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Some New Identities of Second Order Linear Recurrence Sequences

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Abstract: The main purpose of this paper is using the combinatorial method, the properties of the power series and characteristic roots to study the computational problem of the symmetric sums of a certain second-order linear recurrence sequences, and obtain some new and interesting identities. These results not only improve on some of the existing results, but are also simpler and more beautiful. Of course, these identities profoundly reveal the regularity of the second-order linear recursive sequence, which can greatly facilitate the calculation of the symmetric sums of the sequences in practice.

Keywords: the second-order linear recurrence sequence; convolution sums; new identity; recurrence formula

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1. Introduction

The defined of second-order linear recurrence sequence $\{S_n\}$ is

$$S_{n+2} = C_1 S_{n+1} + C_2 S_n, \text{ for all integers } n \geq 0, \text{with } S_0 = a, S_1 = b, \tag{1}$$

where n is integers with $n \geq 0$.

For convenience, we also extend the recursive property of S_n to all negative integers.

We taking $C_1 = x$, $C_2 = 1$, $S_n = F_{n+1}(x)$ with $F_0(x) = 0$, $F_1(x) = 1$ in (1), then $\{S_n\}$ becomes the famous Fibonacci polynomial sequence $\{F_{n+1}(x)\}$. That is,

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \text{ for all integers } n \geq 0.$$

Especially when $x = 1$, $F_n(1) = F_n$ becomes known as the Fibonacci sequence.

Let $\alpha = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta = \frac{x-\sqrt{x^2+4}}{2}$ denote the two roots of the characteristic equation $\lambda^2 - x\lambda - 1 = 0$. Then we have

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \ L_n(x) = \alpha^n + \beta^n, \ n = 0, 1, 2, \cdots,$$

where $L_n(x)$ denotes the Lucas polynomials, and $L_n(1)$ denotes the Lucas sequence.

If we take $C_1 = 2x$, $C_2 = -1$ in (1), then $S_n = U_n(x)$ is Chebyshov polynomials of the second kind with $U_0(x) = 1$ and $U_1(x) = 2x$. Chebyshov polynomials $T_n(x)$ of the first kind is defined by

$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$ for all $n \geq 0$ with $T_0(x) = 1$ and $T_1(x) = x$. Let $\delta = x + \sqrt{x^2-1}$, $\gamma = x - \sqrt{x^2-1}$ are two characteristic roots of the polynomial $\lambda^2 - 2x\lambda + 1 = 0$, then (see [1])

$$T_n(x) = \frac{1}{2}\left(\delta^n + \gamma^n\right), \quad U_n(x) = \frac{\delta^{n+1} - \gamma^{n+1}}{\delta - \gamma}, \quad n = 0, 1, 2, \cdots.$$

Many scholars have studied S_n, and obtained a series of valuable research results. For example, Yi Yuan and Zhang Wenpeng [2] proved the following conclusion: For any positive integer n and k, one has the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1}(x)\cdot F_{a_2}(x)\cdot F_{a_3}(x)\cdots F_{a_k}(x)$$
$$= \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}\cdot x^{n-2m},$$

where $\sum\limits_{a_1+a_2+\cdots+a_k=n}$ denotes the summation is taken over all k-dimension nonnegative integer coordinates $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

Ma Yuankui and Zhang Wenpeng [3] also studied this problem, and proved the following result:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} F_{a_1}(x)F_{a_2}(x)\cdots F_{a_{h+1}}(x) = \frac{1}{h!}\cdot\sum_{j=1}^{h}\frac{(-1)^{h-j}\cdot S(h,j)}{x^{2h-j}}$$
$$\times\left(\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot\binom{2h+i-j-1}{i}\cdot\frac{(-1)^i\cdot 2^i\cdot F_{n-i+j}(x)}{x^i}\right),$$

where $S(h,i)$ is defined by $S(h,0) = 0$, $S(h,h) = 1$, and

$$S(h+1, i+1) = 2\cdot(2h-1-i)\cdot S(h, i+1) + S(h,i)$$

for all positive integers $1 \leq i \leq h-1$.

On the other hand, Zhang Yixue and Chen Zhuoyu [4] studied the properties of Chebyshov polynomials, and proved the following identity:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} U_{a_1}(x)U_{a_2}(x)\cdots U_{a_{h+1}}(x)$$
$$= \frac{1}{2^h\cdot h!}\cdot\sum_{j=1}^{h}\frac{C(h,j)}{x^{2h-j}}\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot\binom{2h+i-j-1}{i}\cdot\frac{U_{n-i+j}(x)}{x^i},$$

where $C(h,i)$ is a second order non-linear recurrence sequence defined by $C(h,0) = 0$, $C(h,h) = 1$, $C(h+1,1) = 1\cdot 3\cdot 5\cdots(2h-1) = (2h-1)!!$ and $C(h+1,i+1) = (2h-1-i)\cdot C(h,i+1) + C(h,i)$ for all $1 \leq i \leq h-1$.

Many other papers related to Fibonacci numbers, Fibonacci polynomials, Chebyshov polynomials and second-order linear recurrence sequences can also be found in references [5–18], here we will no longer list them one by one.

After careful analysis of the research content in [1–4], we think it can be summarized as a sentence: That is, to study the symmetry sum problem of the generalized second-order linear recursive sequence. Of course, they are meaningful to study these problems. It not only reveals the profound properties of the generalized second-order linear recursive polynomials and sequences, but also greatly simplifies the calculation of the symmetry sums of these polynomials and sequences in practice.

Inspired by [1–4], in this paper, we will use a new method to study the computational problem of the symmetry sums of a certain second-order linear recurrence sequences, and give a simple and beautiful generalized conclusion. That is, we will use the elementary methods and the symmetry properties of the characteristic roots to prove the following results:

Theorem 1. *Let $S_n = C_1 \cdot S_{n-1} + C_2 \cdot S_{n-2}$ denotes any second-order linear recurrence sequence with $S_0 = 1$ and $S_1 = C_1$. Then we have the identity*

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} S_{a_1} \cdot S_{a_2} \cdot S_{a_3} \cdots S_{a_k} \\ = \ &\frac{1}{2}\sum_{i=0}^{n} \binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1} \cdot (-C_2)^i \cdot \left(S_{n-2i} + C_2 \cdot S_{n-2-2i}\right).\end{aligned}$$

It is clear that if we taking $C_1 = x$ and $C_2 = 1$, then from Theorem 1 we may immediately deduce the following:

Corollary 1. *For any positive integers n and k, we have the identity*

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot F_{a_3+1}(x) \cdots F_{a_k+1}(x) \\ = \ &\frac{1}{2}\sum_{i=0}^{n} \binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1} \cdot (-1)^i \cdot \left(F_{n+1-2i}(x) + F_{n-1-2i}(x)\right).\end{aligned}$$

Corollary 2. *For any positive integer m, n and k, we have the identity*

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}\left(T_m(x)\right) \cdot U_{a_2}\left(T_m(x)\right) \cdot U_{a_3}\left(T_m(x)\right) \cdots U_{a_k}\left(T_m(x)\right) \\ = \ &\sum_{i=0}^{n} \binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1} \cdot T_{m(n-2i)}(x)\end{aligned}$$

and

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}(x) \cdot U_{a_2}(x) \cdot U_{a_3}(x) \cdots U_{a_k}(x) \\ = \ &\sum_{i=0}^{n} \binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1} \cdot \left(xU_{n-1-2i}(x) - U_{n-2-2i}(x)\right).\end{aligned}$$

It is clear that our Corollary 1 and Corollary 2 are much easier than the results in [1–4]. If $S_{n+2} = C_1 \cdot S_{n+1} + C_2 \cdot S_n$ with $S_0 = 1$ and $S_1 = C_1$ and $H_{n+2} = D_1 \cdot H_{n+1} + D_2 \cdot H_n$ with $H_0 = 1$ and $H_1 = D_1$ are two different second-order linear recurrence sequences, such that the polynomials $x^2 - C_1x - C_2$ and $x^2 - D_1x - D_2$ co-prime. That is, $\left(x^2 - C_1x - C_2, x^2 - D_1x - D_2\right) = 1$. Then we define sequence $\{M_n\}$ as follows:

$$M_n = \sum_{i=0}^{n} S_i \cdot H_{n-i} \quad n = 0,\ 1,\ 2, \cdots. \tag{2}$$

For the sequence $\{M_n\}$ defined in (2), we have the following conclusion:

Theorem 2. *The sequence $\{M_n\}$ is a fourth-order linear recurrence sequence, and it satisfy the fourth-order linear recurrence formula*

$$\begin{aligned}M_{n+4} &= \left(C_1 + D_1\right) M_{n+3} + \left(C_2 + D_2 + C_1D_1\right) M_{n+2} \\ &\quad - \left(C_1D_2 + C_2D_1\right) M_{n+1} - C_2D_2M_n, \quad n \geq 0,\end{aligned}$$

where $M_0 = 1$, $M_1 = C_1 + D_1$, $M_2 = C_1^2 + D_1^2 + C_2 + D_2 + C_1 D_1$ *and*

$$M_3 = C_1^3 + D_1^3 + 2C_1C_2 + 2D_1D_2 + D_1\left(C_1^2 + C_2\right) + C_1\left(D_1^2 + D_2\right).$$

Taking $C_1 = x$, $C_2 = 1$, $D_1 = 2x$ and $D_2 = -1$, from our Theorem 2 we can deduce the following result:

Corollary 3. *For any integer $n \geq 0$, we define the polynomials sequence*

$$M_n(x) = \sum_{i=0}^{n} F_{i+1}(x) \cdot U_{n-i}(x).$$

Then $M_n(x)$ is a fourth-order linear recurrence polynomials, and it satisfy the recurrence formula

$$M_{n+4}(x) = 3xM_{n+3}(x) + 2x^2 M_{n+2}(x) - xM_{n+1}(x) + M_n(x)$$

for all integers $n \geq 0$, where $M_0(x) = 1$, $M_1(x) = 3x$, $M_2(x) = 7x^2$ and $M_3(x) = 15x^3 - x$, $F_n(x)$ and $U_n(x)$ denote the Fibonacci polynomials and Chebyshov polynomials of the second kind respectively.

2. Proof of the Theorem

In this section, we will prove our main results directly. First we prove Theorem 1.

Proof of Theorem 1. It is clear that the characteristic equation of the sequence $\{S_n\}$ is $\lambda^2 - C_1\lambda - C_2 = 0$. Let α and β are the two characteristic roots of the equation $\lambda^2 - C_1\lambda - C_2 = 0$. Then we have

$$S_n = A\alpha^n + B\beta^n \text{ with } A + B = 1 \text{ and } A\alpha + B\beta = C_1.$$

That is,

$$S_n = A\alpha^n + B\beta^n = \frac{\alpha}{\alpha - \beta} \cdot \alpha^n - \frac{\beta}{\alpha - \beta} \cdot \beta^n, \ n \geq 0.$$

The generating function of the sequence $\{S_n\}$ is

$$\frac{1}{1 - C_1 x - C_2 x^2} = \frac{1}{(1 - \alpha x)(1 - \beta x)} = \sum_{n=0}^{\infty} S_n \cdot x^n, \tag{3}$$

where $\alpha \cdot \beta = -C_2$ and $\alpha + \beta = C_1$.

For any positive integer k, we have the identity

$$\begin{aligned} \frac{1}{(1 - C_1 x - C_2 x^2)^k} &= \frac{1}{(1 - \alpha x)^k \cdot (1 - \beta x)^k} \\ &= \sum_{n=0}^{\infty} \left(\sum_{a_1 + a_2 + \cdots + a_k = n} S_{a_1} \cdot S_{a_2} \cdot S_{a_3} \cdots S_{a_k} \right) \cdot x^n. \end{aligned} \tag{4}$$

On the other hand, from the properties of the power series we have

$$\frac{1}{(1 - x)^k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{k - 1} \cdot x^n, \ \ |x| < 1. \tag{5}$$

Thus, from (5) and the properties of the power series we have

$$\begin{aligned}\frac{1}{(1-\alpha x)^k(1-\beta x)^k} &= \left(\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\alpha^n x^n\right)\left(\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\beta^n x^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot\alpha^i\cdot\beta^{n-i}\right)\cdot x^n. \qquad (6)\end{aligned}$$

Combining (4), (6) and note that $\alpha\cdot\beta = -C_2$ and the symmetry of α and β we can deduce the identity

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} S_{a_1}\cdot S_{a_2}\cdot S_{a_3}\cdots S_{a_k}\\ =& \sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot\alpha^i\cdot\beta^{n-i}\\ =& \sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot(-C_2)^i\cdot\beta^{n-2i}\\ =& \sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot(-C_2)^i\cdot\alpha^{n-2i}\\ =& \sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot(-C_2)^i\cdot\frac{1}{2}\left(\alpha^{n-2i}+\beta^{n-2i}\right). \qquad (7)\end{aligned}$$

From the definitions A and B we have

$$A\cdot(\alpha-\beta)=\alpha \text{ and } B\cdot(\beta-\alpha)=\beta.$$

So for any integer r, from the definition of S_n we have

$$\begin{aligned}&\alpha^r+\beta^r = \alpha\cdot\alpha^{r-1}+\beta\cdot\beta^{r-1} = A\cdot(\alpha-\beta)\cdot\alpha^{r-1}+B\cdot(\beta-\alpha)\cdot\beta^{r-1}\\ =& A\cdot\alpha^r+B\cdot\beta^r-\alpha\cdot\beta\left(A\cdot\alpha^{r-2}+B\cdot\beta^{r-2}\right) = S_r+C_2\cdot S_{r-2}. \qquad (8)\end{aligned}$$

Now combining (7) and (8) we may immediately deduce the identity

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} S_{a_1}\cdot S_{a_2}\cdot S_{a_3}\cdots S_{a_k}\\ =& \sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot\alpha^i\cdot\beta^{n-i}\\ =& \frac{1}{2}\sum_{i=0}^{n}\binom{i+k-1}{k-1}\binom{n-i+k-1}{k-1}\cdot(-C_2)^i\cdot\left(S_{2-2i}+C_2\cdot S_{n-2-2i}\right).\end{aligned}$$

This proves Theorem 1. □

Proof of Theorem 2. Note that $\left(x^2-C_1x-C_2, x^2-D_1x-D_2\right)=1$, so from the definitions of α, β, δ and γ we have

$$\begin{aligned}&\left(1-C_1x-C_2x^2\right)\cdot\left(1-D_1x-D_2x^2\right) = 1-(C_1+D_1)\,x\\ &-(C_2+D_2-C_1D_1)\,x^2+(C_1D_2+C_2D_1)\,x^3+C_2D_2x^4\\ =& (1-\alpha x)(1-\beta x)(1-\delta x)(1-\gamma x), \qquad (9)\end{aligned}$$

where α, β, δ and γ are different each others.

It is clear that from the definitions sequences S_n and H_n we have

$$\frac{1}{(1-C_1x-C_2x^2)\cdot(1-D_1x-D_2x^2)}=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}S_i\cdot H_{n-i}\right)\cdot x^n. \tag{10}$$

On the other hand, from the definition and properties of the fourth-order linear recurrence sequence we also have

$$\begin{aligned}&\frac{1}{(1-C_1x-C_2x^2)\cdot(1-D_1x-D_2x^2)}\\ =\;&\frac{1}{(1-\alpha x)(1-\beta x)(1-\delta x)(1-\gamma x)}=\sum_{n=0}^{\infty}M_n\cdot x^n,\end{aligned} \tag{11}$$

where $M_0=1$, $M_1=C_1+D_1$, $M_2=C_1^2+D_1^2C_2+D_2+C_1D_1$,

$$M_3=C_1^3+D_1^3+2C_1C_2+2D_1D_2+D_1\left(D_1^2+D_2\right)+C_1\left(D_1^2+D_2\right)$$

and

$$\begin{aligned}M_{n+4}&=(C_1+D_1)\,M_{n+3}+(C_2+D_2+C_1D_1)\,M_{n+2}\\&-(C_1D_2+C_2D_1)\,M_{n+1}-C_2D_2M_n,\;\; n\geq 0.\end{aligned} \tag{12}$$

From (11) and (12) we know that the sequence

$$M_n=\sum_{i=0}^{n}S_i\cdot H_{n-i}$$

is a fourth-order recurrence sequence, and it satisfy the recurrence Formula (12).

This completes the proof of Theorem 2. □

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