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Identities Involving the Fourth-Order Linear Recurrence Sequence

Lan Qi [1] **and Zhuoyu Chen** [2,*]

[1] School of Mathematics and Statistics, Yulin University, Yulin 719000, China; qilan@yulinu.edu.cn
[2] School of Mathematics, Northwest University, Xi'an 710127, China
* Correspondence: chenzymath@stumail.nwu.edu.cn

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Abstract: In this paper, we introduce the fourth-order linear recurrence sequence and its generating function and obtain the exact coefficient expression of the power series expansion using elementary methods and symmetric properties of the summation processes. At the same time, we establish some relations involving Tetranacci numbers and give some interesting identities.

Keywords: four-order linear recurrence sequence; Tetranacci numbers; generating function; power series; identity

MSC: 11B83

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1. Introduction and Results

Let $n \geq 1$ be an integer, the Fibonacci polynomials $F_n(x)$ are defined by the second-order linear recurrence sequence

$$F_{n+1} = xF_n(x) + F_{n-1}(x),$$

with initial conditions $F_0(x) = 0$, $F_1 = 1$.

The generating function of the Fibonacci polynomials $F_n(x)$ is given by

$$\frac{1}{1 - xt - t^2} = \sum_{n=0}^{\infty} F_n(x)t^n.$$

In particular, for $x = 1, F_n = F_n(x)$ are the famous Fibonacci numbers. These polynomials and numbers play extremely vital roles in the mathematical theories and applications and a significant amount of research has been carried out to obtain a variety of meaningful results by several authors (see [1–15]). For example, Yuan Yi and Wenpeng Zhang (see [16]) researched the computational problem of the summation:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} F_{a_1}(x)F_{a_2}(x)\cdots F_{a_{h+1}}(x).$$

Yuankui Ma and Wenpeng Zhang (see [17]) acquired a different expression about the summation by introducing a new second order non-linear recursive sequence.

In [18], Taekyun Kim and others studied the properties of Fibonacci numbers by introducing the convolved Fibonacci numbers $p_n(x)$, which are given by the generating function

$$\left(\frac{1}{1 - t - t^2}\right)^x = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \quad (x \in \mathbf{R}).$$

The authors gave a new formula for calculating $p_n(x)$ by the elementary and combinatorial methods, and obtained some new and explicit identities of the convolved Fibonacci numbers, including the relationship between $p_n(x)$ and the combination sums about Fibonacci numbers.

In this paper, we consider the Tetranacci numbers H_n (see [19]), which are defined by the fourth-order linear recurrence relation

$$H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4}, \ \ n \geq 4,$$

with $H_0 = H_1 = 0, H_2 = H_3 = 1$.

It is obvious that $H_0 = 0,\ H_1 = 0,\ H_2 = 1,\ H_3 = 1,\ H_4 = 2,\ H_5 = 4,\ H_6 = 8,\ H_7 = 15, H_8 = 29,\ H_9 = 56,\ H_{10} = 108, \cdots\cdots$.

The Tetranacci numbers can be extended to negative index n arising from the rearranged recurrence relation

$$H_{n-4} = H_n - H_{n-1} - H_{n-2} - H_{n-3},$$

which yields the sequence of "nega-Tetranacci" numbers, $H_{-1} = 0,\ H_{-2} = 1,\ H_{-3} = -1,\ H_{-4} = 0,\ H_{-5} = 0,\ H_{-6} = 2,\ H_{-7} = -3,\ H_{-8} = 1,\ H_{-9} = 0, \cdots\cdots$.

The generating function of the Tetranacci sequences H_n is given by

$$\frac{1}{1 - t - t^2 - t^3 - t^4} = \sum_{n=0}^{\infty} H_{n+2} t^n.$$

Tetranacci numbers have important applications in combinatorial counting and graph theory, W. Marcellus E (see [20,21]) studied the arithmetical properties of H_n, Rusen Li (see [22]) obtained some convolution identities for H_n. Moreover, the summation calculation for different sequences is one of the hot topics in number theory, and many scholars have obtained a series of interesting results (see [23,24]). Therefore, it is very meaningful to further study the properties of the Tetranacci sequences. Inspired by the above references, for a real number $x \in \mathbf{R}$, we can define a new function $H_n(x)$, which is given by

$$\left(\frac{1}{1 - t - t^2 - t^3 - t^4}\right)^x = \sum_{n=0}^{\infty} H_n(x) t^n. \tag{1}$$

The main purpose of this paper is to study the relationship between $H_n(x)$ and H_n, and to prove some computational formulas of the fourth-order recurrence sequence by applying the elementary method and the symmetry properties of the summation processes. That is, we shall prove the following:

Theorem 1. *For a real number $x \in \mathbf{R}$ and any integer $n \geq 0$, we have*

$$\begin{aligned} H_n(x) \ &= \ \frac{1}{24} \sum_{a+b+c+d=n} (-1)^d \frac{x^{(a)}}{a!} \frac{x^{(b)}}{b!} \frac{x^{(c)}}{c!} \frac{x^{(d)}}{d!} S_{d-a} S_{d-b} S_{d-c} \\ &\quad - \frac{1}{24} \sum_{a+b+c+d=n} (-1)^d \frac{x^{(a)}}{a!} \frac{x^{(b)}}{b!} \frac{x^{(c)}}{c!} \frac{x^{(d)}}{d!} \left(S_{d-a} S_{2d-b-c}\right. \\ &\quad \left. + S_{d-b} S_{2d-a-c} + S_{d-c} S_{2d-a-b} - 2 S_{4d-n}\right), \end{aligned}$$

where

$$S_r = 4H_{r+2} - 3H_{r+1} - 2H_r - H_{r-1}\,, (r \in \mathbf{Z}),$$

$\sum\limits_{a+b+c+d=n}$ *denotes the summation over all four-dimensional nonnegative integer coordinates (a, b, c, d) such that $a + b + c + d = n$, and $x^{(0)} = 1$, $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$ for all positive integers n.*

According to this theorem, we can obtain the following corollaries:

Corollary 1. *For any integer $n > 0$, we have*

$$\begin{aligned} H_{n+2} &= \frac{1}{24} \sum_{a+b+c+d=n} (-1)^d S_{d-a} S_{d-b} S_{d-c} \\ &\quad -\frac{1}{24} \sum_{a+b+c+d=n} (-1)^d \left(S_{d-a} S_{2d-b-c} + S_{d-b} S_{2d-a-c} \right. \\ &\quad \left. + S_{d-c} S_{2d-a-b} - 2 S_{4d-n} \right). \end{aligned}$$

Corollary 2. *For any integer $k > 0$ and $n > 0$, we have*

$$\begin{aligned} & \sum_{a_1+a_2+\cdots+a_k=n} H_{a_1+2} H_{a_2+2} \cdots H_{a_k+2} \\ = & \frac{1}{24} \sum_{a+b+c+d=n} (-1)^d \frac{k^{(a)}}{a!} \frac{k^{(b)}}{b!} \frac{k^{(c)}}{c!} \frac{k^{(d)}}{d!} S_{d-a} S_{d-b} S_{d-c} \\ & -\frac{1}{24} \sum_{a+b+c+d=n} (-1)^d \frac{k^{(a)}}{a!} \frac{k^{(b)}}{b!} \frac{k^{(c)}}{c!} \frac{k^{(d)}}{d!} \left(S_{d-a} S_{2d-b-c} \right. \\ & \left. + S_{d-b} S_{2d-a-c} + S_{d-c} S_{2d-a-b} - 2 S_{4d-n} \right). \end{aligned}$$

Corollary 3. *For any integer $n > 0$, we have*

$$\begin{aligned} H_n\left(-\frac{1}{2}\right) &= \frac{1}{24 \cdot 2^n} \sum_{a+b+c+d=n} (-1)^d \frac{(2a-3)!!}{a!} \frac{(2b-3)!!}{b!} \frac{(2c-3)!!}{c!} \frac{(2d-3)!!}{d!} \\ &\quad \cdot \left(S_{d-a} S_{2d-b-c} + S_{d-b} S_{2d-a-c} + S_{d-c} S_{2d-a-b} - 2 S_{4d-n} \right) \\ &\quad -\frac{1}{24 \cdot 2^n} \sum_{a+b+c+d=n} (-1)^d \frac{(2a-3)!!}{a!} \frac{(2b-3)!!}{b!} \frac{(2c-3)!!}{c!} \frac{(2d-3)!!}{d!} \\ &\quad \cdot S_{d-a} S_{d-b} S_{d-c}, \end{aligned}$$

where the double factorial is defined as $n!! = 2 \times 4 \times 6 \times \cdots \times n$ for even positive integers and $n!! = 1 \times 3 \times 5 \times \cdots \times n$ for odd positive integers.

Corollary 4. *For any integer $n > 0$, we have*

$$\begin{aligned} H_n\left(\frac{1}{2}\right) &= \frac{1}{24 \cdot 4^n} \sum_{a+b+c+d=n} (-1)^d \frac{(2a)!}{(a!)^2} \frac{(2b)!}{(b!)^2} \frac{(2c)!}{(c!)^2} \frac{(2d)!}{(d!)^2} \\ &\quad \cdot S_{d-a} S_{d-b} S_{d-c} - \frac{1}{24 \cdot 4^n} \sum_{a+b+c+d=n} (-1)^d \frac{(2a)!}{(a!)^2} \frac{(2b)!}{(b!)^2} \\ &\quad \cdot \frac{(2c)!}{(c!)^2} \frac{(2d)!}{(d!)^2} \left(S_{d-a} S_{2d-b-c} + S_{d-b} S_{2d-a-c} \right. \\ &\quad \left. + S_{d-c} S_{2d-a-b} - 2 S_{4d-n} \right). \end{aligned}$$

2. Several Simple Lemmas

To complete the proof of the theorem, we need the following two simple lemmas, which are essential to prove our main results.

Lemma 1. *For any integer $r \in \mathbf{Z}$, we have*

$$S_r = t_1^r + t_2^r + t_3^r + t_4^r = 4H_{r+2} - 3H_{r+1} - 2H_r - H_{r-1},$$

where t_1, t_2, t_3 and t_4 are the four roots of the equation $t^4 - t^3 - t^2 - t - 1 = 0$.

Proof. It is obvious that H_n can be expressed the formula

$$H_n = c_1 t_1^n + c_2 t_2^n + c_3 t_3^n + c_4 t_4^n. \tag{2}$$

Since $H_0 = H_1 = 0, H_2 = H_3 = 1$, so we can get the system of equations

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0, \\ c_1 t_1 + c_2 t_2 + c_3 t_3 + c_4 t_4 = 0, \\ c_1 t_1^2 + c_2 t_2^2 + c_3 t_3^2 + c_4 t_4^2 = 1, \\ c_1 t_1^3 + c_3 t_2^3 + c_3 t_3^3 + c_4 t_4^3 = 1. \end{cases} \tag{3}$$

On the other hand, we observe that $t_1 + t_2 + t_3 + t_4 = 1$, $t_2 t_3 + t_2 t_4 + t_3 t_4 = -1 - (t_1 t_2 + t_1 t_3 + t_1 t_4) = t_1^2 - t_1 - 1$, $t_1 t_2 t_3 t_4 = -1$ and $1 = t_1^4 - t_1^3 - t_1^2 - t_1$.

It is clear that the Equation (3) implies

$$\begin{cases} c_1 = \dfrac{t_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)} = \dfrac{1}{-t_1^3 + 5t_1^2 - 2t_1 - 1}, \\ c_2 = \dfrac{t_2}{(t_2 - t_1)(t_2 - t_3)(t_2 - t_4)} = \dfrac{1}{-t_2^3 + 5t_2^2 - 2t_2 - 1}, \\ c_3 = \dfrac{t_3}{(t_3 - t_1)(t_3 - t_2)(t_3 - t_4)} = \dfrac{1}{-t_3^3 + 5t_3^2 - 2t_3 - 1}, \\ c_4 = \dfrac{t_4}{(t_4 - t_1)(t_4 - t_2)(t_4 - t_3)} = \dfrac{1}{-t_4^3 + 5t_4^2 - 2t_4 - 1}. \end{cases} \tag{4}$$

Then note that (4) can also be written as

$$\begin{cases} c_1 \left(-t_1^3 + 5t_1^2 - 2t_1 - 1\right) = -c_1 t_1^3 + 5c_1 t_1^2 - 2c_1 t_1 - c_1 = 1, \\ c_2 \left(-t_2^3 + 5t_2^2 - 2t_2 - 1\right) = -c_2 t_2^3 + 5c_2 t_2^2 - 2c_2 t_2 - c_2 = 1, \\ c_3 \left(-t_3^3 + 5t_3^2 - 2t_3 - 1\right) = -c_3 t_3^3 + 5c_3 t_3^2 - 2c_3 t_3 - c_3 = 1, \\ c_4 \left(-t_4^3 + 5t_4^2 - 2t_4 - 1\right) = -c_4 t_4^3 + 5c_4 t_4^2 - 2c_4 t_4 - c_4 = 1. \end{cases}$$

Thus, we have

$$\begin{cases} t_1^r = c_1 \left(-t_1^3 + 5t_1^2 - 2t_1 - 1\right) t_1^r = -c_1 t_1^{r+3} + 5c_1 t_1^{r+2} - 2c_1 t_1^{r+1} - c_1 t_1^r, \\ t_2^r = c_2 \left(-t_2^3 + 5t_2^2 - 2t_2 - 1\right) t_2^r = -c_2 t_2^{r+3} + 5c_2 t_2^{r+2} - 2c_2 t_2^{r+1} - c_2 t_2^r, \\ t_3^r = c_3 \left(-t_3^3 + 5t_3^2 - 2t_3 - 1\right) t_3^r = -c_3 t_3^{r+3} + 5c_3 t_3^{r+2} - 2c_3 t_3^{r+1} - c_3 t_3^r, \\ t_4^r = c_4 \left(-t_4^3 + 5t_4^2 - 2t_4 - 1\right) t_4^r = -c_4 t_4^{r+3} + 5c_4 t_4^{r+2} - 2c_4 t_4^{r+1} - c_4 t_4^r. \end{cases} \tag{5}$$

Hence, by (2) and (5), we immediately obtain

$$\begin{aligned} S_r &= t_1^r + t_2^r + t_3^r + t_4^r \\ &= -\left(c_1 t_1^{r+3} + c_2 t_2^{r+3} + c_3 t_3^{r+3} + c_4 t_4^{r+3}\right) \\ &\quad +5\left(c_1 t_1^{r+2} + c_2 t_2^{r+2} + c_3 t_3^{r+2} + c_4 t_4^{r+2}\right) \\ &\quad -2\left(c_1 t_1^{r+1} + c_2 t_2^{r+1} + c_3 t_3^{r+1} + c_4 t_4^{r+1}\right) \\ &\quad -(c_1 t_1^r + c_2 t_2^r + c_3 t_3^r + c_4 t_4^r) \\ &= -H_{r+3} + 5H_{r+2} - 2H_{r+1} - H_r \\ &= 4H_{r+2} - 3H_{r+1} - 2H_r - H_{r-1}. \end{aligned}$$

Now we have completed the proof of Lemma 1. $\square$

Lemma 2. *For a real number $x \in \mathbf{R}$ and any integer $n \geq 0$, we have*

$$\begin{aligned}
&\sum_{a+b+c+d=n} \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\frac{1}{t_1^a t_2^b t_3^c t_4^d} \\
= \; & \frac{1}{24}\sum_{a+b+c+d=n}(-1)^d \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!} S_{d-a}S_{d-b}S_{d-c} \\
& -\frac{1}{24}\sum_{a+b+c+d=n}(-1)^d \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\left(S_{d-a}S_{2d-b-c}\right. \\
& \left. +S_{d-b}S_{2d-a-c} + S_{d-c}S_{2d-a-b} - 2S_{4d-n}\right).
\end{aligned}$$

Proof. For any non-negative integers a, b, c and d, we have

$$\begin{aligned}
& \left(t_1^{d-a}+t_2^{d-a}+t_3^{d-a}+t_4^{d-a}\right)\left(t_1^{d-b}+t_2^{d-b}+t_3^{d-b}+t_4^{d-b}\right)\left(t_1^{d-c}+t_2^{d-c}+t_3^{d-c}+t_4^{d-c}\right) \\
= & \left(t_1^{d-a}+t_2^{d-a}+t_3^{d-a}+t_4^{d-a}\right)\left(t_1^{2d-b-c}+t_2^{2d-b-c}+t_3^{2d-b-c}+t_4^{2d-b-c}\right) \\
& + \left(t_1^{d-b}+t_2^{d-b}+t_3^{d-b}+t_4^{d-b}\right)\left(t_1^{2d-a-c}+t_2^{2d-a-c}+t_3^{2d-a-c}+t_4^{2d-a-c}\right) \\
& + \left(t_1^{d-c}+t_2^{d-c}+t_3^{d-c}+t_4^{d-c}\right)\left(t_1^{2d-a-b}+t_2^{2d-a-b}+t_3^{2d-a-b}+t_4^{2d-a-b}\right) \\
& - 2\left(t_1^{3d-a-b-c}+t_2^{3d-a-b-c}+t_3^{3d-a-b-c}+t_4^{3d-a-b-c}\right) \\
& + t_1^{d-a}t_2^{d-b}t_3^{d-c} + t_1^{d-a}t_2^{d-c}t_4^{d-b} + t_1^{d-a}t_3^{d-b}t_4^{d-c} + \cdots + t_2^{d-c}t_3^{d-a}t_4^{d-b} \\
= & S_{d-a}S_{2d-b-c} + S_{d-b}S_{2d-a-c} + S_{d-c}S_{2d-a-b} - 2S_{3d-a-b-c} + \sum_{\{i,j,k,m\}} \frac{(-1)^d}{t_1^i t_2^j t_3^k t_4^m},
\end{aligned}$$

where $\{i, j, k, m\}$ go through permutations of $\{a, b, c, d\}$. $\square$

Observe that the non-negative integers coordinates (a, b, c, d) with $a + b + c + d = n$ is symmetrical, then we can obtain

$$\begin{aligned}
& \sum_{a+b+c+d=n} \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\left(t_1^{d-a}+t_2^{d-a}+t_3^{d-a}+t_4^{d-a}\right) \\
& \cdot\left(t_1^{d-b}+t_2^{d-b}+t_3^{d-b}+t_4^{d-b}\right)\left(t_1^{d-c}+t_2^{d-c}+t_3^{d-c}+t_4^{d-c}\right) \\
= \; & \sum_{a+b+c+d=n} \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\left(S_{d-a}S_{2d-b-c}\right. \\
& \left. +S_{d-b}S_{2d-a-c} + S_{d-c}S_{2d-a-b} - 2S_{3d-a-b-c}\right) \\
& +24\sum_{a+b+c+d=n} \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\frac{(-1)^d}{t_1^a t_2^b t_3^c t_4^d}.
\end{aligned} \tag{6}$$

On the other hand, we have

$$\begin{aligned}
& \sum_{a+b+c+d=n} \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\left(t_1^{d-a}+t_2^{d-a}+t_3^{d-a}+t_4^{d-a}\right) \\
& \cdot\left(t_1^{d-b}+t_2^{d-b}+t_3^{d-b}+t_4^{d-b}\right)\left(t_1^{d-c}+t_2^{d-c}+t_3^{d-c}+t_4^{d-c}\right) \\
= & \sum_{a+b+c+d=n} \frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!} S_{d-a}S_{d-b}S_{d-c}. \qquad (7)
\end{aligned}$$

Then, applying (6) and (7), we obtain Lemma 2.

3. Proofs of the Main Results

In this section, we will prove our theorem and corollaries.

For any real number $x \in \mathbf{R}$, applying the properties of power series, we have

$$\frac{1}{(1-t)^x} = \sum_{n=0}^{\infty}\binom{-x}{n}(-1)^n t^n = \sum_{n=0}^{\infty}\frac{x^{(n)}}{n!}t^n, (|t|<1),$$

we note that t_1, t_2, t_3 and t_4 satisfy $t_1t_2t_3t_4 = -1$, so

$$\begin{aligned}
& \sum_{n=0}^{\infty} H_n(x)t^n = \left(\frac{1}{1-t-t^2-t^3-t^4}\right)^x \\
= & \frac{(-1)^x}{(t-t_1)^x(t-t_2)^x(t-t_3)^x(t-t_4)^x} \\
= & \frac{(t_1t_2t_3t_4)^x}{(t_1-t)^x(t_2-t)^x(t_3-t)^x(t_4-t)^x} \\
= & \frac{1}{\left(1-\frac{t}{t_1}\right)^x\left(1-\frac{t}{t_2}\right)^x\left(1-\frac{t}{t_3}\right)^x\left(1-\frac{t}{t_4}\right)^x} \\
= & \left(\sum_{n=0}^{\infty}\frac{x^{(n)}}{n!}\frac{t^n}{t_1^n}\right)\left(\sum_{n=0}^{\infty}\frac{x^{(n)}}{n!}\frac{t^n}{t_2^n}\right)\left(\sum_{n=0}^{\infty}\frac{x^{(n)}}{n!}\frac{t^n}{t_3^n}\right)\left(\sum_{n=0}^{\infty}\frac{x^{(n)}}{n!}\frac{t^n}{t_4^n}\right) \\
= & \sum_{n=0}^{\infty}\left(\sum_{a+b+c+d=n}\frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\frac{1}{t_1^a t_2^b t_3^c t_4^d}\right)t^n. \qquad (8)
\end{aligned}$$

Then combining (8) and Lemma 1 and 2, we can obtain

$$\begin{aligned}
& H_n(x) = \sum_{a+b+c+d=n}\frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\frac{1}{t_1^a t_2^b t_3^c t_4^d} \\
= & \frac{1}{24}\sum_{a+b+c+d=n}(-1)^d\frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}S_{d-a}S_{d-b}S_{d-c} \\
& -\frac{1}{24}\sum_{a+b+c+d=n}(-1)^d\frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!} \\
& \cdot\left(S_{d-a}S_{2d-b-c}+S_{d-b}S_{2d-a-c}+S_{d-c}S_{2d-a-b}-2S_{4d-n}\right) \\
= & \frac{1}{24}\sum_{a+b+c+d=n}(-1)^d\frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}\frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\left(4H_{d-a+2}-3H_{d-a+1}\right. \\
& \left.-2H_{d-a}-H_{d-a-1}\right)\left(4H_{d-b+2}-3H_{d-b+1}-2H_{d-b}-H_{d-b-1}\right) \\
& \cdot\left(4H_{d-c+2}-3H_{d-c+1}-2H_{d-c}-H_{d-c-1}\right)-\frac{1}{24}\sum_{a+b+c+d=n}(-1)^d\frac{x^{(a)}}{a!}\frac{x^{(b)}}{b!}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{x^{(c)}}{c!}\frac{x^{(d)}}{d!}\left[\left(4H_{d-a+2}-3H_{d-a+1}-2H_{d-a}-H_{d-a-1}\right)\right.\\
&\cdot\left(4H_{2d-b-c+2}-3H_{2d-b-c+1}-2H_{2d-b-c}-H_{2d-b-c-1}\right)+\left(4H_{d-b+2}-3H_{d-b+1}\right.\\
&\left.-2H_{d-b}-H_{d-b-1}\right)\left(4H_{2d-a-c+2}-3H_{2d-a-c+1}-2H_{2d-a-c}-H_{2d-a-c-1}\right)\\
&+\left(4H_{d-c+2}-3H_{d-c+1}-2H_{d-c}-H_{d-c-1}\right)\left(4H_{2d-a-b+2}-3H_{2d-a-b+1}\right.\\
&\left.\left.-2H_{2d-a-b}-H_{2d-a-b-1}\right)-2\left(4H_{4d-n+2}-3H_{4d-n+1}-2H_{4d-n}-H_{4d-n-1}\right)\right].
\end{aligned}
$$

This completes the proof of Theorem 1.

Since $H_n(1)=H_{n+2}$ and $1^{(n)}=n!$, according to Theorem 1, we can easily obtain Corollary 1.

If we take $x=k\in\mathbf{N}$ in (1), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}H_n(k)t^n=\left(\frac{1}{1-t-t^2-t^3-t^4}\right)^k=\left(\sum_{n=0}^{\infty}H_{n+2}t^n\right)^k\\
=\;&\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_k=0}^{\infty}H_{a_1+2}H_{a_2+2}\cdots H_{a_k+2}t^{a_1+a_2\cdots+a_k}\\
=\;&\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}H_{a_1+2}H_{a_2+2}\cdots H_{a_k+2}\right)t^n,
\end{aligned}
$$

and then by Theorem 1, we can obtain

$$
\begin{aligned}
&\sum_{a_1+a_2+\cdots+a_k=n}H_{a_1+2}H_{a_2+2}\cdots H_{a_k+2}\\
=\;&\frac{1}{24}\sum_{a+b+c+d=n}(-1)^d\frac{k^{(a)}}{a!}\frac{k^{(b)}}{b!}\frac{k^{(c)}}{c!}\frac{k^{(d)}}{d!}S_{d-a}S_{d-b}S_{d-c}\\
&-\frac{1}{24}\sum_{a+b+c+d=n}(-1)^d\frac{k^{(a)}}{a!}\frac{k^{(b)}}{b!}\frac{k^{(c)}}{c!}\frac{k^{(d)}}{d!}\left(S_{d-a}S_{2d-b-c}\right.\\
&\left.+S_{d-b}S_{2d-a-c}+S_{d-c}S_{2d-a-b}-2S_{4d-n}\right).
\end{aligned}
$$

This completes the proof of Corollary 2.

If we take $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ in Theorem 1, since

$$
\left(-\frac{1}{2}\right)^{(n)}=-\frac{(2n-3)!!}{2^n},\quad\left(\frac{1}{2}\right)^{(n)}=-\frac{(2n-1)!!}{2^n}=\frac{(2n)!}{4^n\cdot n!},
$$

we can immediately deduce Corollary 3 and Corollary 4.

4. Conclusions

The main results of this paper are to give some identities involving the fourth-order linear recurrence sequences by using two lemmas. We obtain some identities related to Tetranacci numbers, which gives us a better understanding of the properties of Tetranacci sequences. Although the characteristic equation of the fourth-order linear recurrence sequence has two real and two complex roots, with complicate irrational expressions, the expression in Theorem 1 does not use these roots, and depends only on the Tetranacci numbers. More importantly, we further verify that we can still get the properties of the higher-order linear recurrence sequence using the same method, noting that when the order of the linear recurrence sequence increases, the parameters become more complex, and the coefficients of its expansion will be more complicated.

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