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Complex Asymptotics in λ for the Gegenbauer Functions $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ with $z \in (-1,1)$

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Abstract: We derive asymptotic results for the Gegenbauer functions $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ of the first and second kind for complex z and the degree $|\lambda| \to \infty$, apply the results to the case $z \in (-1,1)$, and establish the connection of these results to asymptotic Bessel-function approximations of the functions for $z \to \pm 1$.

Keywords: Gegenbauer functions; asymptotics

MSC: 33C45; 33C20; 34L10; 30E20

1. Introduction

The Gegenbauer functions $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ of the first and second kinds,

$$C_\lambda^\alpha(z) = \frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)\Gamma(2\alpha)}\, {}_2F_1\left(-\lambda, \lambda+2\alpha; \alpha+\frac{1}{2}; \frac{1-z}{2}\right), \tag{1}$$

$$D_\lambda^\alpha(z) = e^{i\pi\alpha}\,(2(z-1))^{-\lambda-2\alpha}\,\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+\alpha+1)\Gamma(\alpha)}\, {}_2F_1\left(\lambda+2\alpha, \lambda+\alpha+\frac{1}{2}; 2\lambda+2\alpha+1; \frac{2}{1-z}\right), \tag{2}$$

appear frequently in physical problems that involve hyperspherical or hyperbolic geometry. A number of results are known for the asymptotic behavior of these functions for the degree $|\lambda| \to \infty$ with z complex ([1], Section 2.3.2 (17)), ([2], Section 6), ([3], Appendix), ([4], Section 2.3 (1)), ([5], Section 14). However, as usually stated, the simple results presented here in Theorems 1 and 2 exclude the important cases with z real, $-1 < z < 1$ and $1 < z < \infty$, and exclude the limits $z \to \pm 1$. These cases have been of interest in recent problems, for example, in [4] (Private communication from Dr. Howard Cohl).

In the present work, we sketch the derivation of these results and show that they can be extended to include the cases usually excluded. We show also that the results connect smoothly for $|\lambda|$ large and $|1 \pm z|$ small to asymptotic expansions for $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ in terms of Bessel functions, Theorems 3 and 4. Those expansions include the limits $z \to \pm 1$. In particular, corresponding expressions in Theorems 1 and 2 and Theorems 3 and 4 agree in their common ranges of validity where the quantities $|\sqrt{z \pm 1}|$ are much larger than $1/|\lambda|$ but much smaller than $1/|\lambda|^{1/3}$, $1/|\lambda| \ll |\sqrt{z \pm 1}| \ll 1/|\lambda|^{1/3}$.

2. Asymptotic Results for $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$

Theorem 1. *Let $z \in \mathbb{C}$ and define $z_\pm = z \pm \sqrt{z^2-1}$ with $-\pi \le \arg(z \pm 1) \le \pi$. Then for $\Re\lambda \ge 0$, $\Re\alpha > 0$, $-\pi/2 \le \arg\lambda \le \pi/2$, and $|\lambda| \to \infty$,*

$$D_\lambda^\alpha(z) = e^{i\pi\alpha}\frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(z^2-1)^{-\alpha/2}z_+^{-\lambda-\alpha}\left[1+\mathcal{O}(1/|\lambda|)\right], \tag{3}$$

$$C_\lambda^\alpha(z) = \frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(z^2-1)^{-\alpha/2}\left(e^{i\pi\alpha}z_+^{-\lambda-\alpha}+z_-^{-\lambda-\alpha}\right)\left[1+\mathcal{O}(1/|\lambda|)\right]. \tag{4}$$

Theorem 2. *For $z = x$ real with $x \in (-1,1)$, define $x = \cos\theta$, $0 < \theta < \pi$. Then for $|\lambda| \to \infty$ with $\Re\lambda \geq 0$, $\Re\alpha > 0$, $-\pi/2 \leq \arg\lambda \leq \pi/2$, and $\sqrt{1-x^2} = \sin\theta \gg 1/|\lambda|$, the Gegenbauer functions $\mathsf{D}_\lambda^\alpha(\cos\theta)$ and $\mathsf{C}_\lambda^\alpha(\cos\theta) = C_\lambda^\alpha(\cos\theta)$ "on the cut" $(-1,1)$ ([6], 7, 8) have the limiting behavior*

$$\mathsf{D}_\lambda^\alpha(\cos\theta) = -\frac{2^{-\alpha+1}}{\Gamma(\alpha)}\lambda^{\alpha-1}(\sin\theta)^{-\alpha}\sin\left((\lambda+\alpha)\theta-\pi\alpha/2\right)\left[1+\mathcal{O}(1/|\lambda|)\right], \tag{5}$$

$$\mathsf{C}_\lambda^\alpha(\cos\theta) = \frac{2^{-\alpha+1}}{\Gamma(\alpha)}\lambda^{\alpha-1}(\sin\theta)^{-\alpha}\cos\left((\lambda+\alpha)\theta-\pi\alpha/2\right)\left[1+\mathcal{O}(1/|\lambda|)\right]. \tag{6}$$

These functions are proportional to the Ferrers functions of ([5], Section 14.23).

Theorem 3. *For z complex with $z \approx 1$, define $Z = \sqrt{2(\lambda+\alpha)^2(1-z)}$ and $Z' = \sqrt{2(\lambda+\alpha)^2(z-1)}$. Then for $\Re(\lambda+\alpha) \geq 0$, $\Re\alpha \geq -\frac{1}{2}$, $|\sqrt{z-1}| \ll 1/|\lambda|^{1/3}$, and $|\lambda| \to \infty$,*

$$D_\lambda^\alpha(z) = \frac{1}{\sqrt{\pi}}e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}2^{-\alpha+\frac{1}{2}}(\lambda+\alpha)^{\alpha-\frac{1}{2}}(z^2-1)^{-\frac{\alpha}{2}+\frac{1}{4}}K_{\alpha-\frac{1}{2}}(Z')\left[1+\mathcal{O}\left(1/|\lambda|^{2/3}\right)\right], \tag{7}$$

$$C_\lambda^\alpha(z) = \frac{\sqrt{\pi}}{\Gamma(\alpha)}2^{-2\alpha+1}(\lambda+\alpha)^{2\alpha-1}\left(\frac{Z}{2}\right)^{-\alpha+\frac{1}{2}}J_{\alpha-\frac{1}{2}}(Z)\left[1+\mathcal{O}\left(1/|\lambda|^{2/3}\right)\right]. \tag{8}$$

For $x = \cos\theta \in (-1,1)$, $\theta \approx 0$, $X = \sqrt{2(\lambda+\alpha)^2(1-x)}$, and $|\lambda| \to \infty$,

$$\mathsf{D}_\lambda^\alpha(x) = -\frac{\sqrt{\pi}}{\Gamma(\alpha)}2^{-\alpha+\frac{1}{2}}(\lambda+\alpha)^{\alpha-\frac{1}{2}}(1-x^2)^{-\alpha/2+\frac{1}{4}}Y_{\alpha-\frac{1}{2}}(X)\left[1+\mathcal{O}\left(1/|\lambda|^{2/3}\right)\right], \tag{9}$$

$$\mathsf{C}_\lambda^\alpha(x) = \frac{\sqrt{\pi}}{\Gamma(\alpha)}2^{-\alpha+\frac{1}{2}}(\lambda+\alpha)^{\alpha-\frac{1}{2}}(1-x^2)^{-\alpha/2+\frac{1}{4}}J_{\alpha-\frac{1}{2}}(X)\left[1+\mathcal{O}\left(1/|\lambda|^{2/3}\right)\right]. \tag{10}$$

The results in Theorems 2 and 3 match in their common range of validity, $1/|\lambda| \ll |\sqrt{z-1}| \ll 1/|\lambda|^{1/3}$.

Theorem 4. *For z complex with $z \approx -1$, define $Z'' = \sqrt{2(\lambda+\alpha)^2(1+z)}$. Then for $\Re(\lambda+\alpha) \geq 0$, $\Re\alpha \geq -\frac{1}{2}$, $|\sqrt{z+1}| \ll 1/|\lambda|^{1/3}$, and $|\lambda| \to \infty$,*

$$D_\lambda^\alpha(z) = e^{i\pi\alpha}2^{-\alpha}(\lambda+\alpha)^{\alpha-1}\left(2(1+z)\right)^{-\alpha}\frac{1}{\Gamma(\alpha)}\frac{\sqrt{\pi}}{\sin\pi(\alpha-\frac{1}{2})}\left(\frac{Z''}{2}\right)^{\frac{1}{2}}$$
$$\times e^{\mp i\pi(\lambda+2\alpha)}\left[-J_{\alpha-\frac{1}{2}}(Z'')+e^{\pm i\pi(\alpha-\frac{1}{2}}J_{-\alpha+\frac{1}{2}}(Z'')\right]\left[1+\mathcal{O}\left(1/|\lambda|^{2/3}\right)\right], \tag{11}$$

where the + and - signs hold for z on the upper (lower) sides of the cut in $(z-1)^{\alpha-\frac{1}{2}}$.

For $x = \cos\theta \in (-1, 1)$ with $\pi - \theta \ll 1/|\lambda|^{1/3}$ and $|\lambda| \to \infty$,

$$\mathrm{D}_\lambda^\alpha(x) \sim \frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{\lambda+\alpha}{2}\right)^{\alpha-1}(2(1+x))^{-\alpha}\left(\frac{X''}{2}\right)^{\frac{1}{2}}\left[-\sin\pi\lambda J_{\alpha-\frac{1}{2}}(X'') + \cos\pi\lambda Y_{\alpha-\frac{1}{2}}(X'')\right], \quad (12)$$

$$\mathrm{C}_\lambda^\alpha(x) \sim \frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{\lambda+\alpha}{2}\right)^{\alpha-1}(2(1+x))^{-\alpha}\left(\frac{X''}{2}\right)^{\frac{1}{2}}\left[\cos\pi\lambda J_{\alpha-\frac{1}{2}}(X'') + \sin\pi\lambda Y_{\alpha-\frac{1}{2}}(X'')\right], \quad (13)$$

where $X'' = \sqrt{2(\lambda+\alpha)^2(1+x)}$, with uncertainties of relative order $1/|\lambda|^{2/3}$. The results in Theorems 2 and 4 match for $1/|\lambda| \ll |\sqrt{1+z}| \ll 1/|\lambda|^{1/3}$.

Derivation of Theorem 1:

Start with the following integral representation for $D_\lambda^\alpha(z)$ for $z \in \mathbb{C}$ ([2], Section 1 (5)):

$$D_\lambda^\alpha(x) = \frac{1}{2\pi i}e^{2\pi i\alpha}\int_{\mathcal{C}_+} dt\, t^{-\lambda-1}(t-z_+)^{-\alpha}(t-z_-)^{-\alpha}, \quad \Re\lambda \geq 0, \quad \Re(\lambda+2\alpha) > 0, \quad (14)$$

where the integration contour $\mathcal{C}_+$ in the t plane runs from $+\infty$, around the point z_+ in the positive sense, and back to $+\infty$, $\mathcal{C}_+ = (\infty, z_++, \infty)$. The factors $(t-z_\pm)^{-\alpha}$ are taken as cut in the t plane from $z_\pm$ to ∞ along the directions defined by the lines from $t = 0$ to $t = z_\pm$. The phases of the factors $t - z_\pm$ are defined as zero on the upper sides of the cuts for $\Im z > 0$, and elsewhere by continuation in z.

The function $C_\lambda^\alpha(z)$ has a similar integral representation ([2], Section 1 (3)):

$$C_\lambda^\alpha(z) = \frac{1}{2\pi i}e^{2\pi i\alpha}\int_{\mathcal{C}} dt\, t^{-\lambda-1}(t-z_+)^{-\alpha}(t-z_-)^{-\alpha}, \quad \Re\lambda \geq 0, \quad \Re(\lambda+2\alpha) > 0, \quad (15)$$

where the contour $\mathcal{C} = (-\infty - i\epsilon, 0+, -\infty + i\epsilon)$ runs around the negative t axis in the positive sense. In this representation the phases of the factors $(t-z_\pm)^{-\alpha}$ are defined separately for $\Im z \gtrless 0$, with, in both cases, the factors cut in the t plane as above from $t = z_\pm$ to ∞, $0 < \arg(t - z_\pm) < 2\pi$. See ([2], Section 1 (3)) or ([1], Section 3.15.2 (2)).

In these expressions, $z_\pm = z \pm \sqrt{z^2-1}$ with $\sqrt{z^2-1}$ cut in the z plane from $z = 1$ to $-\infty$, $-\pi < \arg z < \pi$. For z in the upper (lower) half plane, z_+ is in the upper (lower) half plane outside the unit circle, while $z_- = 1/z_+$ is in the lower (upper) half plane inside the unit circle. For $z \in (-1, 1)$, $z_\pm$ lie on the unit circle. The singularities at $z_\pm$ pinch the contour $\mathcal{C}_+$ for $z \to \pm 1$, so $D_\lambda^\alpha(z)$ has branch points at ± 1 and can be taken as cut from ± 1 to $-\infty$. Similarly, the singularities at $z_\pm$ pinch the contour $\mathcal{C}$ for $z \to -1$, so $C_\lambda^\alpha(z)$ has a branch point there and can be taken as cut from -1 to $-\infty$.

In treating the asymptotic properties of $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ in λ, we will take $\Re\alpha > 0$ and $\Re\lambda \geq 0$. The integrands in Equations (14) and (15) are then singular at $t = 0$, z_+, and z_- and smaller in magnitude between, and vanish for $|t| \to \infty$, so there will be saddle points in the region of the singularities. If the contours $\mathcal{C}_+$ or $\mathcal{C}$ can be distorted to run through the saddle points in the directions in which the integrands decrease most rapidly, the method of steepest descents provides an estimate of the integrals. This is valid provided the integrands are small on the remainder of the contour and decrease rapidly for $|t| \to \infty$.

To determine the location of the saddle points, write the integrands in Equations (15) and (14) as $e^{\Phi(t)}$, with

$$\Phi(t) = -(\lambda+1)\ln t - \alpha\ln(t-z_+) - \alpha\ln(t-z_-), \quad (16)$$

and require that $d\Phi/dt$ vanish as required for a stationary point. This gives the condition

$$\frac{\lambda+1}{t} + \frac{\alpha}{t-z_+} + \frac{\alpha}{t-z_-} = 0. \quad (17)$$

For $|\lambda|$ large, the solutions must be close to z_+ or z_-. If those points are well separated, the solutions to order $1/|\lambda|$ are

$$t_+ = z_+\left(1-\frac{\alpha}{\lambda+1}\right)+\mathcal{O}\left(\frac{\alpha^2}{(\lambda+1)^2}\right), \quad \text{and} \quad t_- = z_-\left(1-\frac{\alpha}{\lambda+1}\right)+\mathcal{O}\left(\frac{\alpha^2}{(\lambda+1)^2}\right). \tag{18}$$

In general,

$$t_\pm = \frac{1+\alpha'}{1+2\alpha'}\left[z\pm\sqrt{z^2-1+(\alpha'/(1+\alpha'))^2}\right], \quad \alpha' = \frac{\alpha}{\lambda+1}, \tag{19}$$

so there are only the two saddle points $t_\pm$.

Next, expand the exponent function $\Phi(t)$ in a Taylor series around the saddle points. To second order, for $|\lambda|$ large,

$$\Phi(t) \approx \Phi(t_\pm) + \frac{\lambda^2}{\alpha z_\pm^2}(t-t_\pm)^2 \tag{20}$$

near $t_\pm$. This gives the approximation

$$\frac{1}{2\pi i}e^{2\pi i\alpha}\left[z_+\left(1-\frac{\alpha}{\lambda+1}\right)\right]^{-\lambda-1}\left(e^{i\pi}\frac{\alpha}{\lambda+1}z_+\right)^{-\alpha}(z_+-z_-)^{-\alpha}\int_{\mathcal{C}_+}dt\, e^{\frac{1}{2}\frac{\lambda^2}{\alpha z_+^2}(t-t_+)^2+\cdots} \tag{21}$$

for the integral in the neighborhood of the saddle point at t_+. The factors $e^{i\pi\alpha}\,(\alpha/(\lambda+1))\,z_+$ and $(z_+-z_-)^{-\alpha}$ in this expression arise from the factors $(t_+-z_+)^\alpha$ and $(t_+-z_-)^{-\alpha}$ in the limit of large $|\lambda|$. A similar result holds near t_- with a different phase, $(t_--z_+)^{-\alpha} = e^{-i\pi\alpha}(z_+-z_-)^{-\alpha}[1+\mathcal{O}(1/|\lambda|]$.

The coefficient of $(t-t_+)^2$ in the exponential in the last factor in Equation (21) has phase $e^{2i\vartheta_+}$, where

$$\vartheta_+ = \arg\lambda - \arg z_+ - \frac{1}{2}\arg\alpha \tag{22}$$

with $-\pi/2 \le \arg\lambda \le \pi/2$, $-\pi < \arg z_+ < \pi$, and $-\pi/2 < \arg\alpha < \pi/2$. With these ranges, the contour $\mathcal{C}_+$ can be distorted to run through the saddle point in the direction with $\arg(t-t_+) = \frac{\pi}{2}-\vartheta_+$. The exponent is then real and negative, and the integration proceeds in the direction of steepest descent away from the saddle.

The convergence of the integral away from the saddle point is rapid for $|\lambda^2/\alpha z_+^2| \gg 1$. Since the exact integrand remains small on $\mathcal{C}_+$ away from the saddle point, we can extend the integration on t to $\pm\infty$ without changing the integral significantly. The result of the remaining Gaussian integral is just a factor $i\sqrt{2\pi\alpha z_+^2/\lambda^2}$, where the factor $e^{-i\vartheta_+}$ from dt has been absorbed. Thus, taking $|\lambda|$ large,

$$D_\lambda^\alpha(z) = 2^{-\alpha}\left(e^\alpha\alpha^{-\alpha+1/2}/\sqrt{2\pi}\right)\lambda^{\alpha-1}(z^2-1)^{-\alpha/2}e^{i\pi\alpha}z_+^{-\lambda-\alpha}\left[1+\mathcal{O}(1/|\lambda|)\right]. \tag{23}$$

The factor in parentheses is just Stirling's approximation for $1/\Gamma(\alpha)$, a known factor, in $D_\lambda^\alpha(z)$, so

$$D_\lambda^\alpha(z) = \frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(z^2-1)^{-\alpha/2}e^{i\pi\alpha}z_+^{-\lambda-\alpha}\left[1+\mathcal{O}(1/|\lambda|)\right], \quad |\lambda|\to\infty, \tag{24}$$

$$\Re\lambda \ge 0, \quad \Re\alpha > 0, \quad -\pi/2 \le \arg\lambda \le \pi/2, \quad 0 \le \arg(z\pm 1) \le \pi,$$

in agreement with Equations (6.3) and (A5) in [2], but without the restriction on λ noted there. This result holds in the complex z plane cut from $z=1$ to $-\infty$.

In the case of $C_\lambda^\alpha(z)$, we must distinguish the cases $\Im z > 0$ and $\Im z < 0$. For $\Im z > 0$, the integral on the contour $\mathcal{C}_+$ reproduces the result for $D_\lambda^\alpha(z)$ in Equation (24). The integral on the $\mathcal{C}_-$ contour gives a similar result, with the replacement of z_+ by z_- and an extra factor $e^{-i\pi\alpha}$ from the phase of the factor $(t_--z_+)^{-\alpha} = e^{-i\pi\alpha}(z_+-z_-)^{-\alpha}\,(1+\mathcal{O}(1/\lambda))$ in the integrand.

For $\Im z < 0$, $(t_+ - z_-)^{-\alpha} \to e^{-2\pi i\alpha}(z_+ - z_-)^{-\alpha}$ for $|\lambda|$ large, and the factor $e^{i\pi\alpha}$ in Equation (24) from the $\mathcal{C}_+$ contour is replaced by $e^{-i\pi\alpha}$. The contribution from $\mathcal{C}_-$ is unchanged.

Combining the results for the $\mathcal{C}_+$ and $\mathcal{C}_-$ integrations, we find that

$$\mathrm{C}^\alpha_\lambda(z) = \frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(z^2-1)^{-\alpha/2}\left(e^{\pm i\pi\alpha}z_+^{-\lambda-\alpha} + z_-^{-\lambda-\alpha}\right)[1+\mathcal{O}(1/|\lambda|)] \tag{25}$$

$$= \frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(z^2-1)^{-\alpha/2}\left(e^{\pm i\pi\alpha}z_+^{-\lambda-\alpha} + z_+^{\lambda+\alpha}\right)[1+\mathcal{O}(1/|\lambda|)], \quad \Im z \gtrless 0, \tag{26}$$

$$|\lambda| \to \infty, \quad \Re\lambda \geq 0, \quad \Re\alpha > 0, \quad -\pi/2 \leq \arg\lambda \leq \pi/2, \quad 0 \leq \arg(z\pm 1) \leq \pi.$$

This agrees with Eq. (A8) in [2] and with Watson's result for $C^\alpha_\lambda(z)$, ([1], Section 2.3.2 (17)).

The earlier results for $C^\alpha_\lambda(x)$ and $D^\alpha_\lambda(x)$ were derived for $|\lambda| \to \infty$ along rays in the right-half t plane with $|\Im\lambda| \to \infty$, $0 < |\arg\lambda| < \pi/2$. The restrictions are not necessary, and the results continue to hold for $\arg\lambda = 0$ and $|\arg\lambda| = \pi/2$.

The result for $\mathrm{C}^\alpha_\lambda(z)$ must be interpreted with care. Since $|z_+| > 1$ and $|z_-| = 1/|z_+| < 1$ for $z \notin (-1,1)$, one of the two terms in Equation (26) usually becomes exponentially small relative to the other for $|\lambda| \to \infty$ and should be dropped relative to the uncertainties of $O(1/|\lambda|)$ in the dominant term. Thus, for $\Re\lambda \to \infty$ with $\Im\lambda$ fixed, $\Re z > 1$, and $\Im z \to 0$ from either above or below, the first, discontinuous, term in Equation (26) becomes exponentially small and should be dropped relative to the second. In fact, for $|\Im z| \to 0$, the saddle point at z_+ lies inside the contour for the z_- integral, is inaccessible and does not contribute to the final result. The asymptotic estimate for $\mathrm{C}^\alpha_\lambda(z)$ is therefore continuous across the real axis as it should be.

This completes the derivation of the results in Theorem 3.

Derivation of Theorem 2:

The functions $\mathrm{D}^\alpha_\lambda(x)$ and $\mathrm{C}^\alpha_\lambda(x)$ "on the cut", $x \in (-1,1)$, can be defined in terms of $D^\alpha_\lambda(z)$ for the z complex by [6]

$$\mathrm{D}^\alpha_\lambda(x) = -ie^{-i\pi\alpha}\left(e^{i\pi\alpha}D^\alpha_\lambda(x+i0) - e^{-i\pi\alpha}D^\alpha_\lambda(x-i0)\right), \tag{27}$$

$$\mathrm{C}^\alpha_\lambda(x) = e^{-i\pi\alpha}\left(e^{i\pi\alpha}D^\alpha_\lambda(x+i0) + e^{-\pi i\alpha}D^\alpha_\lambda(x-i0)\right) \tag{28}$$

$$= \mathrm{C}^\alpha_\lambda(x \pm i0). \tag{29}$$

For $z \in (-1,1)$, take $z = \cos\theta$, $0 < \theta < \pi$. Then $z_\pm = e^{\pm i\theta}$ and $\sqrt{z^2-1} = e^{\pm i\pi/2}\sin\theta$ for $\Im z \gtrless 0$, and Equations (24) and (26) give

$$\mathrm{D}^\alpha_\lambda(\cos\theta) = i\frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(\sin\theta)^{-\alpha}\left(-e^{-i(\lambda+\alpha)\theta+i\pi\alpha/2} + e^{i(\lambda+\alpha)\theta-i\pi\alpha/2}\right)[1+\mathcal{O}(1/|\lambda|)]$$

$$= -\frac{2^{-\alpha+1}}{\Gamma(\alpha)}\lambda^{\alpha-1}(\sin\theta)^{-\alpha}\sin((\lambda+\alpha)\theta - \pi\alpha/2)[1+\mathcal{O}(1/|\lambda|)] \tag{30}$$

$$\mathrm{C}^\alpha_\lambda(\cos\theta) = \frac{2^{-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}(\sin\theta)^{-\alpha}\left(e^{i\pi\alpha/2}e^{-i(\lambda+\alpha)\theta} + e^{-i\pi\alpha/2}e^{i(\lambda+\alpha)\theta}\right)[1+\mathcal{O}(1/|\lambda|)]$$

$$= \frac{2^{-\alpha+1}}{\Gamma(\alpha)}\lambda^{\alpha-1}(\sin\theta)^{-\alpha}\cos((\lambda+\alpha)\theta - \pi\alpha/2)[1+\mathcal{O}(1/|\lambda|)]. \tag{31}$$

A question now is how large $|\lambda|$ must actually be for this behavior to hold. It follows from the expressions for $t_\pm$ in Equation (19) that the asymptotic limit for the saddle points that used in the calculations requires that $\sqrt{z^2-1} \gg \alpha' \approx \alpha/\lambda$. Furthermore, the points $t_\pm$ or $z_\pm$ must be separated widely enough that the integration over one saddle is not influenced by the presence of the second.

The convergence of the saddle point integrals is determined by the coefficient in the exponential in the integral in Equation (21). Convergence on the right scale requires that the distance between the points be much larger than the sum of the distances over which the saddle point integrations converge, given by the scale factors $\sqrt{|2\alpha z_\pm^2/\lambda^2|}$ in the Gaussian integrands. This gives the condition

$$|z_+ - z_-| = 2\sin\theta \gg \sqrt{|2\alpha z_+^2/\lambda^2|} + \sqrt{|2\alpha z_-^2/\lambda^2|} = 2\sqrt{|2\alpha/\lambda^2|} \tag{32}$$

for $z \in (-1, 1)$, so requires that

$$|\lambda| \gg \left|\sqrt{\alpha}/\sqrt{z^2-1}\right|. \tag{33}$$

This is the same as the condition used in the derivation of $t_\pm$ given above up to a factor $\sqrt{\alpha}$.

For fixed large $|\lambda|$, Equation (33) bounds $|z^2-1|$ away from 1. The saddle points $t_\pm$ merge for $\theta \to 0$ ($z \to 1$) and cannot be treated as independent in the steepest-descent calculations which lead to the results above. For $z \to 1$, the saddle points coalesce into a single saddle between $t = 0$ and $t = 1$, and an integration as in Equation (21) with $\alpha \to 2\alpha$ reproduces the correct asymptotic limit $C_\lambda^\alpha(1) \sim \lambda^{2\alpha-1}/\Gamma(2\alpha)$. For $z \to -1$, the points $z_\pm$ pinch the contour $\mathcal{C}$, and the result is singular, $C_\lambda^\alpha(z) \propto ((z+1)/2)^{-\alpha+1/2}$.

Derivation of Theorem 3:

To treat the limit $z \to 1$, we use a different technique developed in ([7], Sections IIA and IIB). We start with the standard hypergeometric expression for $C_\lambda^\alpha(z)$ in Equation (1) written in a more useful form,

$$C_\lambda^\alpha(z) = \frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)\Gamma(2\alpha)}\left(\frac{1+z}{2}\right)^{-\alpha+\frac{1}{2}} {}_2F_1\left(\lambda+\alpha+\frac{1}{2}, -\lambda-\alpha+\frac{1}{2}; \alpha+\frac{1}{2}; \frac{1-z}{2}\right). \tag{34}$$

We next introduce the Barnes-type representation ([1], Section 2.3.3 (15)) for the type of hypergeometric function that appears in Equation (34) and will be encountered again in Eq. (56),

$$\begin{aligned} {}_2F_1\left(b+\frac{1}{2}, -b+\frac{1}{2}; \nu+1; \frac{u}{2}\right) = \Gamma(\nu+1)\,\frac{1}{2\pi i}\int_{\mathcal{C}_B} ds \frac{\Gamma(b+\frac{1}{2}+s)\Gamma(-b+\frac{1}{2}+s)}{\Gamma(b+\frac{1}{2})\Gamma(-b+\frac{1}{2})} \\ \times \frac{\Gamma(-s)}{\Gamma(\nu+1+s)}\left(-\frac{u}{2}\right)^s. \end{aligned} \tag{35}$$

The contour $\mathcal{C}_B$ in the Barnes' representation initially runs from $-i\infty$ to $+i\infty$ in the s plane, staying to the right of the poles of the factors $\Gamma(b+\frac{1}{2}+s)$ and $\Gamma(-b+\frac{1}{2}+s)$ in the integrand, and to the left of the poles of $\Gamma(-s)$, but it can be deformed to run around the positive real axis, $\mathcal{C}_B = (\infty, 0-, \infty)$ with the same restrictions.

Expanding the ratios of b-dependent gamma functions in the first line of Equation (35) in inverse powers of b, assumed large, using Stirling's approximation for the gamma function, and writing the powers of s that appear in terms of combinations of the form $1 \cdot s(s-1)\cdots(s-k)$ gives a series

$$\begin{aligned} {}_2F_1\left(b+\frac{1}{2}, -b+\frac{1}{2}; \nu+1; \frac{u}{2}\right) &= \Gamma(\nu+1)\frac{1}{2\pi i}\int_{\mathcal{C}_B} ds \frac{\Gamma(-s)}{\Gamma(\nu+1+s)}\left(\frac{U^2}{4}\right)^s \\ &\times \left\{1 - \frac{1}{b^2}\left(\frac{s}{4} + s(s-1) + \frac{1}{3}s(s-1)(s-2)\right)\right. \\ &\left.+\mathcal{O}\left(\frac{1}{b^4}\right)\right\} \end{aligned} \tag{36}$$

where $U = \sqrt{2b^2u}$.

With the choice of the deformed contour $\mathcal{C}_B$ above, the integrals that remain in Equation (36), are expressible in terms of Bessel functions through a Barnes' representation for the latter which uses the same contour $\mathcal{C}_B$,

$$\left(\frac{U}{2}\right)^{-\nu} J_\nu(U) = \frac{1}{2\pi i}\int_{\mathcal{C}_B} ds \frac{\Gamma(-s)}{\Gamma(\nu+s+1)}\left(\frac{U^2}{4}\right)^s. \tag{37}$$

The first term in the series in Equation (36) gives $(U/2)^{-\nu}J_\nu(U)$. After combining the factors $s(s-1)\cdots(s-k)$, $k = 0, 1, \cdots$, with $\Gamma(-s)$ to get $(-1)^{k+1}\Gamma(-s+k+1)$, we can shift the contour of integration to the right to run just to the left of the pole at $s = k+1$. The replacement of s by $s' = s-k-1$ then gives

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\mathcal{C}_B} ds\, s(s-1)\cdots(s-k)\frac{\Gamma(-s)}{\Gamma(s+\nu+1)}\left(\frac{U}{2}\right)^s \\ &= \frac{(-1)^{k+1}}{2\pi i}\int_{\mathcal{C}_B'} ds' \frac{\Gamma(-s')}{\Gamma(s'+\nu+k+2)}\left(\frac{U^2}{4}\right)^{s'+k+1} = (-1)^{k+1}\left(\frac{U}{2}\right)^{-\nu+k+1} J_{\nu+k+1}(U) \end{aligned} \tag{38}$$

for the following terms, with $k = 0, 1, \cdots$.

The use of Stirling's approximation, itself only an asymptotic expansion, is not justified on the entire integration contour, and the result from Equation (36) gives only an asymptotic series for the hypergeometric function,

$$\begin{aligned} {}_2F_1\left(b+\frac{1}{2}, -b+\frac{1}{2}; \nu+1; \frac{u}{2}\right) &= \Gamma(\nu+1)\left(\frac{U}{2}\right)^{-\nu}\left\{J_\nu(U) + \frac{1}{b^2}\left[\frac{1}{4}\frac{U}{2}J_{\nu+1}(U)\right.\right. \\ &\left.\left. - \left(\frac{U}{2}\right)^2 J_{\nu+2}(U) + \frac{1}{3}\left(\frac{U}{2}\right)^3 J_{\nu+3}(U)\right] + \mathcal{O}\left(\frac{1}{b^4}\right)\right\}. \end{aligned} \tag{39}$$

The use of this expression in Equation (36) with $b = \lambda + \alpha$ and $\nu = \alpha - \frac{1}{2}$ gives an asymptotic series for $C_\lambda^\alpha(z)$ in powers of $1/(\lambda+\alpha)^2$. With $Z = \sqrt{2(\lambda+\alpha)^2(1-z)}$,

$$\begin{aligned} C_\lambda^\alpha(z) &= \frac{\Gamma(\lambda+2\alpha)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\lambda+1)\Gamma(2\alpha)}\left(\frac{1+z}{2}\right)^{-\alpha+\frac{1}{2}}\left(\frac{Z}{2}\right)^{-\alpha+\frac{1}{2}}\left\{J_{\alpha-\frac{1}{2}}(Z)\right. \\ &\left.+\frac{1}{(\lambda+\alpha)^2}\left[\frac{1}{4}\frac{Z}{2}J_{\alpha+\frac{1}{2}}(Z) - \left(\frac{Z}{2}\right)^2 J_{\alpha+\frac{3}{2}}(Z) + \frac{1}{3}\left(\frac{Z}{2}\right)^3 J_{\alpha+\frac{5}{2}}(Z)\right] + \mathcal{O}\left(\frac{1}{(\lambda+\alpha)^4}\right)\right\} \end{aligned} \tag{40}$$

for $\lambda + \alpha$ large and Z fixed. This series is useful more generally for $|\lambda+\alpha| \gg 1$ and $|1-z| \ll 1$.

We can obtain a closely-related series using the same technique starting with Equation (34) and expanding in terms of the parameter $\lambda(\lambda+2\alpha) = (\lambda+\alpha)^2 - \frac{1}{4}$. This approach was used in

([7], Section IIA) in our treatment of Bessel-function expansions for the associated Legendre functions $P_j^{-\mu}(z)$. The result is

$$C_\lambda^\alpha(z) = \frac{\Gamma(\lambda+2\alpha)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\lambda+1)\Gamma(2\alpha)}\left(\frac{Y}{2}\right)^{-\alpha+\frac{1}{2}}\left\{J_{\alpha-\frac{1}{2}}(Y)\right.$$
$$+\frac{1}{\lambda(\lambda+2\alpha)}\left[-\frac{2\alpha+1}{2}\left(\frac{Y}{2}\right)^2 J_{\alpha+\frac{3}{2}}(Y)+\frac{1}{3}\left(\frac{Y}{2}\right)^3 J_{\alpha+\frac{5}{2}}(Y)\right]$$
$$\left.+\mathcal{O}\left(\frac{1}{(\lambda(\lambda+2\alpha)^2}\right)\right\} \tag{41}$$

where $Y = \sqrt{2\lambda(\lambda+2\alpha)(1-z)}$.

The series in Equation (40), here obtained directly, is equivalent to that obtained by expanding the powers of $\lambda(\lambda+2\alpha)$ in the coefficients and the argument of the Bessel functions in Equation (41) in terms of the simpler variable $(\lambda+\alpha)^2$, and the prefactor $((1+z)/2)^{-\alpha+\frac{1}{2}}$ in powers of $(1-z)$. The difference in the leading terms is unimportant for $|\lambda| \gg 1$ and $|1-z| \ll 1$.

To connect this result to the asymptotic expression for $C_\lambda^\alpha(z)$ in Equation (31) for $|\lambda| \gg 1$, we consider the case in which only the leading term in the asymptotic series in Equation (40) is important. The result in Equation (31) is valid for $\sqrt{|1-z|} \gg 1/|\lambda|$, which requires that $Z \gg 1$. Despite the appearance of powers of Z in the correction terms, this is allowed provided that the corrections to the leading term are small. The Bessel functions are all of the same general magnitude for Z large, so the term in Z^3 in the second term in the series is dominant for $Z \gg 1$, and the condition for the $|1/(\lambda+\alpha)^2|$ correction to the leading term to be small is

$$\left|\frac{1}{(\lambda+\alpha)^2}\left(\frac{Z}{2}\right)^3\right| = |\lambda+\alpha|\left|\frac{1-z}{2}\right|^{3/2} \ll 1. \tag{42}$$

Under this condition, the following terms in the series in Equation (40) are also initially small.

For $z = \cos\theta$ on $-1 < z < 1$, this requires that θ be small, with $\theta \ll 2/(\lambda+\alpha)^{\frac{1}{3}}$. In the limit of large Z, Hankel's expansion for the Bessel functions ([5], Section 10.17(i)) gives

$$J_\nu(Z) = \sqrt{\frac{2}{\pi Z}}\left\{\cos\left(Z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right)\left[1+\mathcal{O}\left(\frac{1}{Z^2}\right)\right]\right.$$
$$\left.-\sin\left(Z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right)\left[\frac{4\nu^2-1}{8Z}+\mathcal{O}\left(\frac{1}{Z^3}\right)\right]\right\}. \tag{43}$$

The leading term in Equation (40) therefore has the asymptotic limit

$$C_\lambda^\alpha(z) \sim \frac{\Gamma(\lambda+2\alpha)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\lambda+1)\Gamma(2\alpha)}\left(\frac{1+z}{2}\right)^{-\alpha+\frac{1}{2}}\left(\frac{Z}{2}\right)^{-\alpha+\frac{1}{2}}\sqrt{\frac{2}{\pi Z}}\cos\left(Z-\frac{\pi\alpha}{2}\right). \tag{44}$$

The corrections are of relative order $1/Z \propto |\lambda|^{-\frac{2}{3}}$.

Expanding the leading ratio of gamma functions in terms of $\lambda+\alpha$, assumed large, and expressing the result in terms of $\theta \ll 1/|\lambda|^{1/3}$, we obtain

$$C_\lambda^\alpha(\cos\theta) = \left(\frac{\lambda+\alpha}{2}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}(\sin\theta)^{-\alpha}\cos\left((\lambda+\alpha)\theta-\frac{\pi\alpha}{2}\right)\left[1+\mathcal{O}\left(1/|\lambda|^{2/3}\right)\right]. \tag{45}$$

This is equivalent to the expression in Equation (31) for $|\lambda| \gg 1$ as required for the correction terms in Equation (40) to be negligible, so the two expressions connect smoothly in their overlapping region of validity, $1/|\lambda| \ll \theta \ll 1/|\lambda|^{1/3}$.

To obtain an asymptotic Bessel-function series for $D_\lambda^\alpha(z)$ for $z \approx 1$, we use the relation ([2], Section 3 (5))

$$D_\lambda^\alpha(z) = \frac{1}{2}e^{i\pi\alpha}\frac{1}{\cos\pi\alpha}\left\{C_\lambda^\alpha(z) - 2^{-2\alpha+1}(z^2-1)^{\alpha+\frac{1}{2}}\frac{\Gamma(-\alpha+1)\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+1)}C_{\lambda+2\alpha-1}^{-\alpha+1}(z)\right\} \tag{46}$$

to express $D_\lambda^\alpha(z)$ in terms of Gegenbauer functions of the first kind. The function $C_\lambda^\alpha(z)$ can be approximated using the series in Equation (40). The modified indices $\lambda' = +2\alpha - 1$ and $\alpha' = -\alpha + 1$ in the second Gegebauer function give $\lambda' + \alpha' = \lambda + \alpha$ so we may use the same series for this function, but with the index $\alpha - \frac{1}{2}$ on the Bessel functions and their coefficients replaced by $\alpha' - \frac{1}{2} = -\alpha + \frac{1}{2}$.

We begin with $z \in (1, \infty)$, where $Z \to iZ'$, $Z' = \sqrt{2(\lambda+\alpha)^2(z-1)}$ and $Z^{\mp\nu}J_{\pm\nu+n}(Z) \to (-1)^n(Z')^{\mp\nu}I_{\pm\nu+n}(Z')$, with $I_\mu(z)$ the modified or hyperbolic Bessel function of the first kind. This gives as the leading term

$$D_\lambda^\alpha(z) = \frac{\sqrt{\pi}}{2}e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\frac{1}{\cos\pi\alpha}\left\{\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)}2^{-2\alpha+1}\left(\frac{z+1}{2}\right)^{-\alpha+\frac{1}{2}}\left(\frac{Z'}{2}\right)^{-\alpha+\frac{1}{2}}I_{\alpha-\frac{1}{2}}(Z')\right.$$
$$\left. -(z^2-1)^{-\alpha+\frac{1}{2}}\left(\frac{z+1}{2}\right)^{\alpha-\frac{1}{2}}\left(\frac{Z'}{2}\right)^{\alpha-\frac{1}{2}}I_{-\alpha+\frac{1}{2}}(Z') + \mathcal{O}\left(\frac{1}{(\lambda+\alpha)^2}\right)\right\}. \tag{47}$$

The higher-order terms in the series are negligible for $|z-1| \ll |\lambda|^{-\frac{2}{3}}$ for $|\lambda| \to \infty$.

The function $I_{-\alpha+\frac{1}{2}}(Z')$ can be eliminated in terms of the Macdonald function $K_{\alpha-\frac{1}{2}}(Z')$ through the relation ([5], Section 10.27.4)

$$K_\nu(z) = \frac{\pi}{2}\frac{1}{\sin\pi\nu}\left[I_{-\nu}(z) - I_\nu(z)\right] \tag{48}$$

with $\nu = \alpha - \frac{1}{2}$. After making this substitution, extracting the coefficient of $K_\nu(z)$, and using the definition of Z', Equation (47) reduces to

$$D_\lambda^\alpha(z) \sim \frac{1}{\sqrt{\pi}}e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}2^{-\nu}(\lambda+\alpha)^\nu(z^2-1)^{-\nu/2}\left(\frac{z+1}{2}\right)^{\nu/2}$$
$$\times\left\{K_\nu(Z') + \frac{\pi}{2}\frac{1}{\sin\pi\nu}\left[1 - \frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)}(\lambda+\alpha)^{-2\alpha+1}\left(\frac{z+1}{2}\right)^{-\nu}\right]I_\nu(Z') + \cdots\right\}. \tag{49}$$

As $\Gamma(\lambda+2\alpha)/\Gamma(\lambda+1) = (\lambda+\alpha)^{2\alpha-1}[1+\mathcal{O}(1/(\lambda+\alpha)^2]$ while $\left(z+1)/2\right)^{-\nu} \sim 1 - \nu(z-1)/2 + \cdots$, the coefficient of $I_\nu(Z')$ in this expression vanishes up to terms of order $1/(\lambda+\alpha)^2$ and $(z-1)/2 \ll (\lambda+\alpha)^{-\frac{2}{3}}$ over its range of validity. The overall factor $\left((z+1)/2\right)^{-\nu/2}$ can also be dropped to leading order, and

$$D_\lambda^\alpha(z) \sim \frac{1}{\sqrt{\pi}}e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}2^{-\nu}(\lambda+\alpha)^\nu(z^2-1)^{-\nu/2}K_\nu(Z'), \quad \nu = \alpha - \frac{1}{2}. \tag{50}$$

We obtain the asymptotic forms of the Gegenbauer functions on the cut for $x = \cos\theta \approx 1$ using Equations (27) and (28) and the relations

$$K_\nu(e^{\pm i\pi/2}z) = \mp\frac{i\pi}{2}e^{\mp i\pi\nu/2}\left[J_\nu(z) \mp iY_\nu(z)\right]. \tag{51}$$

This gives

$$\mathrm{D}_\lambda^\alpha(x) \sim -\sqrt{\pi}\frac{1}{\Gamma(\alpha)}2^{-\nu}(\lambda+\alpha)^\nu(1-x^2)^{-\nu/2}Y_\nu(Z)+\cdots, \tag{52}$$

$$\mathrm{C}_\lambda^\alpha(x) \sim \sqrt{\pi}\frac{1}{\Gamma(\alpha)}2^{-\nu}(\lambda+\alpha)^\nu(1-x^2)^{-\nu/2}J_\nu(Z)+\cdots, \tag{53}$$

$\nu=\alpha-\frac{1}{2}$, where the uncertainties in these expressions in their range of validity are order $1/\lambda^{2/3}$ for $|\lambda|\to\infty$.

Using Hankel's expansions of $J_\nu(Z)$ and $Y_\nu(Z)$ for Z large ([5], Section 10.17(i)) and expressing the results in terms of θ, with $x=\cos\theta$, $Z=(\lambda+\alpha)\sqrt{2(1-\cos\theta}=(\lambda+\alpha)\theta[1-\theta^2/24+\cdots]$, and $\theta\ll|\lambda|^{-\frac{1}{3}}$, these relations give

$$\mathrm{D}_\lambda^\alpha(\cos\theta) \sim -\frac{1}{\Gamma(\alpha)}2^{-\alpha+1}(\lambda+\alpha)^{\alpha-1}(\sin\theta)^{-\alpha}\sin\left((\lambda+\alpha)\theta-\frac{\alpha\pi}{2}\right)+\cdots, \tag{54}$$

$$\mathrm{C}_\lambda^\alpha(\cos\theta) \sim \frac{1}{\Gamma(\alpha)}2^{-\alpha+1}(\lambda+\alpha)^{\alpha-1}(\sin\theta)^{-\alpha}\cos\left((\lambda+\alpha)\theta-\frac{\alpha\pi}{2}\right)+\cdots, \tag{55}$$

to leading order in $|\lambda|$, in agreement with the results in Equations (30) and (31) for $1/|\lambda|\ll\theta\ll 1/|\lambda|^{1/3}$.

Derivation of Theorem 4:

The standard hypergeometric representation of $D_\lambda^\alpha(z)$ in Equation (2) for $|z|$ large [2,6] can be converted using standard linear transformations ([5], Section 15.8) to a form useful for complex z near -1,

$$\begin{aligned} D_\lambda^\alpha(z) = {} & \frac{1}{\sqrt{\pi}}e^{i\pi\alpha}2^{-2\alpha}\frac{1}{\Gamma(\alpha)}e^{\mp i\pi(\lambda+2\alpha)} \\ & \times\left[\frac{\Gamma(\lambda+2\alpha)\Gamma(-\alpha+\frac{1}{2})}{\Gamma(\lambda+1)}\left(\frac{1-z}{2}\right)^{-\alpha+\frac{1}{2}}{}_2F_1\left(-\lambda-\alpha+\frac{1}{2},\lambda+\alpha+\frac{1}{2};\alpha+\frac{1}{2};\frac{1+z}{2}\right)\right. \\ & \left.+\,e^{\pm i\pi(\alpha-\frac{1}{2})}\Gamma\left(\alpha-\frac{1}{2}\right)\left(\frac{1+z}{2}\right)^{-\alpha+\frac{1}{2}}{}_2F_1\left(-\lambda-\alpha+\frac{1}{2},\lambda+\alpha+\frac{1}{2};-\alpha+\frac{3}{2};\frac{1+z}{2}\right)\right], \end{aligned} \tag{56}$$

where the + and - signs hold for z on the upper (lower) sides of the cut in $(z-1)^{\alpha-\frac{1}{2}}$.

Upon using the asymptotic Bessel-function approximation in Equation (39) for the hypergeometric functions in leading order and expanding the ratio of gamma functions in the first term for $|\lambda+\alpha|\gg 1$, this reduces in leading order to

$$\begin{aligned} D_\lambda^\alpha(z) \sim {} & e^{i\pi\alpha}2^{-\alpha}(\lambda+\alpha)^{\alpha-1}\frac{\sqrt{\pi}}{\sin\pi(\alpha-\frac{1}{2})}\frac{1}{\Gamma(\alpha)}\left(2(1+z)\right)^{-\alpha/2}\left(\frac{Z''}{2}\right)^{\frac{1}{2}} \\ & \times e^{\mp i\pi(\lambda+2\alpha)}\left\{-\left(\frac{1-z}{2}\right)^{-\alpha+\frac{1}{2}}J_{\alpha-\frac{1}{2}}(Z'')+e^{\pm i\pi(\alpha-\frac{1}{2})}J_{-\alpha+\frac{1}{2}}(Z'')\right\}, \end{aligned} \tag{57}$$

where $Z''=\sqrt{2(\lambda+\alpha)^2(1+z)}$. The factor $((1-z)/2)^{-\alpha+\frac{1}{2}}$ differs from 1 only by corrections of order $1/|\lambda|^{\frac{2}{3}}$ in the region in which the leading-order approximation is valid, so it can be replaced by 1 for $|\lambda|\gg 1$.

For $z=x\in(-1,1)$ real and close to -1, with $x=\cos\theta$, $\theta\approx\pi$, the relations in Equations (27), (28), and (57), give the asymptotic forms of the Gegenbauer functions $\mathrm{D}_\lambda^\alpha$ and $\mathrm{C}_\lambda^\alpha(x)$ "on the cut" for

$|\lambda| \gg 1$. Calculating the discontinuities specified in the first two equations and replacing $J_{-\alpha+\frac{1}{2}}(x)$ by the Bessel function of the second kind,

$$Y_{\alpha+\frac{1}{2}}(x) = \frac{1}{\sin \pi(\alpha - \frac{1}{2}} \left(J_{\alpha-\frac{1}{2}}(x) \cos \left(\pi(\alpha - \frac{1}{2} \right) - J_{-\alpha+\frac{1}{2}}(x) \right) \tag{58}$$

gives the relations in Theorem 4,

$$\mathsf{D}_\lambda^\alpha(x) \sim \frac{\sqrt{\pi}}{\Gamma(\alpha)} \left(\frac{\lambda+\alpha}{2} \right)^{\alpha-1} (2(1+x))^{-\alpha/2} \left(\frac{X''}{2} \right)^{\frac{1}{2}} \left[-\sin \pi\lambda J_{\alpha-\frac{1}{2}}(X'') + \cos \pi\lambda Y_{\alpha-\frac{1}{2}}(X'') \right], \tag{59}$$

$$\mathsf{C}_\lambda^\alpha(x) \sim \frac{\sqrt{\pi}}{\Gamma(\alpha)} \left(\frac{\lambda+\alpha}{2} \right)^{\alpha-1} (2(1+x))^{-\alpha/2} \left(\frac{X''}{2} \right)^{\frac{1}{2}} \left[\cos \pi\lambda J_{\alpha-\frac{1}{2}}(X'') + \sin \pi\lambda Y_{\alpha-\frac{1}{2}}(X'') \right], \tag{60}$$

with $X'' = \sqrt{2(\lambda+\alpha)^2(1+x)} \approx (\lambda+\alpha)(\pi-\theta)$.

For $1/|\lambda| \ll \pi - \theta \ll 1/|\lambda|$, the results in Equations (59) and (5) and in (60) and (6) are in their common ranges of validity and should agree. X'' is large in this region, and the agreement is easily shown using Hankel's asymptotic expressions for the Bessel functions ([5], Section 10.17(i)) and noting that $\sin\theta = \sin(\pi-\theta) \approx \pi - \theta$ in this region.

Remarks:

In their discussion of the asymptotics of the associated Legendre functions

$$P_\nu^{-\mu}(z) = \frac{2^\mu}{\sqrt{\pi}} \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \left(z^2 - 1 \right)^{\mu/2} C_{\nu-\mu}^{\mu+\frac{1}{2}}(z) \tag{61}$$

and

$$Q_\nu^{-\mu}(z) = 2^\mu \sqrt{\pi} e^{-2\pi i(\mu+\frac{1}{4})} \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \left(z^2 - 1 \right)^{\frac{\mu}{2}} D_{\nu-\mu}^{\mu+\frac{1}{2}}(z) \tag{62}$$

for $|\nu| \to \infty$, Cohl, Dang, and Dunster ([4], Sections 2.3.1 and 2.4.1) use uniform asymptotic expressions in terms of Bessel functions which hold quite generally ([5], Section14.15 (11–14)). These involve arguments $(\mu + \frac{1}{2})\theta$ in the Bessel functions and pre-factors proportional to $\sqrt{\theta/\sin\theta}$ for $z = \cos\theta$ or $\sqrt{\theta/\sinh\theta}$ for $z = \cosh\theta$. For example, the Ferrers functions $\mathsf{P}_\nu^{-\mu}(\cos\theta)$ and $\mathsf{Q}_\nu^{-\mu}(\cos\theta)$ have the asymptotic forms

$$\mathsf{P}_\nu^{-\mu}(\cos\theta) = \frac{1}{\nu^\mu} \left(\frac{\theta}{\sin\theta} \right)^{\frac{1}{2}} \left[J_\mu \left(\left(\nu + \frac{1}{2} \right) \theta \right) + \mathcal{O}\left(\frac{1}{\nu} \right) \mathrm{env} J_\mu \left(\left(\nu + \frac{1}{2} \right) \theta \right) \right], \tag{63}$$

$$\mathsf{Q}_\nu^{-\mu}(\cos\theta) = -\frac{\pi}{2\nu^\mu} \left(\frac{\theta}{\sin\theta} \right)^{\frac{1}{2}} \left[Y_\mu \left(\left(\nu + \frac{1}{2} \right) \theta \right) + \mathcal{O}\left(\frac{1}{\nu} \right) \mathrm{env} Y_\mu \left(\left(\nu + \frac{1}{2} \right) \theta \right) \right], \tag{64}$$

for $0 < \theta < \pi - \delta$ with δ fixed and $\nu \to \infty$. The envelope functions are treated in ([4], Section 2.3.1).

As may be seen through a comparison with Equations (52) and (53), the results of the two approaches agree for $\nu \gg 1$, with the simple approximations given here in Theorems 2–4 applying in sectors in $z = \cos\theta$ for $0 \le \theta \le \pi$, and the uniform results holding for for θ bounded away from π. The Bessel function expansions derived here also reproduce the first n powers of $(1-z)$ in the Legendre functions properly for $z \to 1$ when the Bessel functions through order $\mu + n$ are included.

The pre-factors and the variable in the uniform approximations are, unfortunately, awkward for physical applications to scattering theory, where, e.g., $\sqrt{2j(j+1)(1-\cos\theta)} = \sqrt{j(j+1)q^2/p^2} = qb$ rather than $(j+\frac{1}{2})\theta$ is the natural variable. Here $j = \nu$ is conserved angular momentum in the scattering, q is the invariant momentum transfer, p is the momentum of the particles in the center-of-mass system, and b the impact parameter or point of closest approach in the free Schrödinger equation. The pre-factors also disrupt the useful connection between partial-wave series in Legendre functions and Fourier–Bessel transforms in the theory of particle scattering; see, for example, ([8], Appendix B). These problems not encountered with the expansions derived here, Equations (40) and (49) for $\theta \approx 0$, and Equations 59) and (60) for $\theta \approx \pi$.

Cohl, Dang, and Dunster ([4], Sections 2.3.1 and 2.4.1) also treat the limits $\nu \to \infty$ and $\nu \to \pm i\infty$ for $z = \cosh\theta \in (1, \infty)$ using uniform expansions. Their results in terms of Bessel functions agree in form and error estimate with the simple asymptotic expressions in Theorems 1 and 2 for $\nu\theta \gg 1$, but also extend smoothly to $\theta = 0$, $z = 1$, the region treated separately in the Bessel function expansions derived here. They do not treat the more complicated cases of complex z and ν, to which the results of Theorems 1 and 2 results apply directly, again away from $z = \pm 1$.

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