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On the Fekete–Szegö Type Functionals for Close-to-Convex Functions

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Abstract: In this paper, we consider two functionals of the Fekete–Szegö type $\Theta_f(\mu) = a_4 - \mu a_2 a_3$ and $\Phi_f(\mu) = a_2 a_4 - \mu a_3{}^2$ for a real number μ and for an analytic function $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, $|z| < 1$. This type of research was initiated by Hayami and Owa in 2010. They obtained results for functions satisfying one of the conditions $\mathrm{Re}\,\{f(z)/z\} > \alpha$ or $\mathrm{Re}\,\{f'(z)\} > \alpha$, $\alpha \in [0, 1)$. Similar estimates were also derived for univalent starlike functions and for univalent convex functions. We discuss $\Theta_f(\mu)$ and $\Phi_f(\mu)$ for close-to-convex functions such that $f'(z) = h(z)/(1-z)^2$, where h is an analytic function with a positive real part. Many coefficient problems, among others estimating of $\Theta_f(\mu)$, $\Phi_f(\mu)$ or the Hankel determinants for close-to-convex functions or univalent functions, are not solved yet. Our results broaden the scope of theoretical results connected with these functionals defined for different subclasses of analytic univalent functions.

Keywords: coefficient problem; close-to-convex function; Fekete–Szegö functional; functional of Fekete–Szegö type

1. Introduction

Let $\mathcal{A}$ be the family of all functions analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1|\}$ having the power series expansion:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \,, \tag{1}$$

and let $\mathcal{S}^*$ denote the class of univalent starlike functions in $\mathcal{A}$ (for the definitions and properties of $\mathcal{S}^*$ and other classes, see [1]). For a given real argument $\beta \in (-\pi/2, \pi/2)$ and a given function $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is called close-to-convex with argument β with respect to g if:

$$Re\left\{\frac{e^{i\beta} z f'(z)}{g(z)}\right\} > 0, \quad z \in \Delta \,.$$

Let $\mathcal{C}_\beta(g)$ be the class of all such functions. Moreover, let:

$$\mathcal{C}_\beta = \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\beta(g).$$

Let $\mathcal{C}$ denote the family of all close-to-convex functions (see [2,3]). It is obvious that:

$$\mathcal{C} = \bigcup_{\beta \in (-\pi/2, \pi/2)} \mathcal{C}_\beta \,.$$

All functions in $\mathcal{C}$ are univalent.

In this paper, we consider the class $\mathcal{C}_0(k)$, where k is the Koebe function:

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots . \tag{2}$$

The class $\mathcal{C}_0(k)$ is sometimes denoted by $\mathcal{CR}^+$. Such functions have a well known geometrical meaning. Namely, for each function f in this class, the set $f(\Delta)$ is a domain such that $\{w + t : t \geq 0\} \subset f(\Delta)$ for every $w \in f(\Delta)$. Such functions f are convex in the positive direction of the real axis.

For a function f analytic in Δ of the form (1), we define two functionals for a fixed real μ:

$$\Theta_f(\mu) = a_4 - \mu a_2 a_3 \tag{3}$$

and:

$$\Phi_f(\mu) = a_2 a_4 - \mu {a_3}^2 . \tag{4}$$

The functionals $\Theta_f(\mu)$ and $\Phi_f(\mu)$ are the generalizations of two well known expressions: $a_4 - a_2 a_3$ and $a_2 a_4 - {a_3}^2$. Both functionals are symmetric, or invariant, under rotations. The first one is a particular case of the generalized Zalcman functional. It was investigated, among others, by Ma [4] and Efraimidis and Vukotić [5]. The second functional is known as the second Hankel determinant, and it was studied in many papers. The investigation of Hankel determinants for analytic functions was started by Pommerenke (see [6,7]) and continued by many mathematicians in various classes of univalent functions (see, for example [8–16]). The functional $\Phi_f(\mu)$ was first studied by Hayami and Owa [17]. They discussed an even more general functional $a_n a_{n+2} - \mu {a_{n+1}}^2$ for the classes $\mathcal{Q}(\alpha)$ and $\mathcal{R}(\alpha)$, $\alpha \in [0, 1)$, of functions $f \in \mathcal{A}$ such that Re $\{f(z)/z\} > \alpha$ and Re $\{f'(z)\} > \alpha$, respectively. The functionals $\Phi_f(\mu)$ and $\Theta_f(\mu)$ for the classes $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively, were discussed in [18].

It is worth pointing out a particularly interesting property of $\Phi_f(\mu)$. The sharp estimates of this functional are often symmetric with respect to a certain point. It was shown in [18] that such points for $\mathcal{S}^*$ and $\mathcal{K}$ are $8/9$ and one, respectively. We have:

$$|\Phi_f(\mu)| \leq \max\{|9\mu - 8|, 1\} \quad \text{for} \quad \mathcal{S}^* \tag{5}$$

and:

$$|\Phi_f(\mu)| \leq \max\{|\mu - 1|, 1/8\} \quad \text{for} \quad \mathcal{K}.$$

A similar situation occurs for $\mathcal{Q}(1/2)$ and for the class $\mathcal{C}_0(h)$, where $h(z) = z/(1-z^2)$; this point is $1/2$ (see [17,19]). This situation appears even in the class $\mathcal{T}$ of typically real functions, which do not necessarily have to be univalent (see [19]).

In this work, we derive bounds of $\Theta_f(\mu)$ and $\Phi_f(\mu)$ for functions in $\mathcal{C}_0(k)$.

2. Preliminary Results

Let $\mathcal{P}$ denote the class of all analytic functions h with a positive real part in Δ satisfying the normalization condition $h(0) = 1$. Let $h \in \mathcal{P}$ have the Taylor series expansion:

$$h(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots . \tag{6}$$

We shall need here three results. The first one is known as Caratheodory's lemma (see, for example, ref. [1]). The second one is due to Libera and Złotkiewicz ([20,21]), and the third one is the result of Hayami and Owa.

Lemma 1 ([1])**.** *If $h \in \mathcal{P}$ is given by (6), then the sharp inequality $|p_n| \leq 2$ holds for $n \geq 1$.*

Lemma 2 ([20,21]). *Let h be given by (6) and p_1 be a given real number, $p_1 \in [-2,2]$. Then, $h \in \mathcal{P}$ if and only if:*

$$2p_2 = {p_1}^2 + (4 - {p_1}^2)x$$

and:

$$4p_3 = {p_1}^3 + 2(4 - {p_1}^2)p_1 x - (4 - {p_1}^2)p_1 x^2 + 2(4 - {p_1}^2)(1 - |x|^2)y$$

for some complex numbers x, y such that $|x| \le 1$, $|y| \le 1$.

Lemma 3 ([17]). *If $h \in \mathcal{P}$ is given by (6), then:*

$$|p_3 - \mu p_1 p_2| \le \max\{2, |2 - 4\mu|\} \ .$$

The next lemma is an improvement of Lemma 3 for $\mu \in [1/2, 1]$.

Lemma 4 ([22]). *If $h \in \mathcal{P}$ is given by (6) and $\mu \in [1/2, 1]$, then:*

$$|p_3 - \mu p_1 p_2| \le \begin{cases} \frac{1}{4}\mu^2 p^3 - \frac{1}{2}\mu(2-\mu)p^2 + 2, & p \in [0, 2/(2-\mu)] \ , \\ (3 - 2\mu)p - (1-\mu)p^3, & p \in [2/(2-\mu), 2] \ , \end{cases} \tag{7}$$

where $p = |p_1|$. The inequality is sharp.

The following lemma was proven by Lecko (see Corollary 2.3 in [23]).

Lemma 5 ([23]). *If $h \in \mathcal{P}$ is given by (6), then:*

$$|p_{n+1} + 2p_n + p_{n-1}| \le 2(2 + Re\{p_1\}) \ . \tag{8}$$

We have proven the next lemma.

Lemma 6. *If $h \in \mathcal{P}$ is given by (6), then:*

$$|p_1 p_3 - {p_2}^2| \le 4 - |p_1|^2 \ . \tag{9}$$

The inequality is sharp.

Proof. By Lemma 2,

$$4(p_1 p_3 - {p_2}^2) = (4 - {p_1}^2)\left[2p_1(1 - |x|^2)y - 4x^2\right] \ .$$

Applying the invariance of $|p_1 p_3 - {p_2}^2|$ under rotation, we can assume that p_1 is a non-negative real number. Writing $r = |x| \in [0,1]$ and $p = p_1 \in [0,2]$, we get by the triangle inequality and the assumption $|y| \le 1$:

$$4|p_1 p_3 - {p_2}^2| \le (4 - p^2)[2p(1 - r^2) + 4r^2] = (4 - p^2)[2p + (4 - 2p)r^2] \le 4(4 - p^2) \ ,$$

which gives the desired bound. The equality (9) holds for:

$$h(z) = \left(1 - \frac{p}{2}\right)\frac{1 + z^2}{1 - z^2} + \frac{p}{2}\frac{1 + z}{1 - z} = 1 + pz + 2z^2 + pz^3 + \dots \ , \tag{10}$$

which means that there is equality in (9) for rotations of (10). □

The next lemma is a special case of more general results due to Choi et al. [24] (see also [9]). Let $\overline{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$. Define:

$$Y(a,b,c) = \max_{z\in\overline{\Delta}} \left(|a + bz + cz^2| + 1 - |z|^2\right), \quad a,b,c \in \mathbb{R}\,.$$

Lemma 7. *If $ac < 0$, then:*

$$Y(a,b,c) = \begin{cases} 1 + |a| + \dfrac{b^2}{4(1+|c|)}, & |b| < 2(1+|c|) \text{ and } b^2 < -4a(1-c^2)/c\,, \\ 1 - |a| + \dfrac{b^2}{4(1-|c|)}, & |b| < 2(1-|c|) \text{ and } b^2 \ge -4a(1-c^2)/c\,, \\ R(a,b,c), & \text{otherwise,} \end{cases}$$

where:

$$R(a,b,c) = \begin{cases} |a| + |b| - |c|, & |ab| \ge |c|\,(|b| + 4|a|)\,, \\ -|a| + |b| + |c|, & |ab| \le |c|\,(|b| - 4|a|)\,, \\ (|c| + |a|)\sqrt{1 - \dfrac{b^2}{4ac}}, & \text{otherwise.} \end{cases}$$

If $ac \ge 0$, then:

$$Y(a,b,c) = \begin{cases} |a| + |b| + |c|, & |b| \ge 2(1-|c|)\,, \\ 1 + |a| + \dfrac{b^2}{4(1-|c|)}, & |b| < 2(1-|c|)\,. \end{cases}$$

Applying the correspondence between the functions in $\mathcal{C}_0(k)$ and $\mathcal{P}$:

$$(1-z)^2 f'(z) = h(z), \quad f \in \mathcal{C}_0(k), \quad h \in \mathcal{P} \tag{11}$$

and Expansions (1) and (6) we get:

$$2a_2 = 2 + p_1, \quad 3a_3 = 3 + 2p_1 + p_2, \quad 4a_4 = 4 + 3p_1 + 2p_2 + p_3\,. \tag{12}$$

Moreover, by Lemma 1, Re $\{a_2\} \ge 0$ with equality if and only if $p_1 = -2$. The equality is possible only for the function $h(z) = \frac{1-z}{1+z} \in \mathcal{P}$, and then, $f(z) = \frac{1}{2}\log\frac{1+z}{1-z} \in \mathcal{C}_0(k)$.

Hence, we can express $\Theta_f(\mu)$ and $\Phi_f(\mu)$ for $f \in \mathcal{C}_0(k)$ as coefficients of a corresponding function $h \in \mathcal{P}$ in the following way:

$$\Theta_f(\mu) = \tfrac{1}{4}p_3 + (\tfrac{1}{2} - \tfrac{1}{3}\mu)p_2 + (\tfrac{3}{4} - \tfrac{7}{6}\mu)p_1 - \tfrac{1}{6}\mu p_1 p_2 - \tfrac{1}{3}\mu {p_1}^2 + 1 - \mu \tag{13}$$

and:

$$\begin{aligned} \Phi_f(\mu) = {} & \tfrac{1}{8}p_1p_3 - \tfrac{1}{9}\mu {p_2}^2 + \tfrac{1}{4}p_3 + \left(\tfrac{1}{4} - \tfrac{4}{9}\mu\right) p_1p_2 + \left(\tfrac{1}{2} - \tfrac{2}{3}\mu\right) p_2 \\ & + \left(\tfrac{3}{8} - \tfrac{4}{9}\mu\right) {p_1}^2 + \left(\tfrac{5}{4} - \tfrac{4}{3}\mu\right) p_1 + 1 - \mu\,. \end{aligned} \tag{14}$$

3. Example

Let us consider the function:

$$F(z) = \tfrac{1}{2}(1-\alpha)\log\frac{1+z}{1-z} + \alpha\,\frac{z}{(1-z)^2}, \quad \alpha \in [0,1]\,, \tag{15}$$

which has the following Taylor series expansion:

$$F(z) = (1-\alpha)\left(z + \tfrac{1}{3}z^3 + \dots\right) + \alpha\left(z + 2z^2 + 3z^3 + 4z^4 + \dots\right)$$
$$= z + 2\alpha z^2 + \tfrac{1}{3}(1+8\alpha)z^3 + 4\alpha z^4 + \dots .$$

Since:

$$(1-z)^2 F'(z) = (1-\alpha)\,\frac{1-z}{1+z} + \alpha\,\frac{1+z}{1-z} \in \mathcal{P}\,,$$

so $F \in \mathcal{C}_0(k)$. Moreover,

$$F(\Delta) = \mathbb{C} \setminus \left\{ x \pm i(1-\alpha)\tfrac{\pi}{4} : x \le \tfrac{1}{4}[(1-\alpha)\ln\tfrac{1-\alpha}{\alpha} - 1] \right\} .$$

For F, we have:

$$\Theta_F(\mu) = \tfrac{2}{3}\left[-8\mu\alpha^2 + (6-\mu)\alpha\right]$$

and:

$$\Phi_F(\mu) = \tfrac{1}{9}\left[8(9-8\mu)\alpha^2 - 16\mu\alpha - \mu\right] .$$

For $\mu < 0$, we have: $\Theta_F(\mu) \le 4 - 6\mu$ and $\Phi_F(\mu) \le 8 - 9\mu$. We find the estimation of $\Theta_F(\mu)$ and $\Phi_F(\mu)$ for $\mu \ge 0$.

Let us denote:

$$f(\alpha) = \tfrac{2}{3}\left[-8\mu\alpha^2 + (6-\mu)\alpha\right] \quad\text{and}\quad g(\alpha) = \tfrac{1}{9}\left[8(9-8\mu)\alpha^2 - 16\mu\alpha - \mu\right] .$$

The critical point $\alpha_0 = (6-\mu)/(16\mu)$ of $f(\alpha)$ is in $(0,1)$ if $\mu \in (6/17, 6)$. Hence,

$$|\Theta_F(\mu)| \le \max\{|f(\alpha_0)|, |f(1)|, |f(0)|\} = \max\left\{\left|\frac{(6-\mu)^2}{48\mu}\right|, |4-6\mu|, 0, \right\}$$

for $\mu \in (6/17, 6)$ and:

$$|\Theta_F(\mu)| \le \max\{|f(1)|, |f(0)|\} = |4-6\mu| \quad \text{for } \mu \in [0, 6/17] \cup [6, \infty) .$$

Similarly, the critical point $\alpha_1 = \mu/(9-8\mu)$ of $g(\alpha)$ is in $(0,1)$ if $\mu \in (0,1)$. Hence,

$$|\Phi_F(\mu)| \le \max\{|g(\alpha_1)|, |g(1)|, |g(0)|\} = \max\left\{\left|\frac{\mu}{8\mu-9}\right|, |8-9\mu|, \left|-\frac{\mu}{9}\right|\right\}$$

for $\mu \in (0,1)$ and:

$$|\Phi_F(\mu)| \le \max\{|g(1)|, |g(0)|\} = |9\mu - 8| \quad \text{for } \mu \in \{0\} \cup [1, \infty) .$$

Finally, for a function F given by (15), we obtain:

$$\Theta_F(\mu) \le \begin{cases} 4 - 6\mu, & \mu \le 6/17 = 0.352\dots \\ \dfrac{(6-\mu)^2}{48\mu}, & 6/17 \le \mu \le 6(15+16\sqrt{2})/287 = 0.786\dots \\ 6\mu - 4, & \mu \ge 6(15+16\sqrt{2})/287 \end{cases}$$

and:

$$\Phi_F(\mu) \le \begin{cases} 8 - 9\mu, & \mu \le (73-\sqrt{145})/72 = 0.846\dots \\ \dfrac{\mu}{9-8\mu}, & (73-\sqrt{145})/72 \le \mu \le 1 \\ 9\mu - 8, & \mu \ge 1 . \end{cases}$$

4. Bounds of $|\Theta(\mu)|$ for the Class $\mathcal{C}_0(k)$

In the main theorem of this section, we establish the sharp bounds of $|\Theta(\mu)|$ for the class $\mathcal{C}_0(k)$. The proof is divided into six lemmas. The first one is a particular case of the result obtained in [22] (Theorem 3.1 or Theorem 3.3 in [22]), and the second one is obvious.

Lemma 8. *Let $f \in \mathcal{C}_0(k)$. Then, $|\Theta_f(1)| = |a_4 - a_2a_3| \le 2$. The result is sharp.*

Lemma 9. *Let $f \in \mathcal{C}_0(k)$ and $\mu \le 0$. Then, $|\Theta_f(\mu)| \le 4 - 6\mu$. The result is sharp.*

Lemma 10. *Let $f \in \mathcal{C}_0(k)$ and $\mu > 1$. Then, $|\Theta_f(\mu)| \le 6\mu - 4$. The result is sharp.*

Proof. From (13), we can write $\Theta_f(\mu)$ as follows:

$$\Theta_f(\mu) = \tfrac{1}{4}\left(p_3 - \tfrac{2}{3}\mu p_1 p_2\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\mu\right) p_2 - \tfrac{1}{3}\mu {p_1}^2 + \left(\tfrac{3}{4} - \tfrac{7}{6}\mu\right) p_1 + 1 - \mu\,.$$

If $\mu \ge 3/2$, then, taking into account Lemmas 1 and 3, we get:

$$|\Theta_f(\mu)| \le \tfrac{1}{4}\left(\tfrac{8}{3}\mu - 2\right) + 2\left(\tfrac{1}{3}\mu - \tfrac{1}{2}\right) + \tfrac{4}{3}\mu + 2\left(\tfrac{7}{6}\mu - \tfrac{3}{4}\right) + \mu - 1 = 6\mu - 4\,.$$

If $\mu \in (1, 3/2)$, then we have:

$$\Theta_f(\mu) = (3 - 2\mu)(a_4 - a_2a_3) + (2\mu - 2)(a_4 - \tfrac{3}{2}a_2a_3)\,.$$

Now, from Lemma 8 and the first part of this proof (i.e., $|a_4 - \tfrac{3}{2}a_2a_3| \le 5$), we obtain:

$$|\Theta_f(\mu)| \le 2(3 - 2\mu) + 5(2\mu - 2) = 6\mu - 4.$$

It is clear that $\Theta_f(\mu) = 4 - 6\mu$ only when $p_1 = p_2 = p_3 = 2$, which means that this equality holds only for the Koebe function (2). In other words, the Koebe function is the extremal function for $\mu > 1$. □

Taking into account (13) and Lemma 2, we can write $\Theta_f(\mu)$ as follows:

$$\begin{aligned}\Theta_f(\mu) &= 1 + \tfrac{1}{16}{p_1}^3 + \tfrac{1}{4}{p_1}^2 + \tfrac{3}{4}p_1 - \tfrac{1}{12}\mu {p_1}^3 - \tfrac{1}{2}\mu {p_1}^2 - \tfrac{7}{6}\mu p_1 - \mu \\ &\quad + \left[\tfrac{1}{8}p_1 + \tfrac{1}{4} - \tfrac{1}{12}\mu(2 + p_1)\right](4 - {p_1}^2)x - \tfrac{1}{16}(4 - {p_1}^2)p_1 x^2 + \tfrac{1}{8}(4 - {p_1}^2)(1 - |x|^2)y\,.\end{aligned}$$

From the above formula, we can obtain bounds of $|\Theta_f(\mu)|$, while $\mu \in (0, 1)$ and $f \in \mathcal{C}_0(k)$, but only with an additional assumption that a_2 is a positive real number. The assumption of Lemma 2 enforces that $p_1 \in [-2, 2]$. Notice that if $p_1 = 2$, then $f(z) = k(z)$ given by (2), and we have:

$$\Theta_f(\mu) = 4 - 6\mu\,. \tag{16}$$

If $p_1 = -2$, then $f(z) = \tfrac{1}{2}\log\tfrac{1+z}{1-z} = z + \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \ldots$ is in $\mathcal{C}_0(k)$, and so:

$$\Theta_f(\mu) = 0\,. \tag{17}$$

To shorten notation, we write p instead of p_1. One can observe that $\Theta_f(\mu)$ can be written as:

$$8\,\Theta_f(\mu) = (4 - p^2)\left[a + bx + cx^2 + (1 - |x|^2)y\right]\,, \tag{18}$$

where:

$$\begin{aligned} a &= \frac{48(1-\mu)+4(9-14\mu)p+12(1-2\mu)p^2+(3-4\mu)p^3}{6(4-p^2)}\,, \\ b &= (2+p)(1-\tfrac{2}{3}\mu)\,, \\ c &= -\tfrac{1}{2}p\,. \end{aligned} \tag{19}$$

From (18), the triangle inequality, $|y| \leq 1$, and Lemma 2, we get:

$$8\,|\Theta_f(\mu)| \leq (4-p^2)\left[|a+bx+cx^2|+1-|x|^2\right]\,, \tag{20}$$

where a, b, and c are given by (19).

Lemma 11. *Let $f \in \mathcal{C}_0(k)$, a_2 be a real number, $a_2 \in [0,2]$ and $\mu \in (0,1/3]$. Then, $|\Theta_f(\mu)| \leq 4-6\mu$. The result is sharp.*

Proof. For $\mu = 1/3$, we have (20) with:

$$a = \frac{5p^2+2p+48}{18(2-p)}, \quad b = \frac{7(2+p)}{9}, \quad c = -\frac{p}{2}\,.$$

We use Lemma 7. Clearly, $ac < 0$ for $p \in (0,2)$. Note that the inequality $|b| < 2(1+|c|)$ from the first case of Lemma 7 is equivalent to the obviously true inequality:

$$7(2+p) < 18(1+p/2)\,. \tag{21}$$

The inequality $b^2 < -4a(1-c^2)/c$, which can be written as:

$$\frac{4(2+p)(p^2+20p-108)}{81p} < 0\,,$$

holds for all $p \in (0,2)$. Hence, for $p \in (0,2)$, we have:

$$Y(a,b,c) = 1+|a|+\frac{b^2}{4(1+|c|)} = \frac{2(238-36p-p^2)}{81(2-p)}\,. \tag{22}$$

For $p \in (-2,0]$, we have $ac \geq 0$, and the inequality $|b| < 2(1-|c|)$ from the last case of Lemma 7 is equivalent to (21). Therefore, $Y(a,b,c)$ is also given by (22).

Thus, from (20) for $\mu = 1/3$, Lemma 7, (16) and (17), we obtain:

$$|\Theta_f(1/3)| \leq g(p)\,, \tag{23}$$

where $g(p) = (2+p)(238-36p-p^2)/324$ and $p \in [-2,2]$ according to the assumption. The function g is increasing for $p \in [-2,2]$; therefore:

$$|\Theta_f(1/3)| \leq g(2) = 2\,. \tag{24}$$

Moreover, we have by the triangle inequality:

$$|\Theta_f(\mu)| = (1-3\mu)a_4 + 3\mu(a_4 - \tfrac{1}{3}a_2a_3), \quad \mu \in (0,1/3)\,.$$

From Lemma 9 and from (24), we get:

$$|\Theta_f(\mu)| \leq 4(1-3\mu) + 2\cdot 3\mu = 4-6\mu,$$

and the proof is complete. Equality holds for the Koebe function (2). □

Let us denote:

$$p_0 = 2(\sqrt{103} - 10)/3 = 0.099\dots ,$$
$$K = 16(103\sqrt{103} - 910)/2187 = 0.9901\dots . \quad (25)$$

Lemma 12. *Let $f \in \mathcal{C}_0(k)$, a_2 be a real number, $a_2 \in [0,2]$, and K be given by (25). Then, $|\Theta_f(2/3)| \le K$. The result is sharp.*

Proof. For $\mu = 2/3$, we have (20) with:

$$a = \frac{12 - p}{18}, \quad b = \frac{5(2+p)}{9}, \quad c = -\frac{p}{2}.$$

We use Lemma 7. Clearly, $ac < 0$ for $p \in (0,2]$. First, note that the inequality $|b| < 2(1+|c|)$ is equivalent to the obviously true inequality:

$$5(2+p) < 18(1+p/2). \quad (26)$$

The inequality $b^2 < -4a(1-c^2)/c$, which is equivalent to:

$$\frac{8(2+p)(2p^2+22p-27)}{81p} < 0,$$

holds for $p \in \left(0, (5\sqrt{7}-11)/2\right]$. For $p \in \left(0, (5\sqrt{7}-11)/2\right]$, we have:

$$Y(a,b,c) = 1 + |a| + \frac{b^2}{4(1+|c|)} = \frac{8(p+20)}{81}, \quad (27)$$

so from (20) for $\mu = 2/3$ and Lemma 7, we obtain:

$$|\Theta_f(2/3)| \le (4-p^2)(p+20)/81. \quad (28)$$

From Lemma 7, the inequality system consists of $|b| < 2(1-|c|)$, and $b^2 \ge -4a(1-c^2)/c$ is contradictory, because the first inequality gives $p < 4/7$, while the second one yields $p \ge (5\sqrt{7}-11)/2$.

Now, consider the third case of Lemma 7. Let $p \in \left[(5\sqrt{7}-11)/2, 2\right]$. The inequality $|ab| \ge |c|(|b|+4|a|)$ is equivalent to $60 - 128p - 16p^2 \ge 0$, and it is not satisfied for any $p \in \left[(5\sqrt{7}-11)/2, 2\right]$. The inequality $|ab| \le |c|(|b|-4|a|)$, which can be written as $30 + 44p - 17p^2 \le 0$, is also not satisfied for any $p \in \left[(5\sqrt{7}-11)/2, 2\right]$. Thus, for $p \in \left[(5\sqrt{7}-11)/2, 2\right]$, we have:

$$Y(a,b,c) = (|c|+|a|)\sqrt{1 - \frac{b^2}{4ac}} = \frac{4(2p+3)}{27}\sqrt{\frac{(2p+25)(2p+1)}{(12-p)p}}. \quad (29)$$

From (20) for $\mu = 2/3$ and Lemma 7, we obtain:

$$|\Theta_f(2/3)| \le \frac{(4-p^2)(2p+3)}{54}\sqrt{\frac{(2p+25)(2p+1)}{(12-p)p}}. \quad (30)$$

For $p \in [-2,0]$, we have $ac \geq 0$, and the inequality $|b| < 2(1-|c|)$ from the last case of Lemma 7 is equivalent to the inequality in (26).

Thus, $Y(a,b,c)$ is given by (27). Finally, from (16), (28) and (30), we obtain:

$$|\Theta_f(2/3)| \leq g(p)\,,$$

where:

$$g(p) = \begin{cases} \frac{1}{81}(4-p^2)(p+20), & p \in \left[-2, (5\sqrt{7}-11)/2\right) \\ \frac{1}{54}(4-p^2)(2p+3)\sqrt{\frac{(2p+25)(2p+1)}{(12-p)p}}, & p \in \left[(5\sqrt{7}-11)/2, 2\right] \end{cases}.$$

Now, let us consider the function g for $p \in \left[(5\sqrt{7}-11)/2, 2\right]$. We have:

$$g'(p) = \frac{M(p)}{54(12-p)^2p^2}\sqrt{\frac{(12-p)p}{(2p+25)(2p+1)}}\,,$$

where $M(p) = 24p^6 - 52p^5 - 3802p^4 - 4801p^3 + 4242p^2 + 1500p - 1800$ and:

$$M(p) = 24p^5(p-13/6) + 900(p-2) + 3802p^2(1-p^2) + p(-4801p^2 + 440p + 600) < 0$$

for $p \in (1,2]$. Hence, $g'(p) < 0$ for $p \in \left[(5\sqrt{7}-11)/2, 2\right]$.

Taking the above into account, one can check that the function g is increasing for $p \in [-2, p_0)$ and is decreasing for $p \in (p_0, 2]$, where p_0 is given by (25). Therefore,

$$|\Theta_f(2/3)| \leq g\left(p_0\right) = 16(103\sqrt{103} - 910)/2187 = 0.9901\ldots\,,$$

so we have the desired result. □

Lemma 13. *Let $f \in \mathcal{C}_0(k)$, a_2 be a real number, and $a_2 \in [0,2]$.*

1. *If $\mu \in (1/3, 2/3)$, then $|\Theta_f(\mu)| < 3 - 3\mu$.*
2. *If $\mu \in (2/3, 1)$, then $|\Theta_f(\mu)| < 3\mu - 1$.*

Proof. We have:

$$|\Theta_f(\mu)| = |(2-3\mu)(a_4 - \tfrac{1}{3}a_2a_3) + (3\mu-1)(a_4 - \tfrac{2}{3}a_2a_3)|, \quad \mu \in (1/3, 2/3)\ .$$

From Lemmas 11 and 12, and the triangle inequality, we get the first part of Lemma 13, i.e.,:

$$|\Theta_f(\mu)| \leq 2(2-3\mu) + K\cdot(3\mu-1) < 2(2-3\mu) + 1\cdot(3\mu-1) = 3 - 3\mu\ .$$

Since:

$$|\Theta_f(\mu)| = |(3-3\mu)(a_4 - \tfrac{2}{3}a_2a_3) + (3\mu-2)(a_4 - a_2a_3)|, \quad \mu \in (2/3, 1)\ ,$$

from Lemma 12, Lemma 8, and the triangle inequality, we get the second part of Lemma 13, i.e.,:

$$|\Theta_f(\mu)| \leq 1\cdot(3-3\mu) + 2(3\mu-2) = 3\mu - 1 < 1\cdot(3-3\mu) + 2(3\mu-2) = 3\mu - 1\ .$$

□

The results presented in Lemmas 8–13 can be collected as follows.

Theorem 1. *Let $f \in \mathcal{C}_0(k)$, a_2 be a real number, and $a_2 \in [0,2]$. Then:*

$$|\Theta_f(\mu)| \leq \begin{cases} 4-6\mu, & \mu \leq 1/3\,, \\ 3-3\mu, & \mu \in (1/3,2/3)\ , \\ K, & \mu = 2/3\,, \\ 3\mu-1, & \mu \in (2/3,1)\ , \\ 6\mu-4, & \mu \geq 1\,, \end{cases}$$

where K is given by (25). The results are sharp for $\mu \leq 1/3$, $\mu = 2/3$, and $\mu \geq 1$. The equality holds for the Koebe function (2) in the first and the last case. The assumption $a_2 \in [0,2]$ is not necessary for $\mu \leq 0$ and $\mu \geq 1$.

5. Bounds of $|\Phi(\mu)|$ for the Class $\mathcal{C}_0(k)$

At the beginning of this section, we will quote the well known theorem of Marjono and Thomas [14].

Theorem 2 ([14])**.** *If $f \in \mathcal{C}_0(k)$, then:*

$$|\Phi_f(1)| = |a_2a_4 - a_3{}^2| \leq 1\,.$$

Now, we shall prove the bound for $\mu \geq 1$.

Theorem 3. *Let $f \in \mathcal{C}_0(k)$ and $\mu \geq 1$. Then, $|\Phi_f(\mu)| \leq 9\mu - 8$. The result is sharp.*

Proof. Rearranging the components in (14):

$$\begin{aligned}\Phi_f(\mu) = &\tfrac{1}{8}(p_1p_3 - p_2{}^2) - (\tfrac{1}{9}\mu - \tfrac{1}{8})p_2{}^2 + \tfrac{1}{4}(p_3 - p_1p_2) - (\tfrac{4}{9}\mu - \tfrac{1}{2})p_1p_2 \\ &- (\tfrac{2}{3}\mu - \tfrac{1}{2})p_2 - (\tfrac{4}{9}\mu - \tfrac{3}{8})p_1{}^2 - (\tfrac{4}{3}\mu - \tfrac{5}{4})p_1 - (\mu - 1)\,,\end{aligned}$$

and writing p instead of $|p_1|$, by Lemmas 1, 3, and 6, for $\mu \geq 9/8$, we obtain:

$$\begin{aligned}|\Phi_f(\mu)| &\leq \tfrac{1}{8}(4-p^2) + (\tfrac{4}{9}\mu - \tfrac{1}{2}) + \tfrac{1}{2} + (\tfrac{8}{9}\mu - 1)p + (\tfrac{4}{3}\mu - 1) \\ &\quad + (\tfrac{4}{9}\mu - \tfrac{3}{8})p^2 + (\tfrac{4}{3}\mu - \tfrac{5}{4})p + (\mu - 1) \\ &= (\tfrac{4}{9}\mu - \tfrac{1}{2})p^2 + (\tfrac{20}{9}\mu - \tfrac{9}{4})p + \tfrac{25}{9}\mu - \tfrac{3}{2} \\ &\leq 9\mu - 8\,.\end{aligned}$$

If $\mu \in (1, 9/8)$, then:

$$\Phi_f(\mu) = (9-8\mu)\left(a_2a_4 - a_3{}^2\right) + (8\mu - 8)\left(a_2a_4 - \tfrac{9}{8}a_3{}^2\right)\,.$$

From the previous part of this proof $|a_2a_4 - \frac{9}{8}a_3{}^2| \leq \frac{17}{8}$ and from Theorem 2, after using the triangle inequality, we get:

$$|\Phi_f(\mu)| \leq (9-8\mu)\cdot 1 + (8\mu-8)\cdot \tfrac{17}{8} = 9\mu - 8\,.$$

It is easy to verify that for the Koebe function (2), we have $\Phi_k(\mu) = 8 - 9\mu$, so the derived estimate is sharp. □

In the next step, we shall prove that the Koebe function (2) is the extremal function for $\mu \leq 63/92$.

Theorem 4. *Let $f \in \mathcal{C}_0(k)$ and $\mu \leq 63/92$. Then, $|\Phi_f(\mu)| \leq 8 - 9\mu$. The result is sharp.*

Proof. At the beginning, let us discuss the case $\mu = 63/92$. From (14), it follows that:

$$184\Phi_f\left(\tfrac{63}{92}\right) = 14(p_1p_3 - {p_2}^2) + 9p_1p_3 + 20(p_3 - \tfrac{1}{2}p_1p_2) \\ + 4(p_3 + 2p_2 + p_1) + 22p_3 + 58p_1 + 13{p_1}^2 + 58\,.$$

Now, applying Lemmas 1 and 4 for $\mu = 1/2$, Lemma 5 (remembering that $2(2 + \mathrm{Re}p_1) \le 2(2 + |p_1|)$), Lemma 6, and the triangle inequality and writing p instead of $|p_1|$, we obtain:

$$184|\Phi_f\left(\tfrac{63}{92}\right)| \le 14(4 - p^2) + 18p + 20h(p) + 8(2 + p) + 44 + 58p + 13p^2 + 58\,,$$

where:

$$h(p) = \begin{cases} \frac{1}{16}p^3 - \frac{3}{8}p^2 + 2\,, & p \in [0, 4/3]\,, \\ 2p - \frac{1}{2}p^3\,, & p \in [4/3, 2]\,. \end{cases}$$

Hence,

$$184|\Phi_f\left(\tfrac{63}{92}\right)| \le H(p)\,,$$

where:

$$H(p) = \begin{cases} \frac{5}{4}p^3 - \frac{17}{2}p^2 + 84p + 214\,, & p \in [0, 4/3]\,, \\ -10p^3 - p^2 + 124p + 174\,, & p \in [4/3, 2]\,. \end{cases} \tag{31}$$

Is it clear that H is an increasing function for $p \in [0, 2]$, so:

$$|\Phi_f\left(\tfrac{63}{92}\right)| \le H(2) = \tfrac{338}{184} = 8 - 9 \cdot \tfrac{63}{92}\,.$$

If $\mu \in (0, 63/92)$, then:

$$\Phi_f(\mu) = (1 - \tfrac{92}{63}\mu)a_2a_4 + \tfrac{92}{63}\mu\left(a_2a_4 - \tfrac{63}{92}{a_3}^2\right)\,.$$

From the previous part of this proof and the bound $|a_n| \le n$ valid for all functions in $\mathcal{C}_0(k)$,

$$|\Phi_f(\mu)| \le (1 - \tfrac{92}{63}\mu) \cdot 8 + \tfrac{92}{63}\mu \cdot \tfrac{338}{184} = 8 - 9\mu\,.$$

Equality holds for the Koebe function. □

It is worth adding that the function H given by (31) is decreasing for $p > 2$, so the choice $\mu = 63/92$ is important.

Now, we will find the exact bound of $\Phi_f(\mu)$ for μ close to one. Namely, we will discuss the case $\mu \in [\mu_0, 1]$, where:

$$\mu_0 = 18/19 = 0.947\ldots\,. \tag{32}$$

In this result, we need in addition that the coefficient a_2 should be real and $a_2 \in [0, 2]$. From (12), we get $p = p_1 \in [-2, 2]$. In the proof, we are going to apply Lemma 7.

Taking into account (14) and Lemma 2, we can write $\Phi_f(\mu)$ as follows:

$$144\Phi_f(\mu) = A_0 + A_1x + A_2x^2 + B(1 - |x|^2)y\,,$$

where:

$$A_0 = \tfrac{1}{2}\,(9-8\mu)\,p^4 + (27-32\mu)\,p^3 + 2\,(45-56\mu)\,p^2 + 12\,(15-16\mu)\,p + 144(1-\mu)\,,$$
$$A_1 = (4-p^2)\left[12(3-4\mu) + 4(9-8\mu)p + (9-8\mu)p^2\right]\,,$$
$$A_2 = -\tfrac{1}{2}(4-p^2)(2+p)\,[(9-8\mu)p + 16\mu]\,,$$
$$B = 9(4-p^2)(2+p)\,.$$

If $p=-2$ and $p=2$, then $f(z)=\frac{1}{2}\log\frac{1+z}{1-z}$ and $f(z)=\frac{z}{(1-z)^2}$, respectively, so:

$$\Phi_f(\mu) = -\mu/9 \quad \text{and} \quad \Phi_f(\mu) = 8-9\mu\,. \tag{33}$$

We will show that these values are less than or equal to the real bound of $|\Phi_f(\mu)|$ for all $f\in\mathcal{C}_0(k)$. Now and on, we assume that $p\in(-2,2)$. Taking into account (14) and Lemma 2, by the triangle inequality and the assumption $|y|\le 1$, we get:

$$|\Phi_f(\mu)| \le \tfrac{1}{16}(4-p^2)(2+p)\left[\left|a+bx+cx^2\right| + 1 - |x|^2\right]\,, \tag{34}$$

where:

$$\begin{aligned} a &= \frac{1}{9(4-p^2)(2+p)}\left[\tfrac{1}{2}(9-8\mu)p^4 + (27-32\mu)p^3 + 2(45-56\mu)p^2\right.\\ &\quad \left.+12(15-16\mu)p + 144(1-\mu)\right]\,,\\ b &= \frac{1}{9(2+p)}\left[(9-8\mu)p^2 + 4(9-8\mu)p + 12(3-4\mu)\right]\,,\\ c &= -\frac{1}{18}\left[(9-8\mu)p + 16\mu\right]\,. \end{aligned} \tag{35}$$

Now, we are ready to establish the main theorem of this section.

Theorem 5. *Let $f\in\mathcal{C}_0(k)$, a_2 be a real number, $a_2\in[0,2]$, and $\mu\in[\mu_0,1]$, where $\mu_0 = 18/19$. Then:*

$$|\Phi_f(\mu)| \le \frac{\mu}{9-8\mu}\,. \tag{36}$$

Equality holds for the function F given by (15).

In the proof of this theorem, we will need the two lemmas that follow. We assume that a, b, and c are given by (35).

Lemma 14. *If $(p,\mu)\in(-2,2)\times[\mu_0,1]$ are such that $a\le 0$, then (36) holds.*

Lemma 15. *If $(p,\mu)\in(-2,2)\times[\mu_0,1]$ are such that $a>0$, then the following inequalities hold:*

$$b<0\,,\ |b|\ge 2(1-|c|)\,,\ b^2\ge -4a(1-c^2)/c\,,\ |ab|\le |c|(|b|-4|a|)\,.$$

Proof of Lemma 14. At the beginning, observe that if $(p,\mu)\in(-2,2)\times[\mu_0,1]$, then:

$$c = -\tfrac{1}{18}\left[9p + 8(2-p)\mu\right] \le -\tfrac{1}{18}\left[9p + 8(2-p)\cdot\tfrac{18}{19}\right] = -\tfrac{1}{38}(3p+32) < 0\,. \tag{37}$$

According to Lemma 7 from (34), we obtain:

$$|\Phi_f(\mu)| \le \tfrac{1}{16}(4-p^2)(2+p)\cdot Y(a,b,c)\,,$$

where:

$$Y(a,b,c)=\begin{cases}-a+|b|-c & ,|b|\geq 2(1+c)\,,\\ 1-a+\dfrac{b^2}{4(1+c)} & ,|b|<2(1+c)\,.\end{cases}$$

If $|b|<2(1+c)$, then from (34), we get:

$$\begin{aligned}144|\Phi_f(\mu)| &\leq 9(4-p^2)(2+p)\\ &\quad-\left[\tfrac{1}{2}(9-8\mu)p^4+(27-32\mu)p^3+2(45-56\mu)p^2+12(15-16\mu)p+144(1-\mu)\right]\\ &\quad+\frac{\left[(9-8\mu)p^2+4(9-8\mu)p+12(3-4\mu)\right]^2}{2(9-8\mu)}\,.\end{aligned}$$

Because the right hand side of this inequality is constant and equal to $144\mu/(9-8\mu)$; hence, $|\Phi_f(\mu)|\leq \mu/(9-8\mu)$.

If $|b|\geq 2(1+c)$, then:

$$|\Phi_f(\mu)|\leq\begin{cases}\frac{1}{16}(4-p^2)(2+p)(-a+b-c)\,, & b\geq 0\,,\\ \frac{1}{16}(4-p^2)(2+p)(-a-b-c)\,, & b\leq 0\,.\end{cases} \tag{38}$$

The first expression in (38) is equal to:

$$\begin{aligned}&\tfrac{1}{144}\left[-2(9-8\mu)p^4-8(9-8\mu)p^3-24(3-4\mu)p^2+64\mu p+16\mu\right]\\ &=-\tfrac{1}{72}\left[(9-8\mu)p^2(p+2)^2-16\mu(p+1)^2+8\mu\right]\,.\end{aligned}$$

Substituting $q=p+1$, $q\in(-1,3)$, we obtain:

$$W_1(q)=-\tfrac{1}{72}\left[(9-8\mu)q^4-18q^2+9\right]=-\tfrac{1}{72}\left[(3-2\sqrt{2\mu})q^2-3\right]\cdot\left[(3+2\sqrt{2\mu})q^2-3\right]\,.$$

Hence, the maximum value of $W_1(q)$ is achieved for:

$${q_*}^2=\frac{1}{2}\left(\frac{3}{3-2\sqrt{2\mu}}+\frac{3}{3+2\sqrt{2\mu}}\right)=\frac{9}{9-8\mu}\,.$$

This value is equal to $W_1(q_*)=\mu/(9-8\mu)$.

The second expression in (38) is equal to:

$$W_2(p)=\tfrac{1}{18}\left[-(9-8\mu)p^2-4(9-10\mu)p-2(18-25\mu)\right]\,,$$

so:

$$W_2(p)\leq W_2\left(\frac{2(10\mu-9)}{9-8\mu}\right)=\frac{\mu}{9-8\mu}\,.$$

It is easy to check that for $p_*=q_*-1=3/\sqrt{9-8\mu}-1$ and $p_{**}=2(10\mu-9)/(9-8\mu)$, we have $b=2(1+c)$ and $b=-2(1+c)$, respectively. This means that the maximum value of $|\Phi_f(\mu)|$ for $|b|\geq 2(1+c)$ is obtained if $|b|=2(1+c)$. □

Proof of Lemma 15. Let $(p,\mu)\in(-2,2)\times[\mu_0,1]$. At the beginning, we want to constrain the range of variability of p to some subset of $(-2,2)$ for which $a>0$.

From (35) for $a=0$, we have:

$$\tfrac{1}{2}(9-8\mu)p^4+(27-32\mu)p^3+2(45-56\mu)p^2+12(15-16\mu)p+144(1-\mu)=0\,,$$

which is equivalent to:

$$9(p^2+2p+8)(2+p)^2 - 8(p^2+4p+6)^2\mu = 0\,.$$

If $p=0$, $\mu=0$, then from (35), $a=2$. Hence, points for which $a>0$ lie below the curve $a=0$. For the function $M(p)=9(p^2+2p+8)(2+p)^2/8(p^2+4p+6)^2$, $p\in(-2,2)$, there is:

$$M'(p) = \frac{9(2+p)}{4(p^2+4p+6)^3}\cdot(p^3+2p^2-10p-4)\,.$$

Consequently, $M(p)$ is an increasing function if $p\in(-2,p_0)$ and a decreasing function if $p\in(p_0,2)$ for $p_0=-0,376\ldots$, where p_0 is the only solution of $M'(p)=0$ in $(-2,2)$. Since $M(-1)<\mu_0$ and $M(2/3)<\mu_0$, then $M(p)<\mu_0$ for $p\in(-2,-1]\cup[2/3,2)$. This means that $a>0$ and $\mu\in[\mu_0,1]$ hold for $p\in I$, $I\subset(-1,2/3)$ (in other words, if $a>0$ and $\mu\in[\mu_0,1]$, then $-1<p<2/3$).

I. Since $\mu\in[\mu_0,1]$ and:

$$b=\frac{1}{9}\,(9-8\mu)\,(2+p)-\frac{16\mu}{9(2+p)}$$

as a function of $p\in(-1,2/3)$, is increasing, it is enough to estimate this expression taking $p=2/3$ as a limit value. Therefore,

$$b<\tfrac{2}{27}(36-41\mu)<0\,.$$

II. The inequality $-b\geq 2(1+c)$ can be written as $(8\mu-9)p+20\mu-18\geq 0$. For $\mu\in[\mu_0,1]$ and $p\in(-1,2/3)$,

$$(8\mu-9)p+20\mu-18>\tfrac{76}{3}\left(\mu-\tfrac{18}{19}\right)\geq 0\,.$$

III. With the notation $W=b^2+4a(1-c^2)/c$ and:

$$g(p,\mu)=(9-8\mu)\left[(16\mu-9)p^3+18(4\mu-3)p^2+4(25\mu-27)p\right]-8(32\mu^2-117\mu+81)\,,$$

we can write:

$$W=\frac{8g(p,\mu)}{9(2+p)^2[(9-8\mu)p+16\mu]}\,.$$

We shall prove that $g(p,\mu)\geq 0$ for $\mu\in[\mu_0,1]$ and $p\in(-1,2/3)$. We have:

$$\frac{\partial g}{\partial p}(p,\mu)=(9-8\mu)\left[3(16\mu-9)p^2+36(4\mu-3)p+4(25\mu-27)\right]\,.$$

For $\mu\in[\mu_0,1]$, we obtain:

$$\frac{\partial g}{\partial p}\,(-1,\mu)=(4\mu-27)(9-8\mu)<0\,,$$

and:

$$\frac{\partial g}{\partial p}\,(2/5,\mu)=4\,(1033\mu-972)\,(9-8\mu)\,/25>0\,.$$

This means that:

$$\min\{g(p,\mu): p\in(-1,2/3)\,,\ \mu\in[\mu_0,1]\}=\min\{g(p,\mu): p\in(-1,2/5]\,,\ \mu\in[\mu_0,1]\}\,.$$

Since $(16\mu - 9)p + 18(4\mu - 3) \geq 0$ for $\mu \in [\mu_0, 1]$ and $p \in (-1, 2/3)$, we have:

$$\begin{aligned}&\min\{g(p,\mu) : p \in (-1,2/3)\,,\ \mu \in [\mu_0,1]\}\\&> \min\left\{4(25\mu-27)(9-8\mu)p - 8(32\mu^2-117\mu+81) : p \in (-1,2/5]\,, \mu \in [\mu_0,1]\right\}\\&\geq 4(25\mu-27)(9-8\mu)\cdot 2/5 - 8(32\mu^2-117\mu+81)\\&= 144(-20\mu^2+57\mu-36)/5 > 0\,.\end{aligned}$$

In this way, we have proven that $b^2 + 4a(1-c^2)/c \geq 0$.

IV. Let us denote $V = c(b+4a) + ab$ and:

$$h(p,\mu) = 32(p^2+4p+6)(2p+5)^2\mu^2 - 36(2+p)(16p^2+71p+82)\mu + 81(2-p)(2+p)^3\,.$$

We have

$$V = \frac{4h(p,\mu)}{81(2+p)^2(4-p^2)}\,.$$

The function h of a variable μ increases for $\mu \in [\mu_0, 1]$. Indeed, for a fixed $p \in (-1, 2/3)$,

$$\begin{aligned}\frac{\partial h}{\partial \mu}(p,\mu) &= 64(p^2+4p+6)(2p+5)^2\mu - 36(2+p)(16p^2+71p+82)\\&\geq 64(p^2+4p+6)(2p+5)^2\cdot\tfrac{18}{19} - 36(2+p)(16p^2+71p+82)\\&= \tfrac{36}{19}\left[32(p^2+4p+6)(2p+5)^2 - 19(2+p)(16p^2+71p+82)\right]\\&= \tfrac{36}{19}\left[351+474(p+1)+395(p+1)^2+336(p+1)^3+128(p+1)^4\right]\end{aligned}$$

and is greater than zero. Finally,

$$\tfrac{361}{81}h(p,\tfrac{18}{19}) = 151p^4 + 732(p+1)p^2 + \tfrac{176}{3}p^2 + \tfrac{4}{3}(6-7p)^2 > 0\,,$$

so h, as well as V are positive for $\mu \in [\mu_0, 1]$ and $p \in (-1, 2/3)$. □

Proof of Theorem 4. From Lemma 14, we know that if $a \leq 0$ and $\mu \in [\mu_0, 1]$, then (36) holds. Assume now that $a > 0$ and $\mu \in [\mu_0, 1]$. By Lemmas 7 and 15, and Formula (37),

$$|\Phi_f(\mu)| \leq \tfrac{1}{16}(4-p^2)(2+p)(-|a|+|b|+|c|) = \tfrac{1}{16}(4-p^2)(2+p)(-a-b-c)\,.$$

This expression is the same as in the second line in (38), and it takes the maximum value $\mu/(9-8\mu)$ for $p = p_{**} = 2(10\mu-9)/(9-8\mu)$. Observe that the function $[\mu_0, 1] \ni \mu \mapsto 2(10\mu-9)/(9-8\mu)$ increases. Hence, $2/3 \leq p_{**} \leq 2$, so p_{**} is not less than $2/3$. For this reason, the maximum value of $|\Phi_f(\mu)|$ is equal to $\mu/(9-8\mu)$, but this value is obtained if $a \leq 0$.

It is easy to check that both values of $\Phi_f(\mu)$ for $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ and $f(z) = k(z)$, which are given in (33), are less than or equal to $\mu/(9-8\mu)$. This completes the proof. □

Theorem 6. *Let $f \in C_0(k)$, $\mu \in [63/92, 18/19]$, and a_2 be a real number, $a_2 \in [0, 2]$. Then:*

$$|\Phi_f(\mu)| \leq (396 - 361\mu)/81\,.$$

Proof. By Theorems 4 and 5, $|\Phi_f(63/92)| \leq 169/92$ and $|\Phi_f(18/19)| \leq 2/3$. Putting $\alpha = 4(414-437\mu)/459$, we can write:

$$\Phi_f(\mu) = \alpha(a_2a_4 - \tfrac{63}{92}{a_3}^2) + (1-\alpha)(a_2a_4 - \tfrac{18}{19}{a_3}^2)\,.$$

Applying the triangle inequality, we obtain our claim. □

The results presented in Theorems 2–6 can be collected as follows.

Corollary 1. *Let $f \in \mathcal{C}_0(k)$ be given by (1), a_2 be a real number, and $a_2 \in [0,2]$. Then:*

$$|\Phi_f(\mu)| \leq \begin{cases} 8-9\mu, & \mu \leq 63/92\,, \\ (396-361\mu)/81, & \mu \in [63/92, 18/19]\,, \\ \dfrac{\mu}{9-8\mu}, & \mu \in [18/19, 1]\,, \\ 9\mu-8, & \mu \geq 1\,, \end{cases}$$

The results are sharp for $\mu \leq 63/92$ and $\mu \geq 18/19$. The equality holds for the Koebe function (2) in the first and the last case. The function F given by (15) is an extremal function when $\mu \in [18/19, 1]$. The assumption $a_2 \in [0,2]$ is not necessary for $\mu \leq 63/92$ and $\mu \geq 1$.

Observe that for $\mu \in (18/19, 1)$, we have $\mu/(9-8\mu) < 1$, so the sharp bound for $\mathcal{C}_0(k)$ is less than the sharp bound for $\mathcal{S}^*$ given by (5).

6. Concluding Remarks

In this paper, we estimated two functionals $\Theta_f(\mu) = a_4 - \mu a_2 a_3$ and $\Phi_f(\mu) = a_2 a_4 - \mu {a_3}^2$ for the family $\mathcal{C}_0(k)$, where μ is a real number. This family is a subset of the class $\mathcal{C}$ of all close-to-convex functions.

The results presented above broaden our knowledge about the behavior of the coefficient functionals defined for functions not only in $\mathcal{C}$, but also generally in the class $\mathcal{S}$ of univalent functions. Unfortunately, there are no good estimates of the discussed functionals in the whole classes $\mathcal{C}$ and $\mathcal{S}$. It seems that further research on the classes of the type $\mathcal{C}_0(f)$, where f is different from k, may result in obtaining some conclusions about $\mathcal{S}$.

In our opinion, the most important problem to be solved now is the estimating of the second Hankel determinant, or in other words $\Phi_f(1)$ for $f \in \mathcal{S}$. Even in the class $\mathcal{C}_0$, the exact bound is unknown. It is only known that for $\mathcal{C}_0$, there is $|a_2 a_4 - {a_3}^2| < 1.242\ldots$ (see [25]). On the other hand, the conjecture posed by Thomas [26] about 30 years ago that $|a_n a_{n+2} - {a_{n+1}}^2| \leq 1$ for $\mathcal{S}$ and $n \geq 2$ was disproven. This means that there are functions in $\mathcal{S}$ for which $|a_n a_{n+2} - {a_{n+1}}^2| > 1$. Finding (even non-sharp) estimates of $\Phi_f(1)$ for $f \in \mathcal{S}$ remains an interesting open problem.

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