Hermite–Hadamard and Fejér Inequalities for Co-Ordinated \((F, G)-\)Convex Functions on a Rectangle

Małgorzata Chudziak † and Marek Żołdak *†

College of Natural Sciences, Institute of Mathematics, University of Rzeszów, Pigonia 1, 35-310 Rzeszów, Poland; mchudziak@ur.edu.pl
* Correspondence: marek_z2@op.pl
† These authors contributed equally to this work.

Received: 14 November 2019; Accepted: 15 December 2019; Published: 19 December 2019

Abstract: We introduce the notion of a co-ordinated \((F, G)-\)convex function defined on an interval in \(\mathbb{R}^2\) and we prove the Hermite–Hadamard and Fejér type inequalities for such functions.

Keywords: Hermite–Hadamard inequality; Fejér inequality; approximate convexity

MSC: 26A51; 26B25

1. Introduction

The celebrated inequality states that, if \(f : [a, b] \to \mathbb{R}\) is a convex function, then
\[
f\left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]
Furthermore, if \(p : [a, b] \to [0, \infty)\) is an integrable function symmetric with respect to \(\frac{a+b}{2}\), that is
\[
p(a + b - x) = p(x) \quad \text{for } x \in [a, b],
\]
then the following weighted generalization of the Hermite–Hadamard inequality is known as the Fejér inequality
\[
f\left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)p(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

Dragomir [1] established a counterpart of the Hermite–Hadamard inequality for co-ordinated convex functions, that is functions \(f : [a, b] \times [c, d] \to \mathbb{R}\) which are convex with respect to each variable separately. It has been proven in [1] that for such functions, the following inequalities hold
\[
f\left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x, \frac{c+d}{2})dx + \frac{1}{d-c} \int_c^d f\left( \frac{a+b}{2}, y \right)dy
\]
\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dydx
\]
\[
\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c)dx + \frac{1}{b-a} \int_a^b f(x, d)dx + \frac{1}{d-c} \int_c^d f(a, y)dy + \frac{1}{d-c} \int_c^d f(b, y)dy \right]
\]
Refinement versions of these inequalities have been presented in [1–3].

A counterpart of the Fejér inequality for co-ordinated convex functions has been formulated by Alomari and Darus [4]. They proved that if \( p : [a, b] \times [c, d] \rightarrow [0, \infty) \) is an integrable function symmetric with respect to the lines \( x = \frac{a+b}{2} \) and \( y = \frac{c+d}{2} \), i.e.,

\[
p(a + b - x, y) = p(x, y) \quad \text{for} \quad x \in [a, b], \ y \in [c, d]
\]

and

\[
p(x, c + d - y) = p(x, y) \quad \text{for} \quad x \in [a, b], \ y \in [c, d],
\]

then for every co-ordinated convex function the following inequalities hold

\[
f\left( \frac{a + b}{2}, \ c + d \right) \leq \frac{1}{\int_d^b \int_c^d f(x, y)p(x, y)dydx} \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

In recent years, several modifications of the notion of convexity were studied by many authors (see e.g., [5–9]). The following general definition was introduced in [10].

**Definition 1.** Let \( F : [0, 1] \times [a, b] \times [a, b] \rightarrow \mathbb{R} \) be a continuous function. A function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be convex with respect to \( F \), or briefly \( F \)-convex, provided

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + F(t, x, y) \quad \text{for} \quad x, y \in [a, b], \ t \in [0, 1].
\]

In particular, if \( F \) is of the form

\[
F(t, x, y) = Ct(1 - t)|x - y| \quad \text{for} \quad x, y \in [a, b], \ t \in [0, 1],
\]

where \( C \in (0, \infty) \), then any function \( f : [a, b] \rightarrow \mathbb{R} \) satisfying (3) is called approximately convex. Furthermore, if \( f : [a, b] \rightarrow \mathbb{R} \) satisfies (3) with \( F \) given by

\[
F(t, x, y) = -Ct(1 - t)(x - y)^2 \quad \text{for} \quad x, y \in [a, b], \ t \in [0, 1],
\]

where \( C \in (0, \infty) \), then it is called strongly convex with modulus \( C \). For some applications of \( F \)-convex functions in the optimization theory and in the theory of partial differential equations we refer to [11] and [12], respectively.

It should be noted here that, although a definition of the \( F \)-convex function does not require any additional properties of \( F \), it is reasonable to assume that \( F \) is symmetric, that is

\[
F(1 - t, y, x) = F(t, x, y) \quad \text{for} \quad x, y \in [a, b], \ t \in [0, 1].
\]

In fact, if \( f \) is \( F \)-convex then there exists a symmetric function \( F_s \) such that \( f \) is \( F_s \)-convex and

\[
F_s(t, x, y) \leq F(t, x, y) \quad \text{for} \quad x, y \in [a, b], \ t \in [0, 1].
\]

To find this, one could take

\[
F_s(t, x, y) := \min\{F(t, x, y), F(1 - t, y, x)\} \quad \text{for} \quad x, y \in [a, b], \ t \in [0, 1].
\]
Note that $F$ given by (4) or (5) is symmetric. Moreover, a symmetry of $F$ is a necessary condition for the existence of an $F$-affine function, i.e., a function satisfying equation

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) + F(t,x,y) \quad \text{for} \quad x,y \in [a,b], \ t \in [0,1].$$

In what follows we deal with the functions of two variables, which are $F$-convex with respect to each variable.

**Definition 2.** Let $F : [c,d] \times [0,1] \times [a,b] \times [a,b] \to \mathbb{R}$, $G : [a,b] \times [0,1] \times [c,d] \times [c,d] \to \mathbb{R}$ be continuous functions. We call a function $f : [a,b] \times [c,d] \to \mathbb{R}$ co-ordinated $(F,G)$-convex, provided

$$f(tx_1 + (1-t)x_2, y) \leq tf(x_1, y) + (1-t)f(x_2, y) + F(y, t, x_1, x_2),$$

$$f(x, ty_1 + (1-t)y_2) \leq tf(x, y_1) + (1-t)f(x, y_2) + G(x, t, y_1, y_2)$$

for $t \in [0,1], x_1, x_2 \in [a,b], y_1, y_2 \in [c,d], x \in [a,b], y \in [c,d]$.

Following the remark formulated above, we restrict our attention to the case where $F(y, \cdot, \cdot, \cdot)$ for $y \in [c,d]$ and $G(x, \cdot, \cdot, \cdot)$ for $x \in [a,b]$ are symmetric functions, i.e.,

$$F(y, 1-t, x_1, x_2) = F(y, t, x_1, x_2) \quad \text{for} \quad x_1, x_2 \in [a,b], \ y \in [c,d], \ t \in [0,1]$$

and

$$G(x, 1-t, y_1, y_2) = G(x, t, y_1, y_2) \quad \text{for} \quad x \in [a,b], \ y_1, y_2 \in [c,d], \ t \in [0,1],$$

respectively. This assumption will not be repeated. Our main aim is to present the Hermite–Hadamard and the Fejér type inequalities for co-ordinated $(F,G)$-convex functions.

## 2. Results

### 2.1. Hermite–Hadamard Type Inequalities

In this section, we prove the Hermite–Hadamard type inequalities for $(F,G)$-convex functions. Our proof is based on some methods used in [1,3]. We begin with the result establishing the Hermite–Hadamard type inequalities for $F$-convex functions. It will be useful in further considerations.

**Theorem 1.** Let $F : [0,1] \times [a,b] \times [a,b] \to \mathbb{R}$ be a continuous symmetric function (cf. (6)). If $f : [a,b] \to \mathbb{R}$ is an integrable $F$-convex function then

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t)dt + \frac{1}{b-a} \int_a^b F \left( \frac{1}{2}, x, a+b-x \right) dx$$

and

$$\frac{1}{b-a} \int_a^b f(t)dt \leq f(a) + f(b) + \int_0^1 F(t, a, b)dt.$$  \hspace{1cm} (8)

**Proof.** Assume that $f : [a,b] \to \mathbb{R}$ is an integrable $F$-convex function. In view of (3), we obtain

$$\frac{1}{b-a} \int_a^b f(s)ds = \int_0^1 f(ta + (1-t)b)dt \leq \frac{1}{2} f(a) + \frac{1}{2} f(b) + \int_0^1 F(t, a, b)dt,$$

which gives (8). Note also that, as $f$ is $F$-convex, we have

$$f \left( \frac{x+y}{2} \right) \leq \frac{f(x) + f(y)}{2} + F \left( \frac{1}{2}, x, y \right) \quad \text{for} \quad x, y \in [a,b].$$
Setting in (9) \( x = ta + (1-t)b, y = tb + (1-t)a \), where \( t \in [0,1] \), and integrating obtained in this way inequality with respect to \( t \), we obtain (7). \( \square \)

Now, we are going to formulate and prove the Hermite–Hadamard type inequalities for co-ordinated \((F,G)\)-convex functions.

**Theorem 2.** Assume that \( f : [a,b] \times [c,d] \rightarrow \mathbb{R} \) is an integrable co-ordinated \((F,G)\)-convex function. Then:

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( \frac{c+d}{2}, \frac{x}{2}, x, a+b-x \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy \right] + R_1, \tag{10}
\]

where

\[
R_1 = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b F \left( \frac{c+d}{2}, \frac{x}{2}, x, a+b-x \right) dx + \frac{1}{d-c} \int_c^d G \left( \frac{a+b}{2}, \frac{y}{2}, y, c+d-y \right) dy \right];
\]

\[
\frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( \frac{c+d}{2}, \frac{x}{2}, y, c+d-y \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy \right] \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx + R_2, \tag{11}
\]

where

\[
R_2 = \frac{1}{2(b-a)(d-c)} \left[ \int_a^b \int_c^d G \left( \frac{x}{2}, y, c+d-y \right) dydx + \int_a^b \int_c^d F \left( \frac{y}{2}, x, a+b-x \right) dydx \right];
\]

\[
\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x,c)dx + \frac{1}{b-a} \int_a^b f(x,d)dx + \frac{1}{d-c} \int_c^d f(a,y)dy + \frac{1}{d-c} \int_c^d f(b,y)dy \right] + R_3, \tag{12}
\]

where

\[
R_3 = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b \int_0^1 G(x,t,c,d)dtdx + \frac{1}{d-c} \int_c^d \int_0^1 F(y,t,a,b)dtdy \right];
\]

and

\[
\frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x,c)dx + \frac{1}{b-a} \int_a^b f(x,d)dx + \frac{1}{d-c} \int_c^d f(a,y)dy + \frac{1}{d-c} \int_c^d f(b,y)dy \right]
\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + R_4, \tag{13}
\]

where

\[
R_4 = \frac{1}{4} \left[ \int_0^1 F(c,t,a,b)dt + \int_0^1 F(d,t,a,b)dt + \int_0^1 G(a,t,c,d)dt + \int_0^1 G(b,t,c,d)dt \right].
\]

**Proof.** Note that, for every \( x \in [a,b] \), the function \( f(x,\cdot) \) is \( G(x,\cdot,\cdot,\cdot) \)-convex. Thus, applying Theorem 1, we obtain

\[
f \left( x, \frac{c+d}{2} \right) \leq \frac{1}{d-c} \int_c^d f(x,y)dy + \frac{1}{d-c} \int_c^d G \left( x, \frac{1}{2}, y, c+d-y \right) dy
\]
Adding up these inequalities, we obtain (11) and (12).

Integrating this inequality with respect to $x$, we find

\[ \frac{1}{b-a} \int_{a}^{b} f \left( x, \frac{c+d}{2} \right) \, dx \]

\leq \frac{1}{(b-a)(d-c)} \left[ \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx + \int_{a}^{b} \int_{c}^{d} G \left( x, \frac{1}{2}, y, c+d-y \right) \, dy \, dx \right]

\leq \frac{1}{2(b-a)} \left[ \int_{c}^{d} f(x,c) \, dx + \int_{a}^{b} f(x,d) \, dx \right]

+ \frac{1}{b-a} \int_{a}^{b} G(x,t,c,d) \, dt \, dx + \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} G \left( x, \frac{1}{2}, y, c+d-y \right) \, dy \, dx.

Moreover, since for every $y \in [c,d]$, $f(\cdot, y)$ is $F(y, \cdot, \cdot)$-convex, using the similar arguments, we conclude that

\[ \frac{1}{d-c} \int_{c}^{d} f \left( \frac{a+b}{2}, y \right) \, dy \]

\leq \frac{1}{(b-a)(d-c)} \left[ \int_{c}^{d} \int_{a}^{b} f(x,y) \, dx \, dy + \int_{c}^{d} \int_{a}^{b} F \left( y, \frac{1}{2}, x, a+b-x \right) \, dx \, dy \right]

\leq \frac{1}{2(d-c)} \left[ \int_{c}^{d} f(a,y) \, dy + \int_{a}^{b} f(b,y) \, dy \right]

+ \frac{1}{d-c} \int_{c}^{d} \int_{0}^{1} F(y,t,a,b) \, dt \, dy + \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} G \left( y, \frac{1}{2}, x, a+b-x \right) \, dx \, dy.

Adding up these inequalities, we obtain (11) and (12).

Since $f(\cdot, c+d)$ is $F(\frac{c+d}{2}, \cdot, \cdot)$-convex and $f(\frac{a+b}{2}, \cdot)$ is $G(\frac{a+b}{2}, \cdot, \cdot)$-convex, taking into account the first inequality in Theorem 1, we have

\[ f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f \left( x, \frac{c+d}{2} \right) \, dx + \frac{1}{b-a} \int_{a}^{b} F \left( \frac{c+d}{2}, \frac{1}{2}, x, a+b-x \right) \, dx \]

and

\[ f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{d-c} \int_{c}^{d} f \left( \frac{a+b}{2}, y \right) \, dy + \frac{1}{d-c} \int_{c}^{d} G \left( \frac{a+b}{2}, \frac{1}{2}, y, c+d-y \right) \, dy. \]

Adding them up we obtain (10).

Finally, as $f(\cdot, c)$, $f(\cdot, d)$, $f(a, \cdot)$ and $f(b, \cdot)$ are $F(c, \cdot, \cdot)$, $F(d, \cdot, \cdot)$, $G(a, \cdot, \cdot)$ and $G(b, \cdot, \cdot)$-convex, respectively, applying the second inequality in Theorem 1, we find

\[ \frac{1}{b-a} \int_{a}^{b} f(x,c) \, dx \leq \frac{f(a,c) + f(b,c)}{2} + \int_{0}^{1} F(c,t,a,b) \, dt, \]

\[ \frac{1}{b-a} \int_{a}^{b} f(x,d) \, dx \leq \frac{f(a,d) + f(b,d)}{2} + \int_{0}^{1} F(d,t,a,b) \, dt, \]
Adding up these inequalities, we obtain (13). □

2.2. Fejér Type Inequalities

In order to prove the Fejér type inequalities for co-ordinated \((F, G)\)-convex functions we need the following auxiliary result.

**Lemma 1.** Assume that \( f : [a, b] \times [c, d] \to \mathbb{R} \) is a co-ordinated \((F, G)\)-convex function.

(i) If \([x_1, x_2] \subset [x_1', x_2'] \subset [a, b]\) and \(x_1 + x_2 = x_1' + x_2'\) then

\[
\frac{1}{d - c} \int_c^d f(a, y) dy \leq \frac{f(a, c) + f(a, d)}{2} + \int_0^1 G(a, t, c, d) dt
\]

and

\[
\frac{1}{d - c} \int_c^d f(b, y) dy \leq \frac{f(b, c) + f(b, d)}{2} + \int_0^1 G(b, t, c, d) dt.
\]

Adding up these inequalities, we obtain (13).

(ii) If \([y_1, y_2] \subset [y_1', y_2'] \subset [c, d]\) and \(y_1 + y_2 = y_1' + y_2'\) then

\[
f(x, y_1) + f(x, y_2) \leq f(x, y_1') + f(x, y_2') + G \left( x, \frac{y_2' - y_1'}{y_2' - y_1}, y_1', y_2' \right) + G \left( x, \frac{y_2 - y_1}{y_2 - y_1'}, y_1, y_2 \right)
\]

for \(x \in [a, b]\).

**Proof.** We prove only the first part of the lemma since the proof of the second part is similar. Assume that \([x_1, x_2] \subset [x_1', x_2'] \subset [a, b]\) and \(x_1 + x_2 = x_1' + x_2'\). Since

\[
x_1 = \frac{x_2' - x_1}{x_2' - x_1'} x_1' + \frac{x_1 - x_1'}{x_2 - x_1} x_2
\]

and

\[
x_2 = \frac{x_2' - x_2}{x_2' - x_1} x_1' + \frac{x_2 - x_1'}{x_2 - x_1} x_2',
\]

for every \(y \in [c, d]\), we obtain

\[
f(x_1, y) + f(x_2, y) \leq \frac{x_2' - x_1}{x_2' - x_1'} f(x_1', y) + \frac{x_1 - x_1'}{x_2 - x_1} f(x_2', y) + F \left( y, \frac{x_2' - x_1}{x_2' - x_1'}, x_1', x_2' \right)
\]

\[
+ \frac{x_2' - x_2}{x_2' - x_1} f(x_1', y) + \frac{x_2 - x_1'}{x_2 - x_1} f(x_2', y) + F \left( y, \frac{x_2' - x_2}{x_2' - x_1}, x_1', x_2' \right)
\]

\[
= \frac{2x_2' - (x_1 + x_2)}{x_2' - x_1} f(x_1', y) + \frac{x_1 + x_2 - 2x_1'}{x_2' - x_1} f(x_2', y) + F \left( y, \frac{x_2' - x_1}{x_2' - x_1'}, x_1', x_2' \right) + F \left( y, \frac{x_2' - x_2}{x_2' - x_1}, x_1', x_2' \right)
\]

\[
= f(x_1', y) + f(x_2', y) + F \left( y, \frac{x_2' - x_1}{x_2' - x_1'}, x_1', x_2' \right) + F \left( y, \frac{x_2' - x_2}{x_2' - x_1}, x_1', x_2' \right),
\]
In the next theorem we establish the Fejér type inequalities for \((F, G)\)-convex functions.

**Theorem 3.** Assume that \(p : [a, b] \times [c, d] \to \mathbb{R}\) is a positive integrable function symmetric with respect to the lines \(x = \frac{a+b}{2}\) and \(y = \frac{c+d}{2}\) (cf. (1) and (2)). If \(f : [a, b] \times [c, d] \to \mathbb{R}\) is a continuous co-ordinated \((F, G)\)-convex function such that \(fp\) is integrable on \([a, b] \times [c, d]\) then

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{\int_a^b \int_c^d f(x, y) p(x, y) dy dx + K}{\int_c^d f(x, y) dy dx},
\]

where

\[
K = 2 \int_a^b \int_c^{c+d} G \left( x, \frac{1}{2}, y, c + d - y \right) p(x, y) dy dx + 2 \int_a^b \int_c^{c+d} G \left( a + b - x, \frac{1}{2}, y, c + d - y \right) p(x, y) dy dx + 4 \int_a^b \int_c^{c+d} F \left( \frac{c + d}{2}, \frac{1}{2}, x, a + b - x \right) p(x, y) dy dx
\]

and

\[
\frac{\int_a^b \int_c^d f(x, y) p(x, y) dy dx - L}{\int_c^d f(x, y) dy dx} \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4},
\]

where

\[
L = L_1 + L_2 + L_3.
\]

\[
L := \int_a^b \int_c^{c+d} \left[ F \left( y, \frac{b - a}{b - a}, a, b \right) + F \left( y, \frac{a - b}{b - a}, a, b \right) + F \left( c + d - y, \frac{b - a}{b - a}, a, b \right) + F \left( c + d - y, \frac{a - b}{b - a}, a, b \right) \right] p(x, y) dy dx + \int_a^b \int_c^{c+d} \left[ G \left( a, \frac{d - y}{d - c}, c, d \right) + G \left( a, \frac{y - c}{d - c}, c, d \right) \right] p(x, y) dy dx + \int_a^b \int_c^{c+d} \left[ G \left( b, \frac{d - y}{d - c}, c, d \right) + G \left( b, \frac{y - c}{d - c}, c, d \right) \right] p(x, y) dy dx.
\]

**Proof.** Assume that \(f : [a, b] \times [c, d] \to \mathbb{R}\) is an integrable co-ordinated \((F, G)\)-convex function such that \(fp\) is integrable. Then, for every \(x \in [a, b]\) and \(y \in [c, d]\), we have

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2} f \left( x, \frac{c + d}{2} \right) + \frac{1}{2} f \left( a + b - x, \frac{c + d}{2} \right) + f \left( \frac{c + d}{2}, \frac{c + d}{2}, x, a + b - x \right) \leq \frac{1}{4} f(x, y) + \frac{1}{4} f(x, c + d - y) + \frac{1}{4} f(a + b - x, y) + \frac{1}{4} f(a + b - x, c + d - y) + \frac{1}{2} G \left( x, \frac{1}{2}, y, c + d - y \right) + \frac{1}{2} G \left( a + b - x, \frac{1}{2}, y, c + d - y \right) + f \left( \frac{c + d}{2}, \frac{c + d}{2}, x, a + b - x \right).
\]
Therefore, as $p$ is symmetric with respect to the lines $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, we obtain

$$f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \int_a^b \int_c^d p(x,y)dydx = 4 \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) p(x,y)dydx$$

$$\leq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)] p(x,y)dydx + \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,c+d-y) + f(a+b-x,y)] p(x,y)dydx + K$$

$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)] p(x,y)dydx + K$$

Thus, (14) holds.

Furthermore, using again the symmetry of $p$ and applying Lemma 1 to $[y, c + d - y] \subset [c, d]$ and $[x, a + b - x] \subset [a, b]$, where $x \in [a, \frac{a+b}{2}]$, $y \in [c, \frac{c+d}{2}]$, we have

$$f(a,c) + f(a,d) + f(b,c) + f(b,d) \frac{1}{4} \int_a^b \int_c^d p(x,y)dydx$$

$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(a,c) + f(a,d) + f(b,c) + f(b,d)] p(x,y)dydx$$

$$\geq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[ f(a,y) + f(a,c + d - y) - G \left( a, \frac{d-y}{d-c}, c, d \right) \right] - G \left( a, \frac{y-c}{d-c}, c, d \right) - \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(b,c) + f(b,d) - G \left( b, \frac{d-y}{d-c}, c, d \right) - \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y)dydx$$

$$\geq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[ f(x,y) + f(a+b-x, y) - F \left( y, \frac{b-x}{b-a}, a, b \right) \right] - F \left( b, \frac{d-y}{d-c}, c, d \right) - \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,c+d-y) + f(a+b-x, c+d-y)$$

$$- F \left( c+d-y, \frac{b-x}{b-a}, a, b \right) - F \left( c+d-y, \frac{x-a}{b-a}, a, b \right) p(x,y)dydx - (L_2 + L_3)$$
which gives (15). □

3. Discussion

In this paper the Hermite–Hadamard and Fejér type inequalities for co-ordinated \((F, G)\)-convex functions are proved. Since every co-ordinated convex function is co-ordinated \((F, G)\)-convex (with \(F\) and \(G\) being identically 0), from our results, one can easily deduce the results by Dragomir [1] and Alomari and Darus [4]. Furthermore, applying Theorems 2 and 3, one can obtain the Hermite–Hadamard and Fejér type inequalities for co-ordinated \((C, D)\)-approximately convex functions and co-ordinated \((C, D)\)-strongly convex functions defined by

\[
f(t x_1 + (1 - t) x_2, y) \leq t f(x_1, y) + (1 - t) f(x_2, y) + D(y) t (1 - t) |x_1 - x_2|,
\]

\[
f(x, t y_1 + (1 - t) y_2) \leq t f(x, y_1) + (1 - t) f(x, y_2) + C(x) t (1 - t) |y_1 - y_2|
\]

for \(t \in [0, 1]\), \(x_1, x_2 \in [a, b]\), \(y_1, y_2 \in [c, d]\), \(x \in [a, b]\), \(y \in [c, d]\); and

\[
f(t x_1 + (1 - t) x_2, y) \leq t f(x_1, y) + (1 - t) f(x_2, y) - D(y) t (1 - t) (x_1 - x_2)^2,
\]

\[
f(x, t y_1 + (1 - t) y_2) \leq t f(x, y_1) + (1 - t) f(x, y_2) - C(x) t (1 - t) (y_1 - y_2)^2
\]

for \(t \in [0, 1]\), \(x_1, x_2 \in [a, b]\), \(y_1, y_2 \in [c, d]\), \(x \in [a, b]\), \(y \in [c, d]\), respectively, where \(C : [a, b] \to (0, \infty)\) and \(D : [c, d] \to (0, \infty)\) are given functions.

Note also that from Theorem 1 the Hermite–Hadamard inequalities for approximately convex functions and strongly convex functions can be derived. Finally, applying Theorem 1, with \(F \equiv 0\), we obtain the classical Hermite–Hadamard inequality.

**Author Contributions:** M.C. and M.Z. have contributed equally to this paper. All authors have read and agree to the published version of the manuscript.
**Funding:** This research received no external funding.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**


