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Some Integral Inequalities for *h*-Godunova-Levin Preinvexity

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Abstract: In this study, we define new classes of convexity called *h*-Godunova–Levin and *h*-Godunova–Levin preinvexity, through which some new inequalities of Hermite–Hadamard type are established. These new classes are the generalization of several known convexities including the *s*-convex, *P*-function, and Godunova–Levin. Further, the properties of the *h*-Godunova–Levin function are also discussed. Meanwhile, the applications of *h*-Godunova–Levin Preinvex function are given.

Keywords: Hermite–Hadamard inequality; *h*-convexity; *h*-Godunova–Levin function; *h*-Godunova–Levin preinvex function

1. Introduction

Recently, the theory of convexity has become a broad area of study since it is related to the theory of inequalities. Many such inequalities are frequently reported in the literature as a result of applications of convexity in both pure and applied sciences (see [1–4]). Considering its many applications in different branches of mathematics, convexity can provide a basis for estimating error bounds in a large class of problems [5]. One example of these is how the convexity was applied to estimate errors when using a trapezoidal formula for numerical integration [6,7]. Others include studying problems in nonlinear programming and applying them to special means [8]. Among them, an interesting inequality for convex function is of Hermite–Hadamard type, which can be stated as follows:

Let S be a nonempty subset in $\mathbb{R}$, $\psi : S \to \mathbb{R}$ be a convex function on S, and $u_1, u_2 \in S, u_1 < u_2$, then we have

$$\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(x)dx \leq \frac{\psi(u_1)+\psi(u_2)}{2}. \tag{1}$$

If ψ is a concave function, the two inequalities can be held in the reverse direction. These inequalities have been extensively improved and generalized. For example, see [1,9–12].

Definition 1. *[13] A positive function $\psi : S \subseteq \mathbb{R} \to \mathbb{R}$ is said to be a Godunova–Levin if*

$$\psi(\delta u_1 + (1-\delta)u_2) \leq \frac{\psi(u_1)}{\delta} + \frac{\psi(u_2)}{1-\delta}, \forall u_1, u_2 \in S, \delta \in (0,1).$$

Several other properties related to this class of functions are given in [14–16]. For example, both the positive monotone and positive convex functions belong to this class.

This concept has been recently extended to s-Godunova–Levin type of convexity by Dragomir [17]. Furthermore, studies were conducted on s-Godunova–Levin type convexity and can be found in the literature [6]. Another important class of convex function is h-convexity, which was introduced by Varošanec [18], through which several generalizations and extensions were made.

Definition 2. *[18] Let $\psi, \omega : S \subseteq \mathbb{R} \to \mathbb{R}$ be two functions, such that $u_1, u_2 \in S$, the inequality $(\psi(u_1) - \psi(u_2))(\omega(u_1) - \omega(u_2)) \geq 0$ is called similarly ordered for ψ and ω on S.*

Now, the following preliminaries on invexity analysis are necessary since they can be frequently used throughout this study. Therefore, we let S be the nonempty subset in $\mathbb{R}$ and let $\psi : S \to \mathbb{R}$ and $\zeta(.,.) : S \times S \to \mathbb{R}$ be a continuous function.

Definition 3. *[19,20] A set S is said to be an invex set with respect to $\zeta(.,.)$ if, for every $u_1, u_2 \in S$, and $\delta \in [0,1]$*

$$u_1 + \delta\zeta(u_2, u_1) \in S.$$

Definition 4. *[20] A function ψ on the invex set S is said to be preinvex with respect to ζ if*

$$\psi(u_1 + \delta\zeta(u_2, u_1)) \leq (1-\delta)\psi(u_1) + \delta\psi(u_2); \forall u_1, u_2 \in S; \delta \in [0,1]. \tag{2}$$

Usually, the preinvex functions can be convexity if $\zeta(u_2, u_1) = u_2 - u_1$ holds in (2). Other properties of preinvex functions are given in [21,22].

We arrange this paper as follows. Section 2 introduces the new classes of h-Godunova–Levin, denoted by $SGX(\frac{1}{h}, t)$ and $SGV(\frac{1}{h}, t)$, together with their properties. This class of function unifies different classes of convexity: s-Godunova–Levin, P-functions, s-convexity, and Godunova–Levin. In Section 3, we prove new Hermite–Hadamard inequalities via h-Godunova–Levin preinvexity. Section 4 introduces a new definition of h-Godunova–Levin preinvexity, which can be the generalization of preinvexity. This Section also presents new Hermite–Hadamard type inequalities for h-Godunova–Levin preinvexity. Section 5 gives some applications to special means, as well as an application to numerical integration.

2. The h-Godunova–Levin Functions and Their Properties

This section introduces the notion of h-Godunova–Levin function together with their properties. This class of function can be denoted by $SGX(\frac{1}{h}, t)$ and $SGV(\frac{1}{h}, t)$ for h-Godunova–Levin convex and h-Godunova–Levin concave, respectively.

Definition 5. *Suppose $h : (0,1) \to \mathbb{R}$. A non-negative function $\psi : S \to \mathbb{R}$ is said to be h-Godunova–Levin, or that ψ belongs to the class $SGX(\frac{1}{h}, S)$, for all $u_1, u_2 \in S$ and $\delta \in (0,1)$, we have*

$$\psi(\delta u_1 + (1-\delta)u_2) \leq \frac{\psi(u_1)}{h(\delta)} + \frac{\psi(u_2)}{h(1-\delta)}. \tag{3}$$

Remark 1. *If $h(\delta) = \delta$, $h(\delta) = \frac{1}{\delta}$, $h(\delta) = \delta^s, h(\delta) = 1$, $h(\delta) = \frac{1}{\delta^s}$ in inequality (3), the definition of h-Godunova–Levin function can be clearly reduced to different types of convexity, such as Godunova–Levin function, classical convex, s-Godunova–Levin function, P-function, and s-convex function. This indicates that h-Godunova–Levin function is the generalization of these different classes.*

Proposition 1. *Suppose that h_1, h_2 are two positive functions defined on the interval S satisfying the property*

$$\frac{1}{h_1(\delta)} \leq \frac{1}{h_2(\delta)}, \ \ \delta \in (0,1).$$

If $\psi \in SGX(\frac{1}{h_1}, S)$, then $\psi \in SGX(\frac{1}{h_2}, S)$. If $\psi \in SGV(\frac{1}{h_1}, S)$, then $\psi \in SGV(\frac{1}{h_2}, S)$, where $h_1(t) \neq 0$ and $h_2(t) \neq 0$.

Proof. If $\psi \in SGX(\frac{1}{h_1}, S)$, then for any $u_1, u_2 \in S$ and $\delta \in (0,1)$ we get

$$\begin{aligned}\psi(\delta u_1 + (1-\delta)u_2) &\leq \frac{1}{h_1(\delta)}\psi(u_1) + \frac{1}{h_1(1-\delta)}\psi(u_2)\\ &\leq \frac{1}{h_2(\delta)}\psi(u_1) + \frac{1}{h_2(1-\delta)}\psi(u_2),\end{aligned}$$

i.e., $\psi \in SGX(\frac{1}{h_2}, S)$. □

Proposition 2. *If $\psi, \omega \in SGX(\frac{1}{h}, S)$ and $\lambda > 0$, then $\psi + \omega, \lambda\psi \in SGX(\frac{1}{h}, S)$. If $\psi, \omega \in SGV(\frac{1}{h}, S)$ and $\lambda > 0$, then $\psi + \omega, \lambda\psi \in SGV(\frac{1}{h}, S)$.*

Proof. The proof is clear from the definition of the classes h-Godunova–Levin convex and h-Godunova–Levin concave, $SGX(\frac{1}{h}, S)$ and $SGV(\frac{1}{h}, S)$, respectively. □

Proposition 3. *Suppose that ψ and ω are two h-Godunova–Levin functions and satisfying the property given in Definition 2. Then, the product of these two functions satisfies*

$$\psi(\delta u_1 + (1-\delta)u_2)\omega(\delta u_1 + (1-\delta)u_2) \leq \left[\frac{1}{h(\delta)}\psi(u_1)\omega(u_1) + \frac{1}{h(1-\delta)}\psi(u_2)\omega(u_2)\right]\left[\frac{1}{h(\delta)} + \frac{1}{h(1-\delta)}\right].$$

Proof. Given that ψ and ω are h-Godunova–Levin functions, we have

$$\begin{aligned}\psi(\delta u_1 + (1-\delta)u_2)\omega(\delta u_1 + (1-\delta)u_2) \leq& \left(\frac{\psi(u_1)}{h(\delta)} + \frac{\psi(u_2)}{h(1-\delta)}\right)\left(\frac{\omega(u_1)}{h(\delta)} + \frac{\omega(u_2)}{h(1-\delta)}\right)\\ =& \frac{1}{(h(\delta))^2}\psi(u_1)\omega(u_1) + \frac{1}{h(\delta)h(1-t)}[\psi(u_1)\omega(u_2)\\ &+ \psi(u_2)\omega(u_1)] + \frac{1}{(h(1-\delta))^2}\psi(u_2)\omega(u_2)\\ =& \left[\frac{1}{h(\delta)}\psi(u_1)\omega(u_1) + \frac{1}{h(1-\delta)}\psi(u_2)\omega(u_2)\right]\left[\frac{1}{h(\delta)} + \frac{1}{h(1-\delta)}\right].\end{aligned}$$

□

Proposition 4. *Suppose that $\psi : S_1 \to [0,\infty)$, $\omega : S_2 \to [0,\infty)$ are two functions such that $\omega(S_2) \subseteq S_1$. If the function ω is convex (concave), and the function ψ is increasing (decreasing), $\psi \in SGX(\frac{1}{h}, S_1)$, then the composition $\psi \circ \omega$ belongs to $SGX(\frac{1}{h}, S_2)$. Meanwhile, if the function ω is convex (concave) and the function ψ is decreasing (increasing), $\psi \in SGV(\frac{1}{h}, S_1)$, then the composition $\psi \circ \omega$ belongs to $SGV(\frac{1}{h}, S_2)$.*

Proof. Suppose that ω is a convex function, ψ is increasing, and $\psi \in SGX(\frac{1}{h}, S_1)$. Then, we have

$$(\psi \circ \omega)((\delta u_1 + (1-\delta)u_2)) \leq \psi(\delta\omega(u_1) + (1-\delta)\omega(u_2)) \leq \frac{1}{h(\delta)}(\psi \circ \omega)(u_1) + \frac{1}{h(1-\delta)}(\psi \circ \omega)(u_2),$$

for all $u_1, u_2 \in S_2$, and $\delta \in (0,1)$. □

3. New Hermite–Hadamard Inequality for h-Godunova–Levin Convex Function

The following generalization of the Hermite–Hadamard inequalities for h-Godunova–Levin convex function can be proved in this section.

Theorem 1. *Let $\psi \in SGX(\frac{1}{h}, S)$, $u_1, u_2 \in S$, with $u_1 < u_2$ and $\psi \in L_1([u_1, u_2])$, where $h : (0,1) \to \mathbb{R}$ is a positive function and $h(\delta) \neq 0$, we have*

$$\frac{h\left(\frac{1}{2}\right)}{2}\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(x)dx \leq [\psi(u_1)+\psi(u_2)]\int_0^1 \frac{1}{h(\delta)}d\delta. \tag{4}$$

Proof. Since ψ is h-Godunova–Levin, we have

$$\psi(\delta u_1 + (1-\delta)u_2) \leq \frac{\psi(u_1)}{h(\delta)} + \frac{\psi(u_2)}{h(1-\delta)}. \tag{5}$$

Considering $v_1 = \alpha u_1 + (1-\alpha)u_2$, $v_2 = (1-\alpha)u_1 + \alpha u_2$, and $\delta = \frac{1}{2}$ in (5), we obtain

$$\begin{aligned}\psi\left(\frac{u_1+u_2}{2}\right) &\leq \frac{1}{h(\frac{1}{2})}\psi(\alpha u_1 + (1-\alpha)u_2) + \frac{1}{h(\frac{1}{2})}\psi((1-\alpha)u_1 + \alpha u_2)\\ &\leq \frac{1}{h(\frac{1}{2})}[\psi(\alpha u_1 + (1-\alpha)u_2) + \psi((1-\alpha)u_1 + \alpha u_2)].\end{aligned} \tag{6}$$

Thus, after integrating (6), we get the following

$$\begin{aligned}\psi\left(\frac{u_1+u_2}{2}\right) &\leq \frac{1}{h(\frac{1}{2})}\left[\int_0^1 \psi(\alpha u_1 + (1-\alpha)u_2)d\alpha + \int_0^1 \psi((1-\alpha)u_1 + \alpha u_2)d\alpha\right]\\ &\leq \frac{2}{h(\frac{1}{2})(u_2-u_1)}\int_{u_1}^{u_2}\psi(x)dx.\end{aligned}$$

This ends the proof of the first inequality. Now, taking $v_1 = u_1$ and $v_2 = u_2$ in (5) and integrating the result over the interval $[0,1]$ with respect to δ, we obtain

$$\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(x)dx \leq [\psi(u_1)+\psi(u_2)]\int_0^1 \frac{1}{h(\delta)}d\delta.$$

This completes the proof of the second inequality (4).

□

Remark 2. *In Theorem 1, choosing $h(\delta) = \delta^s$, we obtain the Hermite–Hadamard inequalities for s-convexity in the second sense, Theorem 2.1. in [23]. If we choose $h(\delta) = 1$, Theorem 1 can be reduced to the result for P-function [12]. Taking $h(\delta) = \frac{1}{\delta}$, the theorem reduces the result for classical Hermite–Hadamard inequalities given in inequality (1).*

4. Hermite–Hadamard Inequalities for h-Godunova–Levin Preinvex Function

The definition of h-Godunova–Levin preinvex is introduced in this section. The inequalities of Hermite–Hadamard type for functions whose first derivatives absolute values are h-Godunova–Levin preinvex are also presented here.

Definition 6. *A function $\psi : S \to \mathbb{R}$ is said to be h-Godunova–Levin preinvex function with respect to ζ if, for all $u_1, u_2 \in S, \delta \in (0,1)$,*

$$\psi(u_1 + \delta\zeta(u_2, u_1)) \leq \frac{\psi(u_1)}{h(1-\delta)} + \frac{\psi(u_2)}{h(\delta)}$$

holds.

The following lemma can be used to prove the generalization of the Hermite–Hadamard inequalities for *h*-Godunova–Levin preinvexity.

Lemma 1. *[24] Suppose that $\psi : S = [u_1, u_1 + \zeta(u_2, u_1)] \to (0, \infty)$ is a differentiable function, where $u_1, u_1 + \zeta(u_2, u_1) \in S$ with $u_1 < u_1 + \zeta(u_2, u_1)$. If $\psi' \in L1[u_1, u_1 + \zeta(u_2, u_1)]$, we have*

$$\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx - \frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2} = \frac{\zeta(u_2,u_1)}{2}\left[\int_0^1 (1-2t)\psi'(u_1+\delta\zeta(u_2,u_1))d\delta\right].$$

Theorem 2. *Suppose that $\psi : S = [u_1, u_1 + \zeta(u_2, u_1)] \to (0, \infty)$ is a differentiable mapping on S°, $u_1, \zeta(u_2, u_1) \in S^\circ$, with $u_1 < u_1 + \zeta(u_2, u_1)$. If $|\psi'|$ is a h-Godunova–Levin preinvex on $[u_1, u_1 + \zeta(u_2, u_1)]$, then we get the following inequality:*

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2} - \frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| &\leq \frac{\zeta(u_2,u_1)}{2}[|\psi'(u_1)|+|\psi'(u_2)|] \\ &\times \int_0^1 |1-2\delta|\left[\frac{1}{h(\delta)}+\frac{1}{h(1-\delta)}\right]d\delta. \end{aligned} \tag{7}$$

Proof. We use Lemma 1 to prove inequality (7) as follows:

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2} - \frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| &= \left|\frac{\zeta(u_2,u_1)}{2}\int_0^1 (1-2\delta)\psi'(u_1+\delta\zeta(u_2,u_1))d\delta\right| \\ &\leq \frac{\zeta(u_2,u_1)}{2}\int_0^1 |1-2\delta||\psi'(u_1+\delta\zeta(u_2,u_1))d\delta| \\ &\leq \frac{\zeta(u_2,u_1)}{2}\int_0^1 |1-2\delta|\left|\frac{\psi'(u_1)}{h(\delta)}+\frac{\psi'(u_2)}{h(1-\delta)}\right|d\delta \\ &\leq \frac{\zeta(u_2,u_1)}{2}[|\psi'(u_1)|+|\psi'(u_2)|] \\ &\times \int_0^1 |1-2\delta|\left[\frac{1}{h(\delta)}+\frac{1}{h(1-\delta)}\right]d\delta.\end{aligned}$$

□

Corollary 1. *Since $\int_0^1 \frac{1}{h(1-\delta)}d\delta = \int_0^1 \frac{1}{h(\delta)}d\delta$, subsituting this fact in inequality (7), we get*

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2} - \frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| &\leq \frac{\zeta(u_2,u_1)}{2}[|\psi'(u_1)|+|\psi'(u_2)|] \\ &\times \int_0^1 \frac{|1-2\delta|}{h(\delta)}d\delta. \end{aligned} \tag{8}$$

Corollary 2. *Taking $\zeta(u_2, u_1) = u_2 - u_1$ in inequality (8), we obtain the following inequality:*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(x)dx\right| \leq \frac{u_2-u_1}{2}[|\psi'(u_1)|+|\psi'(u_2)|]\int_0^1 \frac{|1-2\delta|}{h(\delta)}d\delta.$$

Theorem 3. *Suppose that $\psi : S = [u_1, u_1 + \zeta(u_2, u_1)] \to (0, \infty)$ is a differentiable function on S°, $u_1, \zeta(u_2, u_1) \in S^\circ$, with $u_1 < u_1 + \zeta(u_2, u_1)$. If $|\psi'|$ is a h-Godunova–Levin preinvex on $[u_1, u_1 + \zeta(u_2, u_1)]$,*

with $p > 1$ such that $q = \frac{p}{p-1}$, we obtain

$$\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2}-\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| \leq \frac{\zeta(u_2,u_1)}{(p+1)^{\frac{1}{p}}}(|\psi'(u_1)|^q + |\psi'(u_2)|^q)^{\frac{1}{q}}\int_0^1\frac{1}{h(\delta)}d\delta.$$

Proof. Applying Lemma 1, we have

$$\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2}-\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| = \left|\frac{\zeta(u_2,u_1)}{2}\int_0^1(1-2\delta)\psi'(u_1+\delta\zeta(u_2,u_1))d\delta\right| \leq \frac{\zeta(u_2,u_1)}{2}\int_0^1|1-2\delta||\psi'(u_1+\delta\zeta(u_2,u_1))d\delta|.$$

We use Hölder's integral inequality as follows:

$$\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2}-\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| \leq \frac{\zeta(u_2,u_1)}{2}\left(\int_0^1|1-2\delta|^pd\delta\right)^{\frac{1}{p}} \times\left(\int_0^1|\psi'(u_1+\delta\zeta(u_2,u_1))|^qd\delta\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Now, since $|\psi'|^p$ is a h-Godunova–Levin preinvex, we obtain

$$\int_0^1|\psi'(u_1+\delta\zeta(u_2,u_1))|^qd\delta \leq \int_0^1\left(\frac{|\psi'(u_1)|^q}{h(\delta)}+\frac{|\psi'(u_2)|^q}{h(1-\delta)}\right)d\delta \leq 2\int_0^1\frac{1}{h(\delta)}d\delta(|\psi'(u_1)|^q+|\psi'(u_2)|^q).$$

Using the basic calculus, we have $\int_0^1|1-2\delta|^pd\delta = \frac{1}{p+1}$. This completes the proof of the Theorem 3. □

Corollary 3. *Choosing $\zeta(u_2,u_1) = u_2-u_1$ in Theorem 3 reduces inequality (9) to the following:*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(x)dx\right| \leq \frac{u_2-u_1}{(p+1)^{\frac{1}{p}}}(|\psi'(u_1)|^q+|\psi'(u_2)|^q)^{\frac{1}{q}}\int_0^1\frac{1}{h(\delta)}d\delta.$$

Theorem 4. *With the assumptions of Theorem 3, we get the following:*

$$\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2}-\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right| \leq \frac{\zeta(u_2,u_1)}{4}\left(\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta\right)^{\frac{1}{q}} \times(|\psi'(u_1)|^q+|\psi'(u_2)|^q). \quad (9)$$

Proof. We use Lemma 1 to show that

$$\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2}-\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right|=\left|\frac{\zeta(u_2,u_1)}{2}\int_0^1(1-2\delta)\psi'(u_1+\delta\zeta(u_2,u_1))d\delta\right|$$
$$\leq\frac{\zeta(u_2,u_1)}{2}\int_0^1|1-2\delta||\psi'(u_1+\delta\zeta(u_2,u_1))d\delta|.$$

Applying power-mean inequality, we get

$$\left|\frac{\psi(u_1)+\psi(u_1+\zeta(u_2,u_1))}{2}-\frac{1}{\zeta(u_2,u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)}\psi(x)dx\right|\leq\frac{\zeta(u_2,u_1)}{2}\left(\int_0^1|1-2\delta|d\delta\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1|1-2\delta||\psi'(u_1+\delta\zeta(u_2,u_1))|^q d\delta\right)^{\frac{1}{q}}.$$

Since $|\psi'|^q$ is a h-Godunova–Levin preinvex, we obtain

$$\int_0^1|1-2\delta||\psi'(u_1+\delta\zeta(u_2,u_1))|^q d\delta\leq\int_0^1\left(\frac{|1-2\delta|}{h(\delta)}|\psi'(u_1)|^q+\frac{|1-2\delta|}{h(1-\delta)}|\psi'(u_2)|^q\right)d\delta$$
$$\leq\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta(|\psi'(u_1)|^q+|\psi'(u_2)|^q).$$

Applying the basic calculus, we have $\int_0^1|1-2\delta|d\delta=\frac{1}{2}$. □

Corollary 4. *Taking $\zeta(u_2,u_1)=u_2-u_1$, $h(\delta)=1$ and $q=1$ in inequality (9), we have*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_1}\psi(x)dx\right|\leq\frac{u_2-u_1}{8}(|\psi'(u_1)|+|\psi'(u_2)|),$$

which is similar to Theorem 2.2 repoted by Dragomir and Agarwal [4].

5. Applications

5.1. Applications to Numerical Integration

As mentioned in the introduction, the convexity can be applied to many areas of studies. Here, we give an example of how the h-Godunova–Levin convex and preinvex functions can be used to estimate the errors accumulated when using the trapezoidal formula for numerical integration.

Let d be a division of the interval $[u_1,u_2]$, i.e., $d: u_1=v_0<v_1<\cdots<v_{n-1}<v_n=u_2$, of a given quadrature formula

$$\int_{u_1}^{u_2}\psi(x)dx\cong T(\psi,d)+E(\psi,d),$$

where

$$T(\psi,d)=\sum_{i=0}^{n-1}\frac{\psi(v_i)+\psi(v_{i+1})}{2}(v_{i+1}-v_i)$$

is the trapezoidal formula. The associated approximation error is denoted by $E(\psi,d)$.

Proposition 5. *Let ψ be a differentiable mapping on S°, $u_1, u_2 \in S^\circ$ with $u_1 < u_2$. If $|\psi'|$ is a h-Godunova–Levin preinvex on $[u_1, u_1 + \zeta(u_2, u_1)]$, then for every division d of $[u_1, u_2]$, we have*

$$\begin{aligned} |E_n(\psi, d)| &\leq \frac{1}{2}\sum_{i=0}^{n-1}(v_{i+1}-v_i)^2\left(|\psi'(v_i)|+|\psi'(v_{i+1})|\right)\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta \\ &\leq \sum_{i=0}^{n-1}(v_{i+1}-v_i)^2\max\left\{|\psi'(u_1)|,|\psi'(u_2)|\right\}\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta. \end{aligned} \tag{10}$$

Proof. We now apply Corollary (2) on the subinterval $[v_i, v_{i+1}]$ $(i = 0, 1, 2, \ldots, n-1)$ of the division d. This gives the following:

$$\left|\frac{\psi(v_i)+\psi(v_{i+1})}{2}(v_{i+1}-v_i)-\int_{v_i}^{v_{i+1}}\psi(x)\mathrm{d}x\right| \leq \frac{(v_{i+1}-v_i)^2\left(|\psi'(v_i)|+|\psi'(v_{i+1})|\right)}{2}\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta.$$

Since $|\psi'|$ is h-Godunova–Levin preinvex, using the triangle inequality and summing the result over i from 0 to $n-1$, we get

$$\begin{aligned} \left|T(\psi,d)-\int_{u_1}^{u_2}\psi(x)\mathrm{d}x\right| &\leq \frac{1}{2}\sum_{i=0}^{n-1}(v_{i+1}-v_i)^2\left(|\psi'(x_i)|+|\psi'(v_{i+1})|\right) \\ &\leq \max\left\{|\psi'(v_i)|,|\psi'(v_{i+1})|\right\}\sum_{i=0}^{n-1}(v_{i+1}-v_i)^2\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta \\ &\leq \max\left\{|\psi'(u_1)|,|\psi'(u_2)|\right\}\sum_{i=0}^{n-1}(v_{i+1}-v_i)^2\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta. \end{aligned}$$

□

The above inequality is an error bound of numerical integration obtained by h-Godunova–Levin preinvex. Choosing different functions of $h(\delta)$ in inequality (10) can give different results (see [6]).

5.2. Applications to Special Means

We finally use Hermite–Hadamard inequalities for h-Godunova–Levin preinvex function to form the inequalities for special means. Thus, the means of two positive numbers u_1, u_2, and $u_1 \neq u_2$ can be considered as follows:

1. The arithmetic mean:
   $A = A(u_1, u_2) = \dfrac{u_1+u_2}{2}$; $u_1, u_2 \in \mathbb{R}$, with $u_1, u_2 > 0$.
2. The generalized log-mean:
   $L_m(u_1, u_2) = \left[\dfrac{u_2^{m+1}-u_1^{m+1}}{(m+1)(u_2-u_1)}\right]^{\frac{1}{m}}$, $m \neq -1, 0$.

The following propositions are obtained from the results in Section 4 and the above applications of special means.

Proposition 6. *Let $0 < u_1 < u_2$, where $m \geq 2$, then we have*

$$\left|A(u_1^m, u_2^m)-L_m^m(u_1,u_2)\right| \leq \frac{m(u_2-u_1)}{2}A(|u_1^{m-1}|,|u_2^{m-1}|)\int_0^1\frac{|1-2\delta|}{h(\delta)}d\delta.$$

Proof. This inequality is obtained from Corollary (2) and applied on the h-Godunova–Levin preinvex function $\psi : \mathbb{R} \to \mathbb{R}, \psi(x) = x^m, m \geq 2$. □

Proposition 7. *Let* $0 < u_1 < u_2$, *where* $p > 1, q = \frac{p}{p-1}$ *and* $m \geq 2$, *then we get*

$$\left| A(u_1^m, u_2^m) - L_m^m(u_1, u_2) \right| \leq \frac{m(u_2 - u_1)}{(p+1)^{\frac{1}{p}}} A(|u_1|^{\frac{(m-1)p}{p-1}}, |u_2|^{\frac{(m-1)p}{p-1}})^{\frac{1}{q}} \int_0^1 \frac{1}{h(\delta)} d\delta.$$

Proof. We derived this inequality from Corollary 3 applied to the h-Godunova–Levin preinvex function $\psi : \mathbb{R} \to \mathbb{R}$. □

6. Conclusions

Since the Hermite–Hadamard type inequalities, due to their importance, can be found in many fields of study, the present study established new generalizations of such inequalities. Thus, two classes of function, h-Godunova–Levin and h-Godunova–Levin preinvex functions, along with some of their properties were established here. The applications to special means and numerical integration were also discussed in this study.

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