_{i+2}}{1-a_{k+1}}$, $i = 2, \ldots, k-1$. Since $a_0 + a_1 + 2\sum_{i=0}^{k-1} a_{i+2} < 1$, we obtain that $\sum_{i=0}^{k-1} b_i < 1$, therefore using Lemma 4 we conclude that ς_n is a Cauchy sequence, and so there is $\varsigma \in Y$ such that $\lim_{n\to\infty} \varsigma_n = \varsigma$. It further shows that ς is the unique fixed point of Ω (as in Theorem 1). □

Remark 5. *From Theorem 2, we obtain Theorem 2.3 of [7] (case $k = 2$).*

Finally, as an open problem, we may ask the following question:
Let (Y, η, s) be a b-metric space. Find the conditions for the constants a_0, a_1, b_i, c_i such that the mapping $\Omega : Y \to Y$ has a unique fixed point, if the next condition (Hardy–Rogers type contraction order of k) is fulfilled:

$$\begin{aligned}
\eta(\Omega^k\varsigma, \Omega^k\omega) &\leq a_0\eta(\varsigma, \omega) + a_1\eta(\Omega\varsigma, \Omega\omega) \\
&+ \sum_{i=0}^{k-1} b_i[\eta(\Omega^i\varsigma, \Omega^{i+1}\varsigma) + \eta(\Omega^i\omega, \Omega^{i+1}\omega)] \\
&+ \sum_{i=0}^{k-1} c_i[\eta(\Omega^i\varsigma, \Omega^{i+1}\omega) + \eta(\Omega^i\omega, \Omega^{i+1}\varsigma)],
\end{aligned}$$

for all $\varsigma, \omega \in Y$.

4. Conclusions

We generalized the Istratescu's result for convex contractions. Particulary, we considered convex contractions of order k in the setting of b-metric spaces. Some related observations have been made for the classes of almost and quasi b-metric spaces. The presented results have been supported by some examples.

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