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On Generalized Hardy–Rogers Type α-Admissible Mappings in Cone b-Metric Spaces over Banach Algebras

Wasfi Shatanawi [1,2,3], Zoran D. Mitrović [4,5,*], Nawab Hussain [6] and Stojan Radenović [7]

1 Department of Mathematics and General Courses, Prince Sultan University Riyadh, Riyadh 11586, Saudi Arabia; wshatanawi@psu.edu.sa or wshatanawi@yahoo.com
2 Department of Medical Research, China Medical University Hospital China Medical University, Taichung 40402, Taiwan
3 Department of M-Commerce and Multimedia Applications, Asia University, Taichung 41354, Taiwan
4 Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
5 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
6 King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; nhusain@kau.edu.sa
7 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd 35, Serbia; radens@beotel.rs
* Correspondence: zoran.mitrovic@tdtu.edu.vn

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Abstract: We introduce the notion of α-admissibility of mappings on cone b-metric spaces using Banach algebra with coefficient s, and establish a result of the Hardy-Rogers theorem in these spaces. Furthermore, using symmetry, we derive many recent results as corollaries. As an application we prove certain fixed point results in partially ordered cone b-metric space using Banach algebra. Also, we use our results to derive and prove some real world problems to show the usability of our obtained results. Moreover, it is worth noticing that fixed point theorems for monotone operators in partially ordered metric spaces are widely investigated and have found various applications in differential, integral and matrix equations.

Keywords: fixed points; cone b-metric space (CnMs); Banach algebra

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1. Introduction

The notion of α-admissibility plays one of the main idea in the filed of mathematics to launch many contractions for self-mappings on a set X with a metric d (see [1–3] and references therein). This amazing concept was defined by Samet et al. [4]. Thereafter, many authors studied a lot of fixed point results related to contractions depending on α-admissibility (see for instance [5–11]).

In this our paper we launch the notion of $\alpha - \psi$-contractive type mappings in the context of cone b-metric spaces over Banach algebra. For other interesting results in the context of metric and b-metric spaces (see [12–16]).

2. Preliminaries

We begin with the definitions of known notions as well as known results from cone metric and cone b-metric spaces (for more details see [17–32]) over Banach algebra, respectively.

Definition 1. *A real Banach algebra (in short BA) $\mathcal{B}$ is a real Banach space $\mathcal{B}$ with a product that satisfies*

1. $s(mk) = (sm)k$,

2. $s(m+k) = sm + sk$,
3. $\alpha(sm) = (\alpha s)m = s(\alpha m)$,
4. $||sm|| \leq ||s||\,||m||$,

for all $s, m, k \in \mathcal{B}, \alpha \in \mathbb{R}$.

In the rest of this paper $\mathcal{B}$ stands to a real Banach algebra unless otherwise stated. We call $\mathcal{B}$ unital if there is $e \in \mathcal{B}$ such that $ei = ie = i$ for all $i \in \mathcal{B}$. In this case e is called the unit of $\mathcal{B}$. An element $i \in \mathcal{B}$ is said to be invertible if there is a $j \in \mathcal{B}$ such that $ij = ji = e$. In such case the inverse of i is unique and denoted by i^{-1} (see [17]).

Let $\mathcal{B}$ be unital with zero θ. A non-empty closed set $P \subset \mathcal{B}$ is said to be a cone if

1. $e \in P$,
2. $P + P \subset P$,
3. $\lambda P \subset P$ for all $\lambda \geq 0$,
4. $P \cdot P \subset P$,
5. $P \cap (-P) = \{\theta\}$.

Define $\preceq$ on $\mathcal{B}$ with respect to the cone P by $b_1 \preceq b_2$ if and only if $b_2 - b_1 \in P$ and we write $b_1 \prec b_2$ if $b_1 \preceq b_2$ and $b_1 \neq b_2$ while $b_1 \ll b_2$ will stand for $b_2 - b_1 \in intP$, where $intP$ stands for the interior of P. We say that P is solid if $intP \neq \emptyset$. A cone P is called normal if there there exists $M > 0$ such that for all $b_1, b_2 \in \mathcal{B}$, we have

$$\theta \preceq b_1 \preceq b_2 \text{ implies } \|b_1\| \leq M\|b_2\|.$$

The cone b-metric space over a BA with constant $s \geq 1$ was introduced in [18] as a generalization of a b-metric space. Mitrović and Hussain [19] initiated the notion of cone b-metric space over a BA with constant $s \succeq e$.

Definition 2 ([19])**.** *Over a nonempty set W, we let $d : W \times W \to \mathcal{B}$ be a mapping satisfying:*
(CbM1) $\theta \preceq d(t,m)$ for all $t, m \in W$, $d(t,m) = \theta$ if and only if $t = m$;
(CbM2) $d(t,m) = d(m,t)$ for all $t, m \in W$;
(CbM3) there exists $s \in C$, $e \preceq s$ such that $d(t,m) \preceq s[d(t,k) + d(k,m)]$ for all $t, m, k \in W$.
Then we call d a cone b-metric on W. The space (W,d) is called a cone b-metric space over a BA with coefficient s (in short CnMs-BA). If $s = e$, we call (W,d) is CMS over BA (in short CMS-BA).

Definition 3 ([20])**.** *Let $\{i_n\}$ be a sequence in $\mathcal{B}$.*

(i) We call $\{i_n\}$ a c-sequence, if for each $c \gg \theta$, there exists is $n_0 \in \mathbb{N}$ such that $i_n \ll c$ for all $n \geq n_0$.
(ii) We call $\{i_n\}$ a θ-sequence if $i_n \to \theta$ as $n \to \infty$.

Definition 4 ([19])**.** *Let (W,d) be a CnMs-BA with coefficient s and $\{i_n\}$ a sequence in W. Then*

(i) $\{i_n\}$ b-converges to $w \in W$, if $\{d(i_n, w)\}$ is a c-sequence.
(ii) $\{i_n\}$ is b-Cauchy if for each $c \in \mathcal{B}$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that $d(i_n, i_m) \ll c$ for all $n, m > n_0$.
(iii) (W,d) is called a b-complete CnMs, if whenever (i_n) is b-Cauchy in W, then $\{i_n\}$ is b-convergent.

Definition 5. *Let $h : W \to W$ be a mapping. We call h continuous at $w \in W$, if whenever $\{i_n\}$ in W such that $i_n \to w$ as $n \to \infty$, we have $hi_n \to hw$ as $n \to \infty$.*

Lemma 1 ([17])**.** *Let e be the unit of $\mathcal{B}$. Then for $b \in \mathcal{B}$, $\lim\limits_{n\to\infty} ||b^n||^{\frac{1}{n}}$ exists. Moreover the spectral radius $r(b)$ satisfies*

$$r(b) = \lim_{n\to\infty} ||b^n||^{\frac{1}{n}} = \inf_{n\geq 1} ||b^n||^{\frac{1}{n}}.$$

If there exists a constant λ such that $r(b) < |\lambda|$, then $\lambda e - b$ is invertible in $\mathcal{B}$. Moreover,

$$(\lambda e - b)^{-1} = \sum_{j=0}^{\infty} \frac{b^j}{\lambda^{j+1}}$$

and

$$r((\lambda e - b)^{-1}) \leq \frac{1}{|\lambda| - r(b)}.$$

Lemma 2 ([24]). *Assume that P is a solid cone in $\mathcal{B}$, and $\{m_n\}$ and $\{s_n\}$ are c-sequences in $\mathcal{B}$. Let $\alpha, \beta \in P$ be arbitrarily given vectors. Then $\{\alpha m_n + \beta s_n\}$ is a c-sequence in $\mathcal{B}$.*

Lemma 3 ([25]). *Let $P \subset \mathcal{B}$ be a cone.*

(a) *If $i, j \in \mathcal{B}, a \in P$ and $i \preceq j$, then $ai \preceq aj$,*
(b) *If $i, a \in P$ are such that $r(a) < 1$ and $i \preceq ai$, then $i = \theta$,*
(c) *If $a \in P$ and $r(a) < 1$, then for any fixed $t \in \mathbb{N}$ we have $r(a^t) < 1$.*

Lemma 4 ([20]). *Let P be a solid cone in $\mathcal{B}$.*

1. *Let $a \in P$. Then $r(a) < 1$ if and only if $\{a^n\}$ is a θ-sequence.*
2. *Every θ-sequence in $\mathcal{B}$ is c-sequence.*
3. *Each c-sequence in P is a θ-sequence if and only if P is normal.*

Lemma 5 ([17]). *Let $i, j \in \mathcal{B}$. If i commutes with j, then*

$$r(i + j) \leq r(i) + r(j), \; r(ij) \leq r(i)r(j).$$

Lemma 6 ([19]). *Let $\{i_n\}$ be a sequence in a CnMs-BA (X, d) over $\mathcal{B}$ with coefficient s and P be solid cone in $\mathcal{B}$. Suppose that there exists $a \in \mathcal{B}$ which commute with s such that $r(a) \in [0, 1)$ and satisfying $d(i_{n+1}, i_n) \preceq ad(i_n, i_{n-1})$ for any $n \in \mathbb{N}$. Then $\{i_n\}$ is b-Cauchy.*

Lemma 6 plays a crucial role generalize a lot of results exist in the literature.

In this paper, we introduce the notion of α-admissibility of mappings [4] defined on CnMs-BA and give a result of Hardy-Rogers [33] in CnMs-BA with coefficient s.

3. Main Results

Definition 6. *Let (W, d) be a CnMs-BA with coefficient $s, (e \preceq s)$, P be a solid cone, $h : W \to W$ and $\alpha : W \times W \to P$ be two mappings. We say that h is α-admissible Hardy-Rogers contraction with vectors $a_i \in P$, $i \in \{1, \ldots, 5\}$ such that $\sum_{j=1}^{5} r(a_j) < 1$. If*

$$\alpha(k, m) \succeq e \text{ implies } \alpha(hk, hm) \succeq e, \tag{1}$$

and

$$\alpha(k, m)d(hk, hm) \preceq a_1 d(k, m) + a_2 d(k, hk) + a_3 d(m, hm) + a_4 d(k, hm) + a_5 d(m, hk), \tag{2}$$

for all $k, m \in W$ with $\alpha(k, m) \succeq e$.

Definition 7 ([5]). *Let (W, d) be a CnMs-BA with coefficient $s, (e \preceq s)$, P be a solid cone, $h : W \to W$ and $\alpha : W \times W \to P$ be two mappings. Then (W, d) is α-regular if for any sequence $\{w_n\}$ in W, with $\alpha(w_{n+1}, w_n) \succeq e$ for all $n \in \mathbb{N}$ and $w_n \to w^*$ as $n \to \infty$, it follows that $\alpha(w_n, w^*) \succeq e$ for all $n \in \mathbb{N}$.*

Lemma 7. *Let (W, d) be a CnMs-BA with coefficient $s, (e \preceq s)$ and $h : W \to W$ be a α-admissible Hardy-Rogers contraction with vectors $a_i, i \in \{1, \ldots, 5\}$. Assume the following conditions:*

1. *there is $w_0 \in W$ such that $e \preceq \alpha(hw_0, w_0)$;*
2. *$a_1, a_2, a_3, a_4, a_5, s$ a commute with each other;*
3. *$\sum_{j=1}^{3} r(a_j) + 2r(a_5)r(s) < 1$.*

Then sequence $\{w_n\}$ defined by $w_0 \in W$ and $w_{n+1} = hw_n, n \geq 0$ is a b-Cauchy sequence.

Proof. From condition (1) we have that $\alpha(w_{i+1}, w_i) \succeq e$ for all $i \in \mathbb{N}$. Also, from condition (2) we obtain

$$\begin{aligned} \alpha(w_i, w_{i-1})d(w_{i+1}, w_i) &\preceq a_1 d(w_i, w_{i-1}) + a_2 d(w_i, w_{i+1}) + a_3 d(w_{i-1}, w_i) \\ &+ a_4 d(w_i, w_i) + a_5 d(w_{i-1}, w_{i+1}). \end{aligned}$$

Since, $e \preceq \alpha(w_i, w_{i-1})$, we obtain $d(w_{i+1}, w_i) \preceq \alpha(w_i, w_{i-1})d(w_{i+1}, w_i)$ for all $i \in \mathbb{N}$. Therefore,

$$\begin{aligned} (e - a_2)d(w_{i+1}, w_i) &\preceq (a_1 + a_3)d(w_{i-1}, w_i) + a_5 d(w_{i-1}, w_{i+1}) \\ &\preceq (a_1 + a_3)d(w_{i-1}, w_i) + a_5 s[d(w_{i-1}, w_i) + d(w_i, w_{i+1})]. \end{aligned}$$

So,

$$(e - a_2 - a_5 s)d(w_{i+1}, w_i) \preceq (a_1 + a_3 + a_5 s)d(w_{i-1}, w_i),$$

since $r(a_2) + r(a_5)r(s) < 1$ from Lemma 1, we have,

$$d(w_{i+1}, w_i) \preceq [e - (a_2 + a_5 s)]^{-1}(a_1 + a_3 + a_5 s)d(w_i, w_{i-1}). \tag{3}$$

Put

$$\lambda = [e - (a_2 + a_5 s)]^{-1}(a_1 + a_3 + a_5 s).$$

From Lemma 1 we have that

$$r(\lambda) \leq \frac{r(a_1) + r(a_3) + r(a_5)r(s)}{1 - r(a_2) - r(a_5)r(s)}.$$

So, $r(\lambda) \in [0, 1)$. From Lemma 4 we deduce that $\{w_i\}$ is b-Cauchy in (W, d). □

Theorem 1. *Let (W, d) be a b-complete CnMs-BA with coefficient s, $(e \preceq s)$ and $h : W \to W$ be a α-admissible Hardy-Rogers contraction with vectors $a_i, i \in \{1, \ldots, 5\}$. Assume the following:*

1. *there is $w_0 \in W$ such that $e \preceq \alpha(hw_0, w_0)$;*
2. *h is continuous;*
3. *$a_1, a_2, a_3, a_4, a_5, s$ commute with each other;*
4. *$\sum_{j=1}^{3} r(a_j) + 2r(a_5)r(s) < 1$.*

Then h has a fixed point.

Proof. Choose $w_0 \in W$ with $e \preceq \alpha(hw_0, w_0)$. Define the sequence $\{w_i\}$ by $w_{i+1} = hw_i$ for all $n \geq 0$. Lemma 7 implies that $\{w_i\}$ is b-Cauchy in (W, d). The completeness of (W, d) ensures that there is $w^* \in W$ such that

$$\lim_{i \to \infty} w_i = w^*. \tag{4}$$

Since h is continuous, $w_{i+1} = hw_i \to hw^*$as $i \to \infty$. Hence $w^* = hw^*$ from the uniqueness of limit. So w^* is a fixed point of h. □

Remark 1. *Due to the symmetry, we can replace the condition (1) in Theorem 1 and Lemma 7 by $(1')$ there exists $w_0 \in W$ such that $e \preceq \alpha(w_0, hw_0)$ and condition (4) in Theorem 1 and condition (3) in Lemma 7 by $(3')$ $\sum_{j=1}^{3} r(a_j) + 2r(a_4)r(s) < 1$.*

From the previous theorem, we obtain Reich type theorem [34] in CnMs-BA with coefficient s.

Theorem 2 ([19])**.** *Let (W,d) be a CnMs-BA with coefficient $s,(e \preceq s)$ and $h : W \to W$ be a continuous. Assume:*

$$d(hi,hj) \preceq a_1 d(i,j) + a_2 d(i,hi) + a_3 d(j,hj) \tag{5}$$

for all $i,j \in W$, where $a_1,a_2,a_3 \in P$ commutes such that $\sum_{j=1}^{3} r(a_j) < 1$. Then h possess a unique fixed point.

Example 1. *Take $\mathcal{B} = \{b = (b_{nm})_{3\times 3} : b_{nm} \in \mathbb{R}, 1 \leq n,m \leq 3\}$ such that*

$$\|b\| = \frac{1}{3} \sum_{1 \leq n,m \leq 3} |b_{nm}|.$$

Consider the cone $P = \{b \in B : b_{nm} \geq 0, 1 \leq n,m \leq 3\}$ over $\mathcal{B}$. Take $W = \{1,2,3\}$. Define $d : W \times W \longrightarrow \mathcal{B}$ by

$$d(1,1) = (0)_{3\times 3} = d(2,2) = d(3,3),$$

$$d(1,2) = d(2,1) = \begin{pmatrix} 0 & 4 & 8 \\ 4 & 8 & 12 \\ 32 & 16 & 28 \end{pmatrix},$$

$$d(3,1) = d(1,3) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 8 & 4 & 7 \end{pmatrix},$$

and

$$d(2,3) = d(3,2) = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 6 \\ 16 & 8 & 14 \end{pmatrix}.$$

Then (W,d) is a CnMs-BA with coefficient $s = \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$. Let $h : W \to W$ be a mapping defined by $h1 = 1, h2 = 3, h3 = 1$ and let $a_1 = a_2 = a_3 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$. Then h satisfies:

$$d(hs,hm) \preceq a_1 d(s,m) + a_2 d(s,hs) + a_3 d(m,hm)$$

for all $s,m \in W$, where $a_i, i = 1,2.3$ commute with $\sum_{i=1}^{3} r(a_i) < 1$. So h possess its unique fixed point at $w = 1$.

Furthermore, we may obtain Banach, Kannan, and Chatterjea type results (see in [35]) as immediate consequences of Theorem 1 in CnMs-BA with coefficient s.

The continuity assumption in Theorem 1 can be skipped by adding a suitable condition.

Theorem 3. *Let (W,d) be a b-complete CnMs-BA with coefficient $s,(e \preceq s)$ and $h : W \to W$ be a α-admissible Hardy-Rogers contraction with vectors $a_i, i \in \{1,\ldots,5\}$. Assume the following:*

1. *There is $w_0 \in W$ such that $e \preceq \alpha(hw_0,w_0)$;*
2. *a_1,a_2,a_3,a_4,a_5,s a commute with each other;*
3. *$r(sa_3 + s^2 a_4) < 1$;*
4. *$\sum_{j=1}^{3} r(a_i) + 2r(a_5)r(s) < 1$.*

5. (W,d) *is* α*-regular.*

Then h *has a fixed point.*

Proof. Choose $w_0 \in W$ such that $e \preceq \alpha(hw_0, w_0)$ and $\{w_i\}$ by $w_{i+1} = hw_i$ for all $n \geq 0$. From Lemma 7 it follows that $\{w_i\}$ is a b-Cauchy sequence in (W,d). By the completeness of (W,d), there exists $w^* \in W$ such that

$$\lim_{i\to\infty} w_i = w^*. \tag{6}$$

Now we obtain that w^* is the fixed point of h. Namely, we have

$$\begin{aligned} d(w^*, hw^*) &\preceq sd(w^*, w_{i+1}) + sd(w_{i+1}, hw^*) \\ &= sd(w^*, w_{i+1}) + sd(hw_i, hw^*). \end{aligned}$$

Since (W,d) is α-regular and h is α-admissible Hardy-Rogers contraction we obtain

$$\begin{aligned} d(hw_i, hw^*) &\preceq \alpha(hw_i, hw^*)d(hw_i, hw^*) \\ &\preceq d(w^*, w_{i+1}) + a_1 d(w_i, w^*) + a_2 d(w_i, w_{i+1}) + a_3 d(w^*, hw^*) \\ &+ a_4 d(w_i, hw^*) + a_5 d(w^*, w_{i+1}). \end{aligned}$$

So,

$$\begin{aligned} d(hw_i, hw^*) &\preceq sd(w^*, w_{i+1}) + sa_1 d(w_i, w^*) + sa_2 d(w_i, w_{i+1}) + sa_3 d(w^*, hw^*) \\ &+ sa_4 d(w_i, hw^*) + sa_5 d(w^*, w_{i+1}) \\ &\preceq sd(w^*, w_{i+1}) + sa_1 d(w_i, w^*) + sa_2 d(w_i, w_{i+1}) + sa_3 d(w^*, hw^*) \\ &+ sa_4 [s(d(w_i, w^*) + d(w^*, hw^*)) + a_5 d(w^*, w_{i+1})]. \end{aligned}$$

Because $\lim_{i\to\infty} d(w^*, w_i) = \theta$, $\lim_{i\to\infty} d(w_i, w_{i+1}) = \theta$, we obtain

$$d(hw^*, w^*) \preceq (sa_3 + s^2 a_4) d(hw^*, w^*).$$

Because, $r(sa_3 + s^2 a_4) < 1$, from Lemma 3 we claim that $d(w^*, hw^*) = \theta$, that is, $hw^* = w^*$. □

Next, to assure the uniqueness of fixed points in above theorems, we use the following property (see [26,27]).

(H) For all $s, m \in Fix(h)$, there exists $k \in W$ with $\alpha(s,k) \succeq e$ and $\alpha(m,k) \succeq e$, where $Fix(h)$ denotes the set of all fixed points of h.

Theorem 4. *Assume condition (H) holds together of all conditions of Theorem 1 (resp. Theorem 3). Then we guarantee the uniqueness of the fixed point of* h*.*

Proof. Let w^* is a fixed point of h. Let m^* be another fixed point of h. Then it follows from (2) that

$$\begin{aligned} d(w^*, m^*) &= d(hw^*, hm^*) \preceq a_1 d(w^*, m^*) + a_2 d(w^*, hw^*) + a_3 d(m^*, hm^*) \\ &+ a_4 d(w^*, hm^*) + a_5 d(m^*, hw^*) \\ &\preceq (a_1 + a_4 + a_5) d(w^*, m^*). \end{aligned}$$

Now from Lemma 3, we obtain $d(w^*, m^*) = \theta$, i.e., $w^* = m^*$. □

Please note that Theorem 3, due to symmetry, improves and generalizes Theorem 2.1 in [18] and Theorem 3.3 in [5].

Theorem 5 ([18]). *Consider a complete CBM-BA (W, d) coefficient $s \geq 1$. Let P be a solid which is not necessarily normal cone of the BA $\mathcal{B}$. Assume that $h : W \to W$ is a mapping. Also, suppose that there is $p \in P$ such that, for all $i, j \in W$, one of the following conditions holds:*

(i) $d(hi, hj) \preceq pd(i, j)$ *and* $r(p) < \frac{1}{s}$;
(ii) $d(hi, hj) \preceq p(d(hi, i) + d(hj, j))$ *and* $r(p) < \frac{1}{1+s}$;
(iii) $d(hi, hj) \preceq p(d(hi, j) + d(hj, i))$ *and* $r(p) < \frac{1}{s+s^2}$. *Then h possess unique fixed point.*

Also, Theorem 3 improves and generalizes Theorem 2.1 in [28].

Theorem 6 ([28]). *Consider a complete CBM-BA (W, d) with coefficient $s \geq 1$. Let P be a solid cone of BA $\mathcal{B}$ which is not necessarily normal. Assume that $h : W \to W$ is a mapping. Also, assume that there is $p \in P$ such that, for all $i, j \in P$, the following conditions hold:*

$$d(hi, hj) \preceq pd(i, j),$$

and $r(p) < 1$. Then h has a unique fixed point in W and for any $w \in W$, the iterative sequence $\{h^n w\}$ b-converges to the fixed point.

Remark 2. *In (i) of Theorem 5, the condition $r(p) < \frac{1}{s}$ can be replaced by a weaker condition $r(p) < 1$. Similarly, in condition (ii), for $r(p) < \frac{1}{1+s}$ we can relax with $r(p) < \min\{\frac{1}{2}, \frac{1}{r(s)}\}$, and in condition (iii) instead of $r(p) < \frac{1}{s+s^2}$ put $r(p) < \min\{\frac{1}{2r(s)}, \frac{1}{r^2(s)}\}$.*

In the next result, we generalize and unify the results of Ran and Reurings [36], Liu and Xu [29] and Nieto, Rodríguez-López [3] and many others.

Theorem 7. *Let $(W, \sqsubseteq)$ be a partially ordered set. Suppose that (W, d) is a complete CBM-BA $\mathcal{B}$. Let P be the underlying solid cone. Let $h : W \to W$ be nondecreasing mapping with respect to $\sqsubseteq$. Suppose condition (2) in Lemma 7 is satisfied together with the following assumptions:*

(i) $d(hi, hj) \preceq a_1 d(i, j) + a_2 d(i, hi) + a_3 d(j, hj) + a_4 d(i, hj) + a_5 d(j, hi)$ *for all $i, j \in W$ with $i \sqsubseteq j$;*
(ii) *there exists $w_0 \in W$ such that $w_0 \sqsubseteq hw_0$;*
(iii) *either $(W, \sqsubseteq)$ is regular or $h : W \to W$ is continuous*
Then h possess a fixed point in W.

Remark 3. *Using Lemma 6 we can improve the following results:*

1. *Theorem 2.5 in [18].*
2. *Theorem 2.9 in [24].*
3. *Theorems 3.3 and 3.5 in [5].*
4. *Theorem 12 in [30].*
5. *Theorem 2.3 in [31].*
6. *Theorem 3.3 in [37].*
7. *Theorem 3.2 in [32].*

Remark 4. *In Lemma 2.5. in paper [19] we consider that k and s are commutative.*

4. Examples and Applications

By using the main facts of C*-algebra (see [38–44]) enough researchers obtained the new results in the framework of it (that is, in algebra-valued metric spaces and in C*-algebra-valued b-metric spaces). In the fact, under Definition 2. we get so-called cone metric space over Banach algebra $(s = e)$, that is, cone b-metric spaces over Banach algebra $(e \preceq s$ and $e \neq s)$.

Before of all, we give the main notions in C*-algebra. A vector space $\mathcal{V}$ (real or a complex) is an algebra if it become a ring under vector addition and vector multiplication and if for each scalar

γ and each pair of elements $u, v \in \mathcal{V}$, the next it is true: $\gamma(uv) = (\gamma u) v = u(\gamma v)$. If $\mathcal{V}$ admits with a so-called submultiplicative norm $\|.\|$, that is, $\|uv\| \leq \|u\| \|v\|$ for each $u, v \in \mathcal{V}$, then $(\mathcal{V}, \|.\|)$ is called a normed algebra. A name of complete normed algebra is Banach algebra. An involution mapping $*$ on the algebra vector $\mathcal{V}$ is a conjugate linear mapping $* : \mathcal{V} \to \mathcal{V}$ given with $u^{**} = u$ and $(uv)^* = v^* u^*$ for each $u, v \in \mathcal{V}$. Then, we say that the pair $(\mathcal{V}, *)$ is called a $*$-algebra. A Banach $*$-algebra $\mathcal{V}$ is a $*$-algebra $\mathcal{V}$ with a complete submultiplicative norm where is $\|u^*\| = \|u\|$ for each $u \in \mathcal{V}$. Hence, a C*-algebra is a Banach $*$-algebra with $\|u^* u\| = \|u\|^2$. Obviously examples of C*-algebras are: the set $\mathbb{C}$ of all complex numbers, further the set $L(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$, and the set $\mathcal{M}_n(\mathbb{C})$ of $n \times n$-matrices. If normed algebra $\mathcal{V}$ admits a unit e, that is, there exists an element $e \in \mathcal{V}$ such that $eu = ue = u$ for each $u \in \mathcal{V}$, and $\|u\| = 1$, we say that $\mathcal{V}$ is a unit normed algebra. A complete unital normed algebra $\mathcal{V}$ is called a unital Banach algebra. Here, $\mathcal{V}$ will denote a unital C*-algebra with a unit e. For an element a of a unital algebra $\mathcal{V}$, we say that a is invertible if there is an $b \in \mathcal{V}$ such that $ab = ba = e$. We denote $\mathcal{I}nv(\mathcal{V})$ the set of all invertible elements of $\mathcal{V}$. The set

$$S(b) = \{z \in \mathbb{C} : ze - b \notin \mathcal{I}nv(\mathcal{V})\},$$

is the spectrum of b.

Let $\mathcal{V}_h = \{g \in \mathcal{V} : g = g^*\}$. A positive element, $b \in \mathcal{V}$, denote by $e \sqsubseteq b$,if $b \in \mathcal{V}_h$ and $S(b) \subset \mathbb{R}_+ = [0, +\infty)$. Now, we introduce a partial ordering $\sqsubseteq$ on $\mathcal{V}_h$ as follows: $u \sqsubseteq v$ if and only if $e \sqsubseteq v - u$. Now, put $\mathcal{V}_+ = \{u \in \mathcal{V} : e \sqsubseteq u\}$ and $|u| = (uu^*)^{\frac{1}{2}}$.

In the sequel we give the main properties for this framework:

Lemma 8 ([39,43]). *Let $\mathcal{V}$ be a unital C*-algebra with a unit e.*

(1) For each $u \in \mathcal{V}_+$ we have $u \sqsubseteq e$ if and only if $\|u\| < 1$.
(2) $u \in \mathcal{V}_+$ with $\|u\| < \frac{1}{2}$, implies $e - u$ has a inverse and $\left\| u(e-u)^{-1} \right\| < 1$.
(3) Let $u, v \in \mathcal{V}$ in which $e \sqsubseteq u, v$ and $uv = vu$, then $e \sqsubseteq uv$.
(4) Consider $\mathcal{V}' = \{u : uv = vu, \text{ for all } v \in \mathcal{V}\}$. Assume that $u \in \mathcal{V}'$, if $v, w \in \mathcal{V}$ with $e \sqsubseteq w \sqsubseteq v$ and $e - u \in \mathcal{V}'_+$ is an invertible operator, so $(e-u)^{-1} v \sqsubseteq (e-u)^{-1} w$.
(5) Let $\mathcal{V}$ be unital and $u \in \mathcal{V}$ is Hermitian. If $\|u - \lambda e\| \leq \lambda$ for some $\lambda \in \mathbb{R}$, then u is positive. In reverse direction, for every $\lambda \in \mathbb{R}$, if $\|u\| \leq \lambda$ and u is positive, then $\|u - \lambda e\| \leq \lambda$.
(6) For every $u, v, w \in \mathcal{V}_h$, $u \sqsubseteq v$ implies $u + w \sqsubseteq v + w$.
(7) if $\alpha, \beta \geq 0$ then for each $u, v \in \mathcal{V}_+$, $\alpha u + \beta v \in \mathcal{V}_+$.
(8) $\mathcal{V}_+ = \{u^* u : u \in \mathcal{V}\}$.
(9) For all $u, v \in \mathcal{V}_+$, if $e \sqsubseteq u \sqsubseteq v$ then $\|u\| \leq \|v\|$.
(10) Assume that $u, v \in \mathcal{V}$ then $u \sqsubseteq v$ implies $u^{\frac{1}{2}} \preceq v^{\frac{1}{2}}$.
(11) Let $u \in \mathcal{V}_+$ and $u \sqsubseteq v$. Then for any $w \in \mathcal{V}$ both $w^* u w$ and $w^* v w$ are positive elements and $w^* u w \sqsubseteq w^* v w$.

Using $\mathcal{V}_+$ on define a C*-algebra-valued metric space that is, C*-valued b-metric spaces as:

Definition 8. *Let $X \neq \emptyset$ and $s \in \mathcal{V}_+$ such that $e \sqsubseteq s$. Assume that the function $d : X \times X \to \mathcal{V}$ satisfies:*

(i) $e \sqsubseteq d(u, v)$ for all $u, v \in X$ and $d(u, v) = e$ if and only if $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, w) \sqsubseteq s(d(u, w) + d(w, v))$ for all $u, v, w \in X$.

Then we say that the triplet $(X, \mathcal{V}, d)$ is C*-algebra-valued b-metric space. If $s = e$ then we have C*-algebra-valued metric space.

The next two examples are particularly important in this framework:

Example 2. *For a Lebegue measurable set M, suppose that $X = L^{\infty}(M)$ and $\mathcal{H} = L^2(M)$. Let $L(\mathcal{H})$ be the set of bounded linear operators on the Hilbert space $\mathcal{H}$. So, $L(\mathcal{H})$ is a C^*-algebra with the usual operator norm. Define $d : X \times X \to L(\mathcal{H})$ by*

$$d(f,g) = \Lambda_{|f-g|} \textit{ for all } f,g \in X = L^{\infty}(M),$$

in which $\Lambda_{\eta} : \mathcal{H} \to \mathcal{H}$ is multiplication operator defined by $\Lambda_{\eta}(\mathcal{T}) = \eta \cdot \mathcal{T}$, for $\mathcal{T} \in X = L^{\infty}(M)$. Now, we have that $(X, L(\mathcal{H}), d)$ is a complete C^*-algebra-valued metric space.

Example 3. *Suppose that $X = \mathbb{R}$ and $\mathcal{V} = \mathcal{M}_n(\mathbb{R})$. Consider*

$$d(u,v) = diag\left(r_1 |u-v|^p, r_2 |u-v|^p, \ldots, r_n |u-v|^p\right),$$

in which diag denotes a diagonal matrix, and, $u,v \in \mathbb{R}, r_i > 0$ $(i = 1,2,\ldots,n)$ are constants and $p > 1$. It is not hard to check that $(X, \mathcal{M}_n(\mathbb{R}), d)$ is a complete C^*-algebra-valued b-metric space. We shall check only (iii) of Definition 8. The inequality,

$$|u-w|^p \leq 2^p \left(|u-v|^p + |v-w|^p\right),$$

implies that $d(u,v) \sqsubseteq s(d(u,w) + d(w,v))$ for all $u,v,w \in X$, where $s = 2^p \cdot e$ and $e \sqsubset s$ because $1 < 2^p$. However, $|u-w|^p \leq |u-v|^p + |v-w|^p$ is impossible for each $u \sqsubset v \sqsubset w$. Therefore, $(X, \mathcal{M}_n(\mathbb{R}), d)$ is not C^*-agebra-valued metric space.

One application of previous example is:

Example 4. *Consider the next well known integral equation:*

$$u(t) = \int_M \mathcal{K}(t, u(r))\, dr + v(t), t \in M,$$

where M is a Lebesgue measurable set. Suppose also that

(1) $\mathcal{K} : M \times \mathbb{R} \to \mathbb{R}$ and $v \in L^{\infty}(M)$;

(2) there exists a continuous function $\psi : M \times M \to \mathbb{R}$ and $\lambda \in (0,1)$ such that

$$|\mathcal{K}(t, u(r)) - \mathcal{K}(t, v(s))| \leq \lambda |\psi(t,r)| \, \|u(r) - v(r)\|$$

for $t \in M$ and $u, v \in L^{\infty}(M)$.

(3) $\sup_{t \in M} \int_M |\psi(t,r)|\, dr \leq 1$.

Then the integral equation has a unique solution $\overline{u}$ in $L^{\infty}(E)$.

Proof. Let $X = L^{\infty}(M)$ and $\mathcal{H} = L^2(M)$. For $f,g \in X$ and $q > 1$, we set $d : X \times X \to L(\mathcal{H})$ by $d(f,g) = \Lambda_{|f-g|^q}$ where $\Lambda_{|f-g|^q} : \mathcal{H} \to \mathcal{H}$ is multiplication operator defined by $\Lambda_{|f-g|^q}(\mathcal{T}) = |f-g|^q \cdot \mathcal{T}$, for $\mathcal{T} \in X = L^{\infty}(M)$. Then $(X, L(\mathcal{H}), d)$ is a complete C^*-algebra-valued metric space (Example 2). Define now $\mathcal{F} : L^{\infty}(M) \to L^{\infty}(M)$ by

$$\mathcal{F}(u(t)) = \int_M \mathcal{K}(t, u(r))\, dr + v(t), t \in M.$$

Set $a = \lambda e$, then $a \in L(\mathcal{H})_+$ and $\|a\| = \lambda < 1$. For any $\mathcal{T} \in \mathcal{H}$, we get

$$\begin{aligned}
\|d(\mathcal{F}(u), \mathcal{F}(v))\| &= \sup_{\|T\|=1} \left(\Lambda_{|f-g|^q}\mathcal{T}, \mathcal{T}\right) = \sup_{\|T\|=1} \int_M \left[\left|\int_M (\mathcal{K}(t, u(r)) - \mathcal{K}(t, v(r)))\right|^q\right] \mathcal{T}(t)\overline{\mathcal{T}(t)}dt \\
&\leq \sup_{\|T\|=1} \int_M \left[\int_M |\mathcal{K}(t, u(r)) - \mathcal{K}(t, v(r))|^q\right] |\mathcal{T}(t)|^2 dt \\
&\leq \lambda^q \sup_{\|T\|=1} \int_M \left[\int_M |\psi(t, r)| dr\right]^q |\mathcal{T}(t)|^2 dt \cdot \|u - v\|_\infty^q \\
&\leq \lambda \sup_{t \in M} \int_M |\psi(t, r)| dr \cdot \sup_{\|T\|=1} \int_M |\mathcal{T}(t)|^2 dt \cdot \|u - v\|_\infty^q \\
&\leq \lambda \|u - v\|_\infty^q = \|a\| \cdot \|d(u, v)\|.
\end{aligned}$$

Since $\|a\| < 1$, the given integral equation has a unique solution $\overline{u}$ in $X = L^\infty(M)$. □

Remark 5. *For more details on other results from C^*-algebra-cone metric spaces that is, from C^*-algebra-cone b-metric spaces the reader can be see [40–42,44].*

Here is another application for our results.

Theorem 8. *For any positive integer n and non-negative real number b with $b \leq n$, $3b < 2n$ and $n < 3b$, the equation*

$$2x^n + b = 3x^{n+1} + nx$$

has a real solution in $[0, 1]$.

Proof. Given $n \in \mathbb{N}$ and a non-negative real number b with $b \leq n$, $3b < 2n$ and $n < 3b$. Let $\mathcal{B} = (-\infty, +\infty)$. Then $\mathcal{B}$ with usual multiplication is a Banach algebra. Let $P = [0, +\infty)$. Then P is a cone on $\mathcal{B}$. Define $\preceq$ on $\mathcal{B}$ with respect to P via $m \preceq t$ if $m \leq t$. Let $W = [0, 1]$. Define $d : W \times W \to \mathcal{B}$ via $d(t, m) = |t - m|$. Then (W, d) is a cone b-metric space over $\mathcal{B}$. Define $h : W \to W$ via

$$ht = \frac{2t^n + b}{3t^n + nt}.$$

Also, define $\alpha : W \times W \to P$ via $\alpha(t, m) = 1$. Then for $t, m \in W$, we have

$$\begin{aligned}
\alpha(t, m)d(ht, hm) &= |ht - hm| \\
&= \left|\frac{2t^n + b}{3t^n + n} - \frac{2m^n + b}{3m^n + n}\right| \\
&= \frac{(2n - 3b)|t^n - m^n|}{(3t^n + n)(3m^n + n)} \\
&\leq \frac{2n - 3b}{n}|t - m| \\
&= \frac{2n - 3b}{n}d(t, m).
\end{aligned}$$

Please note that $\frac{2n-3b}{n} < 1$. So h is α-admissible Hardy-Rogers contraction. Moreover, note that h satisfies all the hypothesis of Theorem 1 with $a_1 = \frac{2n-3b}{n}$ and $a_2 = a_3 = a_4 = a_5 = 0$. Thus h has a fixed point in W say u. Thus $hu = u$ and hence u is a solution of

$$2x^n + b = 3x^{n+1} + nx$$

□

Taking $n = 100$ and $b = 34$ in Theorem 8, we have the following result:

Example 5. *The equation*

$$2x^{100} + 34 = 3x^{101} + 100x$$

has a real solution in $[0, 1]$.

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