symmetry

Article

Total Roman {3}-domination in Graphs

Zehui Shao [1], **Doost Ali Mojdeh** [2,*] and **Lutz Volkmann** [3]

[1] Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China; zshao@gzhu.edu.cn
[2] Department of Mathematics, University of Mazandaran, Babolsar 47416-95447, Iran
[3] Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany; volkm@math2.rwth-aachen.de
* Correspondence: damojdeh@umz.ac.ir; Tel.: +98-1135302448

Received: 4 January 2020; Accepted: 3 February 2020; Published: 9 February 2020

Abstract: For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, a Roman $\{3\}$-dominating function (R$\{3\}$-DF) is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that $\sum_{u \in N_G(v)} f(u) \geq 3$, if $f(v) = 0$, and $\sum_{u \in N_G(v)} f(u) \geq 2$, if $f(v) = 1$ for any vertex $v \in V(G)$. The weight of a Roman $\{3\}$-dominating function f is the sum $f(V) = \sum_{v \in V(G)} f(v)$ and the minimum weight of a Roman $\{3\}$-dominating function on G is the Roman $\{3\}$-domination number of G, denoted by $\gamma_{\{R3\}}(G)$. Let G be a graph with no isolated vertices. The total Roman $\{3\}$-dominating function on G is an R$\{3\}$-DF f on G with the additional property that every vertex $v \in V$ with $f(v) \neq 0$ has a neighbor w with $f(w) \neq 0$. The minimum weight of a total Roman $\{3\}$-dominating function on G, is called the total Roman $\{3\}$-domination number denoted by $\gamma_{t\{R3\}}(G)$. We initiate the study of total Roman $\{3\}$-domination and show its relationship to other domination parameters. We present an upper bound on the total Roman $\{3\}$-domination number of a connected graph G in terms of the order of G and characterize the graphs attaining this bound. Finally, we investigate the complexity of total Roman $\{3\}$-domination for bipartite graphs.

Keywords: Roman domination; Roman $\{3\}$-domination; Total Roman $\{3\}$-domination

1. Introduction

In this paper, we introduce and study a variant of Roman dominating functions, namely, total Roman $\{3\}$-dominating functions. First we present some necessary terminology and notation. Let $G = (V, E)$ be a graph of order n with vertex set $V = V(G)$ and edge set $E = E(G)$. The open neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = N(v) = \{u : uv \in E(G)\}$. The closed neighborhood of a vertex $v \in V(G)$ is $N_G[v] = N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N(v)$. The closed neighborhood of a set $S \subseteq V$ is the set $N_G[S] = N[S] = N(S) \cup S = \bigcup_{v \in S} N[v]$. We denote the degree of v by $d_G(v) = d(v) = |N(v)|$. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the maximum degree and minimum degree of a graph G, respectively. A vertex of degree one is called a leaf and its neighbor a support vertex. We denote the set of leaves and support vertices of a graph G by $L(G)$ and $S(G)$, respectively. We write K_n, P_n and C_n for the complete graph, path and cycle of order n, respectively. A tree T is an acyclic connected graph. The corona $H \circ K_1$ of a graph H is the graph constructed from H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. The union of two graphs G_1 and G_2 ($G_1 \cup G_2$) is a graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

A set $S \subseteq V$ in a graph G is called a dominating set if $N[S] = V$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G, [1]. A set $S \subseteq V$ in a graph G is called a total dominating set if $N(S) = V$.

The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set in G, and a total dominating set of G of cardinality $\gamma_t(G)$ is called a γ_t-set of G, [2].

Given a graph G and a positive integer m, assume that $g : V(G) \to \{0, 1, 2, \ldots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \ldots, V_m)$ is the ordered partition of V induced by g, where $V_i = \{v \in V : g(v) = i\}$ for $i \in \{0, 1, \ldots, m\}$. So we can write $g = (V_0, V_1, V_2, \ldots, V_m)$. A Roman dominating function on graph G is a function $f : V \to \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ such that $w \in V_2$. The weight of a Roman dominating function (RDF) is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every Roman dominating function f on G is called the Roman domination number of G, denoted by $\gamma_R(G)$, see also [3].

Let G be a graph with no isolated vertices. The total Roman dominating function (TRDF) on G, is an RDF f on G with the additional property that every vertex $v \in V$ with $f(v) \neq 0$ has a neighbor w with $f(w) \neq 0$. The minimum weight of any TRDF on G is called the total Roman domination number of G denoted by $\gamma_{tR}(G)$. A TRDF on G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function.

The mathematical concept of Roman domination, is originally defined and discussed by Stewart [4] in 1999, and ReVelle and Rosing [5] in 2000. Recently, Chellali et al. [6] have introduced the Roman $\{2\}$-dominating function f as follows. A Roman $\{2\}$-dominating function is a function $f : V \to \{0, 1, 2\}$ such that for every vertex $v \in V$, with $f(v) = 0$, $f(N(v)) \geq 2$ where $f(N(v)) = \sum_{x \in N(v)} f(x)$, that is, either v has a neighbor u with $f(u) = 2$, or has two neighbors x, y with $f(x) = f(y) = 1$ [7].

In terms of the Roman Empire, this defense strategy requires that every location with no legion has a neighboring location with two legions, or at least two neighboring locations with one legion each.

Note that for a Roman $\{2\}$-dominating function (R$\{2\}$-DF) f, and for some vertex v with $f(v) = 1$, it is possible that $f(N(v)) = 0$. The sum $w_f = \sum_{v \in V(G)} f(v)$ is denoted the weight of a Roman $\{2\}$-dominating function, and the minimum weight of a Roman $\{2\}$-dominating function f is the Roman $\{2\}$-domination number, denoted by $\gamma_{\{R2\}}(G)$. Roman $\{2\}$-domination is a generalization of Roman domination that has also studied by Henning and Klostermeyer [8] with the name Italian domination.

The total Roman $\{2\}$-domination for graphs are defined as follows [9]. Let G be a graph without isolated vertices. Then $f : V \to \{0, 1, 2\}$ is total Roman$\{2\}$-dominating function (TR$\{2\}$-DF) if it is a Roman $\{2\}$-dominating function and the subgraph induced by the positive weight vertices has no isolated vertex. The minimum weight $w_f = \sum_{v \in V(G)} f(v)$ of a any total Roman$\{2\}$-dominating function of a graph G is called the total Roman $\{2\}$-domination number of G and is denoted by $\gamma_{t\{R2\}}(G)$. Beeler et al. [10] have defined double Roman domination.

A double Roman dominating function (DRDF) on a graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are hold:

(a) if $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
(b) if $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a double Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called the double Roman domination number of G. We denote this number with $\gamma_{dR}(G)$ and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-function of G, see also [11].

Hao et al. [12] have recently defined total double Roman domination. The *total double Roman dominating function* (TDRDF) on a graph G with no isolated vertex is a DRDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V(G) : f(v) \neq 0\}$ has no isolated vertices. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a TDRDF on G. A TDRDF on G with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$-*function*. Mojdeh et al. [13] have recently defined the Roman $\{3\}$-dominating function correspondingly to the Roman $\{2\}$-dominating function of graphs. For a graph G, a Roman $\{3\}$-dominating function (R$\{3\}$-DF) is a function $f : V \to \{0, 1, 2, 3\}$ having the property that $f(N[u]) \geq 3$ for every vertex $u \in V$ with $f(u) \in \{0, 1\}$. Formally, a Roman

$\{3\}$-dominating function $f : V \to \{0,1,2,3\}$ has the property that for every vertex $v \in V$, with $f(v) = 0$, there exist at least either three vertices in $V_1 \cap N(v)$, or one vertex in $V_1 \cap N(v)$ and one in $V_2 \cap N(v)$, or two vertices in $V_2 \cap N(v)$, or one vertex in $V_3 \cap N(v)$ and for every vertex $v \in V$, with $f(v) = 1$, there exist at least either two vertices in $V_1 \cap N(v)$, or one vertex in $(V_2 \cup V_3) \cap N(v)$. This notion has been defined recently by Mojdeh and Volkmann [13] as Roman $\{3\}$-domination.

The weight of a Roman $\{3\}$-dominating function is the sum $w_f = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{3\}$-dominating function f is the Roman $\{3\}$-domination number, denoted by $\gamma_{\{R3\}}(G)$.

Now we introduce the total Roman $\{3\}$-domination concept to consider such situation.

Definition 1. *Let G be a graph G with no isolated vertex. The total Roman $\{3\}$-dominating function (TR$\{3\}$-DF) on G is an R$\{3\}$-DF f on G with the additional property that every vertex $v \in V$ with $f(v) \neq 0$ has a neighbor w with $f(w) \neq 0$, in the other words, the subgraph of G induced by the set $\{v \in V(G) : f(v) \neq 0\}$ has no isolated vertices. The minimum weight of a total Roman $\{3\}$-dominating function on G is called the* total Roman $\{3\}$-domination number *of G denoted by $\gamma_{t\{R3\}}(G)$. A $\gamma_{t\{R3\}}(G)$-function is a total Roman $\{3\}$-dominating function on G with weight $\gamma_{t\{R3\}}(G)$.*

In this paper We study of total Roman $\{3\}$-domination versus to other domination parameters. We present an upper bound on the total Roman $\{3\}$-domination number of a connected graph G in terms of the order of G and characterize the graphs attaining this bound. Finally, we investigate the complexity of total Roman $\{3\}$-domination for bipartite graphs.

2. Total Roman $\{3\}$-domination of Some Graphs

First we easily see that $\gamma_{\{R3\}}(G) \leq \gamma_{t\{R3\}}(G) \leq \gamma_{tdR}(G)$, because by the definitions every total Roman $\{3\}$-dominating function is a Roman $\{3\}$-dominating function and every total double Roman dominating function is a total Roman $\{3\}$-dominating function.

In [10] we have.

Proposition 1. ([10] Proposition 2) *Let G be a graph and $f = (V_0, V_1, V_2)$ a γ_R-function of G. Then $\gamma_{dR}(G) \leq 2|V_1| + 3|V_2|$. This bound is sharp.*

As an immediate result we also have:

Corollary 1. *Let G be a graph and $f = (V_0, V_1, V_2)$ a total Roman $\{2\}$-dominating function or a Roman dominating function for which the induced subgraph by $V_1 \cup V_2$ has no isolated vertex. Then $\gamma_{t\{R3\}}(G) \leq 2|V_1| + 3|V_2|$. This bound is sharp.*

For some special graphs we obtain the total Roman $\{3\}$-domination numbers.

Observation 1. *Let $n \geq 2$. Then* $\gamma_{t\{R3\}}(P_n) = \begin{cases} n+2 & \text{if } n \equiv 1 \pmod 3 \\ n+1 & \text{otherwise} \end{cases}$,

Proof. Let $P_n = v_1 v_2 \ldots v_n$. Since by assigning 2 to the vertices v_1 and v_n and value 1 to the other vertices, we have $\gamma_{t\{R3\}}(P_n) \leq n+2$. Since $f(v_1) + f(v_2) \geq 3$ and $f(v_{n-1}) + f(v_n) \geq 3$, $f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 3$ for $4 \leq i \leq n-3$, $f(v_{i-1}) + f(v_i) + f(v_{i+1}) + f(v_{i+2}) \geq 3$ for $4 \leq i \leq n-4$ and $f(v_{i-2}) + f(v_{i-1}) + f(v_i) + f(v_{i+1}) + f(v_{i+2}) \geq 4$ for $5 \leq i \leq n-4$, we observe that $\gamma_{t\{R3\}}(P_n) \geq n+1$ and $\gamma_{t\{R3\}}(P_n) \geq n+2$ if $n \equiv 1 \pmod 3$. If $n = 3k$, then by assigning 1 to v_{3t+1} and v_n, 2 to v_{3t+2}, 0 to v_{3t} except v_n, we have $\gamma_{t\{R3\}}(P_n) \geq 3k+1 = n+1$. If $n = 2+3k$, then by assigning 1 to v_{3t+1}, 2 to v_{3t+2}, 0 to v_{3t}, we have $\gamma_{t\{R3\}}(P_n) \geq 3k+1 = n+1$. Thus the proof is complete. □

In [10], it has been shown that $\gamma_{dR}(C_n) = n$ if $n \equiv 0,2,3,4 \pmod 6$ and otherwise $\gamma_{dR}(C_n) = n+1$ and since $\gamma_{tdR}(G) \geq \gamma_{dR}(G)$, we deduce that $\gamma_{tdR}(C_n) \geq n$.

Here we show that $\gamma_{t\{R3\}}(C_n) = n$ for all $n \geq 3$. If we assign weight 1 to every vertex of C_n, then it is a total Roman $\{3\}$-dominating function of C_n. Hence $\gamma_{t\{R3\}}(C_n) \leq n$. In [13], we have shown that $\gamma_{\{R3\}}(C_n) = n$. Since $\gamma_{\{R3\}}(C_n) \leq \gamma_{t\{R3\}}(C_n)$, we obtain the desired result.

Observation 2. $\gamma_{t\{R3\}}(C_n) = n$

The next result shows another family of graphs G with $\gamma_{t\{R3\}}(G) = |V(G)|$. Let C_n be a cycle with vertices $v_1, v_2, \ldots, v_n$ and P_m be a path with vertices $u_1, u_2, \ldots, u_m$ for which $u_1 = v_1$ and for some $2 \leq i \leq m$, $u_i \neq v_j$. Let H be a graph obtained from a cycle C_n and k paths like $P_{m_1}, P_{m_2}, \ldots, P_{m_k}$ $(1 \leq k \leq n)$ such that the first vertex of any path P_{m_i} must be v_i. Let G be a graph consisting of m graphs like H such that any both of them have at most one common vertex on their cycles. Figure 1 is a sample of graph G is formed of 4 cycles and 15 paths P_{m_i}, where $m_i \equiv 1(\mod 3)$.

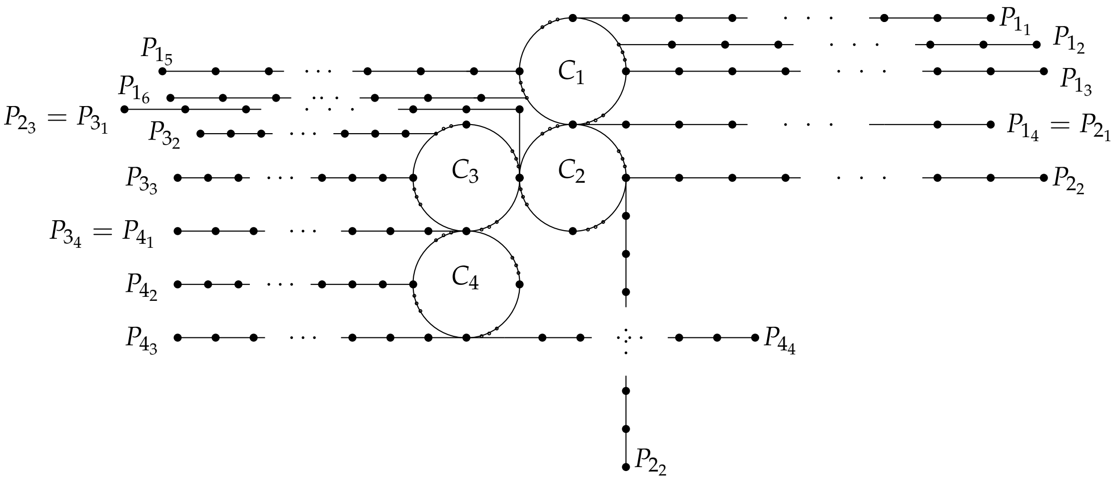

Figure 1. A sample of graph G.

Observation 3. *Let G be the graph constructed as above. If P_{m_i} with vertices $u_{1_i}, u_{2_i}, \ldots, u_{m_i}$ is a path such that $3 \mid (m_i - 1)$, then $\gamma_{t\{R3\}}(G) = |V(G)|$*

Proof. Let f be a function that assign value 1 to every vertex of the cycles and if $3 \mid (m_i - 1)$, we assign value 2 to vertices with indices $3t$, value 1 to vertices with indices $3t + 1$, $(1 \leq t \leq \frac{m_i - 1}{3})$ and value 0 to the other vertices of the path P_{m_i}, except to the common vertex $u_{1_i} = v_i$ of the cycle. Therefore $\gamma_{t\{R3\}}(G) = |V(G)|$. □

Let C_n be a cycle and P_m be a path with m vertices and let the first vertex of P_m be the vertex v_i of C_n. If $3 \mid m_i$ or $3 \mid (m_i - 2)$, then $\gamma_{t\{R3\}}(C_n \cup P_m) = |V(C_n \cup P_m)| + 1$. Therefore we have the following result.

Corollary 2. *In the graphs constructed above, if there are l paths $P_{m_1}, P_{m_2}, \ldots, P_{m_l}$ such that $3 \mid m_i$ or $3 \mid (m_i - 2)$ for $1 \leq m_i \leq l$, then $\gamma_{t\{R3\}}(G) = |V(G)| + l$.*

The Observation 3 and Corollary 2 show that for every nonnegative integer k, there is a graph G such that $\gamma_{t\{R3\}}(G) = |V(G)| + k$.

Proposition 2. *If G is a connected graph of order $n \geq 2$, then $\gamma_{t\{R3\}}(G) \geq 3$ and $\gamma_{t\{R3\}}(G) = 3$ if and only if G has at least two vertices of degree $\Delta(G) = n - 1$.*

Proof. If $n = 2$, then the statement is clear. Let now $n \geq 3$ and let $f = (V_0, V_1, V_2, V_3)$ be a total Roman $\{3\}$-dominating function on G of weight $\gamma_{\{R3\}}(G)$. If $V_0 \neq \emptyset$, then $\sum_{u \in N(v)} f(u) \geq 3$ for a vertex

$v \in V_0$ and thus $\gamma_{t\{R3\}}(G) \geq 3$. If $V_0 = \emptyset$, then $f(x) \geq 1$ for each vertex $x \in V(G)$ and therefore $\gamma_{t\{R3\}}(G) \geq n \geq 3$.

If G has at least two vertices of degree $\Delta(G) = n-1$, then we may assume v and u are two adjacent vertices of maximum degree. Define the function f by $f(v) = 1$, $f(u) = 2$ and $f(x) = 0$ for $x \in V(G) \setminus \{v, u\}$. Then f is a total Roman $\{3\}$-dominating function on G of weight 3 and hence $\gamma_{t\{R3\}}(G) = 3$.

Conversely, assume that $\gamma_{t\{R3\}}(G) = 3$. Then there are two adjacent vertices v, u with weights 1 and 2 respectively, for which $n-2$ vertices with weight 0 are adjacent to them, or there are three mutuality adjacent vertices u, v, w with weights 1 for which $n-3$ vertices with weight 0 are adjacent to them. Therefore there are at least two vertices of degree $n-1$. □

As an immediate result we have:

Corollary 3. *If G has only one vertex of degree $\Delta(G) = n-1$, then $\gamma_{t\{R3\}}(G) = 4$.*

In the follow, total Roman $\{3\}$-domination and total double Roman domination numbers are compared.

Since any partite set of a bipartite graph is an independent set, the weight of total Roman $\{3\}$-domination number of any partite set is positive. Therefore we have the following.

Proposition 3. *For any complete bipartite graph we have.*

1. $\gamma_{t\{R3\}}(K_{1,n}) = 4$,
2. $\gamma_{t\{R3\}}(K_{m,n}) = 5$ *for* $m \in \{2, 3\}$ *and* $n \geq 3$.
3. $\gamma_{t\{R3\}}(K_{m,n}) = \gamma_{dR}(K_{m,n}) = 6$ *for* $m, n \geq 4$.

Proof. In any complete bipartite graph, let $V(G) = U \cup W$, where U is the small partite set and W is the big partite set.

1. This follows from Corollary 3.
2. We consider two cases.

   (i) Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. Let f be a TR{3}DF of $K_{2,n}$. If $f(W) = 2$, then $f(U) \geq 3$. If $f(W) = 3$, then $f(U) \geq 2$. If $f(W) \geq 4$, since $f(U)$ is positive, then $f(V) \geq 5$. Therefore $f(V) \geq 5$. Assigning $f(u_1) = 2$, $f(u_2) = 1$ and $f(w_1) = 2$, shows that $\gamma_{t\{R3\}}(K_{2,n}) \leq 5$.

   (ii) Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. Using sketch of the proof of item 2, $\gamma_{t\{R3\}}(K_{3,n}) \geq 5$. If we assign value 1 to the vertices u_1, u_2, u_3, weight 2 to w_1 and 0 to w_j, for $j \geq 2$, then $\gamma_{t\{R3\}}(K_{3,n}) \leq 5$.

3. The function f with $f(u_1) = 3 = f(w_1)$ and $f(u_i) = 0 = f(u_j)$ for $i, j \neq 1$ is a TR{3}DF for $K_{m,n}$. Therefore $\gamma_{t\{R3\}}(K_{m,n}) \leq 6$.

Now let f be a $\gamma_{t\{R3\}}$ function of $K_{m,n}$ for $m, n \geq 4$. If $m, n \geq 5$, then it is easy to see that f should be assigned 0 to at least one vertex of each partite set. Therefore every partite set must have weight at least 3. If, without loss of generality, $n = 4$, then let $U = \{u_1, u_2, u_3, u_4\}$. If $f(u_i) \geq 1$ for $1 \leq i \leq 4$, then $f(u_i) = 1$ for $1 \leq i \leq 4$ and thus $f(W) \geq 2$. So $f(V) \geq 6$ and therefore $\gamma_{t\{R3\}}(K_{m,n}) \geq 6$, and the proof is complete. □

One can obtain a similar result for complete r-partite graphs for $r \geq 3$.

Proposition 4. *Let $G = K_{n_1,n_2,\ldots,n_r}$ be the complete r-partite graph with $r \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_r$. Then:*

1\. *If $n_1 = n_2 = 1$, then $\gamma_{t\{R3\}}(G) = 3$.*

2. *If* $n_1 = 1$ *and* $n_2 \geq 2$, *then* $\gamma_{t\{R3\}}(G) = 4$.
3. *If* $n_1 = 2$ *or* $n_1 \geq 3$ *and* $r \geq 4$, *then* $\gamma_{t\{R3\}}(G) = 4$.
4. *If* $r = 3$ *and* $n_1 \geq 3$, *then* $\gamma_{t\{R3\}}(G) = 5$.

Proof. Let $V = \bigcup_{i=1}^{r} U_i$ where U_i is the ith partite set with vertices $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{n_i}}\}$.

1. This follows from Proposition 2.
2. This follows from Corollary 3.
3. Let $n_1 \geq 2$. By Proposition 2, we have $\gamma_{t\{R3\}}(G) \geq 4$. If $n_1 = 2$, then define $f(u_{1_1}) = f(u_{1_2}) = f(u_{2_1}) = f(u_{3_1}) = 1$ and $f(v) = 0$ otherwise. Then f is a TR$\{3\}$-DF on G with $f(V) = 4$ and thus $\gamma_{t\{R3\}}(G) = 4$. Now let $n_1 \geq 3$ and $r \geq 4$. Then any TR$\{3\}$-DF f on G with $f(u_{1_1}) = f(u_{2_1}) = f(u_{3_1}) = f(u_{4_1}) = 1$ and $f(v) = 0$ for the other vertices, is a $\gamma_{t\{R3\}}$ function on G. Therefore $\gamma_{t\{R3\}}(G) = 4$.
4. Let $n_1 \geq 3$ and $r = 3$, and let f be a TR$\{3\}$-DF function on G. Since two partite sets must have positive weight, we can assume $f(U_2) \geq 1$. If $f(U_2) = 1$, then $f(U_1 \cup U_3) \geq 4$. If $f(U_2) = 2$, then $f(U_1 \cup U_3) \geq 3$. If $f(U_2) = 3$, then $f(U_1 \cup U_3) \geq 2$. If $f(U_2) \geq 4$, then $f(U_1 \cup U_3) \geq 1$. Thus $f(V) \geq 5$. Conversely, define $f(u_{1_1}) = f(u_{2_1}) = 2$ and $f(u_{3_1}) = 1$ and $f(v) = 0$ otherwise. Then f is a TR$\{3\}$-DF on G with $f(V) = 5$ and so $\gamma_{t\{R3\}}(G) = 5$.

Theorem 4. *If G is a graph with* $\delta(G) = \delta \geq 2$, *then* $\gamma_{t\{R3\}}(G) \leq |V(G)| + 2 - \delta$, *and this bound is sharp.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, and let v be a vertex of degree δ with neighbors $\{u_1, u_2, \ldots, u_\delta\}$. Let $U = \{v, u_{\delta+2}, u_{\delta+3}, \ldots, u_n\} \cup \{u_1, u_2\}$. Define the function f by $f(x) = 1$ for $x \in U$ and $f(x) = 0$ for $x \in V(G) \setminus U$. Then $\sum_{x \in N(u)} f(x) \geq 2$ for $u \in U$ and $\sum_{x \in N(u)} f(x) \geq 3$ for $u \in V(G) \setminus U$. Therefore f is a total Roman $\{3\}$-dominating function on G of weight $n + 2 - \delta$ and thus $\gamma_{t\{R3\}}(G) \leq |V(G)| + 2 - \delta$.

According to Observation 2 and Propositions 3 and 8, we note that $\gamma_{t\{R3\}}(C_n) = n = |V(C_n)| + 2 - \delta(C_n)$, $\gamma_{t\{R3\}}(K_n) = 3 = |V(K_n)| + 2 - \delta(K_n)$ for $n \geq 3$, $\gamma_{t\{R3\}}(K_{3,3}) = 5 = |V(K_{3,3})| + 2 - \delta(K_{3,3})$, $\gamma_{t\{R3\}}(K_{4,4}) = 6 = |V(K_{4,4})| + 2 - \delta(K_{4,4})$, $\gamma_{t\{R3\}}(K_{3,3,3}) = 5 = |V(K_{3,3,3})| + 2 - \delta(K_{3,3,3})$ and $\gamma_{t\{R3\}}(K_{n_1,n_2,\ldots,n_r}) = 4 = |V(K_{n_1,n_2,\ldots,n_r})| + 2 - \delta(K_{n_1,n_2,\ldots,n_r})$ for $r \geq 4$ and $n_1 \leq n_2 \leq \cdots \leq n_r = 2$. All these examples demonstrate that the inequality $\gamma_{t\{R3\}}(G) \leq |V(G)| + 2 - \delta$ is sharp. □

Hao et al. defined in [12] the family of graphs $\mathcal{G}$ as follows and have proved Theorem 5 below. Let $\mathcal{G}$ be the family of graphs that can be obtained from a star $S_t = K_{1,t-1}$ of order $t \geq 2$ by adding a pendant edge to each vertex of $V(S_t)$ and adding any number of edges joining the leaves of S_t.

Theorem 5. [12] *For any connected graph G of order* $n \geq 2$,

$$\gamma_{tdR}(G) \leq 2n - \Delta$$

with equality if and only if $G \in \{P_2, P_3, C_3\} \cup \mathcal{G}$.

This theorem with a little changing may be explored as follows.

Theorem 6. *For any connected graph G of order* $n \geq 2$,

$$\gamma_{t\{R3\}}(G) \leq 2n - \Delta$$

with equality if and only if $G \in \{P_2, P_3\} \cup \mathcal{G}$.

3. Total Roman $\{3\}$-domination and Total Domination

In this section we study the relationship between total domination and total Roman $\{3\}$-domination of a graph.

In [10] (Proposition 8) the authors proved that, if G is a graph, then $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$. If we use the method of the proof of Proposition 8 of [10], then it is easy to show that:
If G is a graph with a $\gamma_{\{R3\}}$-function $f = (V_0, V_2, V_3)$, then $2\gamma(G) \leq \gamma_{\{R3\}}(G) \leq 3\gamma(G)$.
In [13] Proposition 17 authors proved that:
If G is a graph, then $\gamma(G) + 2 \leq \gamma_{\{R3\}}(G) \leq 3\gamma(G)$, and these bounds are sharp. However, we have the following.

Proposition 5. *If G is a graph without isolated vertices, then $\gamma_t(G) + 1 \leq \gamma_{t\{R3\}}(G) \leq 3\gamma_t(G)$.*

Proof. Let S be a γ_t-set of G. Then $(V_0 = V \setminus S, \emptyset, \emptyset, V_3 = S)$ is a $\gamma_{t\{R3\}}$-function of G. Therefore $\gamma_{t\{R3\}}(G) \leq 3\gamma_t(G)$.
For the lower bound, let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{t\{R3\}}$-function of G. We distinguish two cases.

Case 1. Let $|V_2| \geq 1$ or $|V_3| \geq 1$. Then $\gamma_t(G) \leq |V_1| + |V_2| + |V_3| \leq |V_1| + 2|V_2| + 3|V_3| - 1 = \gamma_{t\{R3\}}(G) - 1$.

Case 2. Let $V_2 = V_3 = \emptyset$. By the definition, $\delta(G[V_1]) \geq 2$. Therefore, for each vertex $v \in V_1$, the subgraph $G[V_1 \setminus \{v\}]$ does not contain an isolated vertex. Consequently, $V_1 \setminus \{v\}$ is total dominating set of G and hence $\gamma_t(G) \leq \gamma_{t\{R3\}}(G) - 1$. □

By Proposition 5 the question may arise as whether for any positive integer r, exists a graph G for which $\gamma_{t\{R3\}}(G) = \gamma_t(G) + r$, where $1 \leq r \leq 2\gamma_t(G)$. For $r = 1$ we have. If G is a connected graph of order $n \geq 2$ with at least two vertices of maximum degree $\Delta(G) = n - 1$, then Proposition 2 implies that $\gamma_{t\{R3\}}(G) = 3$. Since $\gamma_t(G) = 2$ for such graphs, we observe that $\gamma_{t\{R3\}}(G) = \gamma_t(G) + 1$.

Proposition 6. *If G is a graph without isolated vertices, then $\gamma_{t\{R3\}}(G) = \gamma_t(G) + 1$ if and only if G has at least two vertices of degree $\Delta = |V(G)| - 1$, in the other words $\gamma_{t\{R3\}}(G) = 3$ and $\gamma_t(G) = 2$.*

Proof. The part "if" has been proved. Part "only if": Let G be a graph with $\gamma_{t\{R3\}}(G) = \gamma_t(G) + 1$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{t\{R3\}}(G)$ function. Therefore $V_1 \cup V_2 \cup V_3$ is a total dominating set for G, and $|V_1| + |V_2| + |V_3| \geq \gamma_t(G) = \gamma_{t\{R3\}}(G) - 1 = |V_1| + 2|V_2| + 3V_3| - 1$. Therefore $|V_2| + 2|V_3| \leq 1$ that is $|V_2| \leq 1$ and $|V_3| = 0$. If $|V_2| = 1 = |V_1|$ or $|V_2| = 0$ and $|V_1| = 3$, then G has at least two vertices of degree $\Delta(G) = |V(G)| - 1$. Now we show that there are not any cases for G. On the contrary, we suppose that there are different cases. (1) $|V_2| = 1$ and $|V_1| \geq 2$. (2) $|V_2| = 0$ and $|V_1| \geq 4$.
Case 1. Let $V_2 = \{v\}$, $|V_1| \geq 2$. Assume first that there exist two vertices $v_1, v_2 \in V_1$ which are adjacent to the vertex v. Then $V_2 \cup V_1 \setminus \{v_1\}$ is a $\gamma_t(G)$-set of size $|V_1|$ and so $\gamma_{t\{R3\}}(G) = 2 + |V_1|$, a contradiction. Assume next that there exists only one vertex, say $v_1 \in V_1$, which is adjacent to v. Then all other vertices of V_1 have at least two neighbors in V_1. If $v_2 \in V_1$ with $v_2 \neq v_1$, then we observe that $V_2 \cup V_1 \setminus \{v_2\}$ is a $\gamma_t(G)$-set of size $|V_1|$. It follows that $\gamma_{t\{R3\}}(G) = 2 + |V_1|$, a contradiction.

Case 2. Let $|V_2| = 0$ and $|V_1| \geq 4$. Then there exist two vertices v_1, v_2 in which each of them has neighbors in $V_1 \setminus \{v_1, v_2\}$ and $G(V_1 \setminus \{v_1, v_2\})$ has no isolated vertex. Therefore $V_1 \setminus \{v_1, v_2\}$ is a $\gamma_t(G)$-set that is also a contradiction. □

Now we show that for any positive integer n and integer $2 \leq r \leq 2n$, there exists a graph G for which $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = n + r$.

Proposition 7. *Let n and r be positive integers with $2 \leq r \leq 2n$. Then there exists a graph G for which $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = n + r$.*

Proof. For graph G with $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = n + 2$, we consider the following graph. Let H be the graph consisting of a cycle C_{n+2} with $n \geq 3$ and a vertex set V_0 of $\binom{n+2}{3}$ further vertices. Let each vertex of V_0 be adjacent to 3 vertices of $V(C_{n+2})$ such that the neighborhoods of every two distinct vertices of V_0 are different. Let $V_1 = V(C_{n+2})$. Then $\gamma_{t\{R3\}}(H) = n + 2$ and $\gamma_t(H) = n$ (Figure 2).

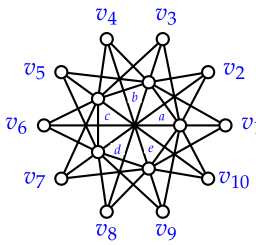

Figure 2. A graph H with $n = 3$.

For $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = n + k$, where $3 \leq k \leq n - 1$. Let $k = 3$. For $\gamma_t(G) = 4$ and $\gamma_{t\{R3\}}(G) = 7$, we consider the cycle C_7. For $\gamma_t(G) \geq 5$, let H be the above graph where $\gamma_t(H) = n \geq 3$ and $\gamma_{t\{R3\}}(H) = n + 2 \geq 5$. Now we consider $G = H \cup K_3$. Then $\gamma_t(G) \geq n \geq 4$ and $\gamma_{t\{R3\}}(G) = n + 3$.

Let $k = 4$. For $\gamma_t(G) = 5$ and $\gamma_{t\{R3\}}(G) = 9$, we consider the cycle C_9. For $\gamma_t(G) \geq 6$, consider the graphs G' with $\gamma_{t\{R3\}}(G') = \gamma_t(G') + 3$ for $\gamma_t(G') \geq 4$. Now we let $G = G' \cup K_3$. Then we have $\gamma_{t\{R3\}}(G) = \gamma_t(G) + 4$.

For $5 \leq k \leq n - 1$ we use induction on k. Let for any integer $4 \leq m \leq k - 1$ there exist graphs G' such that $\gamma_{t\{R3\}}(G) = \gamma_t(G) + m$ for $\gamma_t(G) \geq m + 1$. Let $m = k$. For $\gamma_t(G) = k + 1$ and $\gamma_{t\{R3\}}(G) = 2k + 1$, we consider the cycle C_{2k+1}. For graphs G with $\gamma_{t\{R3\}}(G) = \gamma_t(G) + k$ for $\gamma_t(G) \geq k + 2$, using hypothesis of induction, let G' be the graphs with $\gamma_{t\{R3\}}(G') = \gamma_t(G') + k - 1$ with $\gamma_t(G') \geq k$. Now we let $G = G' \cup K_3$. It can be seen $\gamma_{t\{R3\}}(G) = \gamma_t(G) + k$ for $\gamma_t(G) \geq k + 2$.

We now verify the case of $\gamma_{t\{R3\}}(G) = 2\gamma_t(G) + r$ for $0 \leq r \leq \gamma_t(G)$, that is, we wish to show the existence of graphs G, so that $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 2n + r$ for $0 \leq r \leq n$. Let $r = 0$. For even n, let $G = C_{2n}$. Then $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 2n$.
For odd $n = 2k + 1$, if $2n \equiv 1\ (mod\ 3)$ or $2n \equiv 0\ (mod\ 3)$, then we let $G = P_{2n-1}$, and by Observation 1, it can be seen that $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 2n$.
If $2n \equiv 2\ (mod\ 3)$, consider a cycle C_{2n-1} with an additional vertex a that is adjacent to two vertices v_1 and v_2. Then $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 2n$.

For $r = 1$ and positive even integer n, consider $G = (\frac{n}{2} - 1)P_3 \cup C_5^+$, where $(\frac{n}{2} - 1)P_3$ is the union of $\frac{n}{2} - 1$ of path P_3 and C_5^+ is the cycle C_5 with a chord, then $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 2n + 1$. For $r = 1$ and positive odd integer n, consider $G = (\frac{n-1}{2} - 1)P_3 \cup P_5^+$ where P_5^+ is the path P_5 with an additional vertex adjacent to the second or fourth vertex of P_5, then $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 2n + 1$.

For $2 \leq r \leq n - 1$, we do as follows. Let $r = 2$ and so $n \geq 3$. Let $n = 3$ and $2n + 2 = 8$. Let G_1 be a graph constructed from path P_5 with vertices v_1, v_2, v_3, v_4, v_5 with additional vertices $u_{12}, u_{13}, u_{14}, u_{52}, u_{53}, u_{54}, u_{24}$ such that the given vertex $u_{i,j}$ is adjacent to vertices v_i and v_j of P_5. Then $\gamma_t(G_1) = 3$ and $\gamma_{t\{R3\}}(G_1) = 8$.

Let $n = 4$ and $2n + 2 = 10$. Then say $G_2 = 2C_5^+$. Let $n = 5$ and so $2n + 2 = 12$. Then say $G_3 = C_5^+ \cup P_5^+$. For $\gamma_t(G) = k$ and $\gamma_{t\{R3\}}(G) = 2k + 2$, where $r + 1 \leq k \leq n$, there consider three cases.
1. If $k \equiv 0\ (mod\ 3)$, then we say $G = \frac{k-3}{3}P_5 \cup G_1$.
2. If $k \equiv 1\ (mod\ 3)$, then we say $G = \frac{k-4}{3}P_5 \cup G_2$.
3. If $k \equiv 2\ (mod\ 3)$, then we say $G = \frac{k-5}{3}P_5 \cup G_3$.

It is easy to verifiable, $\gamma_t(G) = k$ and $\gamma_{t\{R3\}}(G) = 2k + 2$.

Let $r = 3$ and so $n \geq 4$. For graph G_1' with $\gamma_t(G_1') = 4$ and $\gamma_{t\{R3\}}(G_1') = 11$, we let $G_1' = P_4 \cup C_5^+$. For graph G_2' with $\gamma_t(G_2') = 5$ and $\gamma_{t\{R3\}}(G_2') = 13$, we let $G_2' = P_4 \cup P_5^+$. And for graph G_3' with $\gamma_t(G_3') = 6$ and $\gamma_{t\{R3\}}(G_3') = 15$, we let $G_3' = 3C_5^+$. For $\gamma_t(G) = k$ and $\gamma_{t\{R3\}}(G) = 2k+3$, where $r+1 \leq k \leq n$, there consider three cases.
1. If $k \equiv 1 \ (mod\ 3)$, then we say $G = \frac{k-4}{3}P_5 \cup G_1'$.
2. If $k \equiv 2 \ (mod\ 3)$, then we say $G = \frac{k-5}{3}P_5 \cup G_2'$.
3. If $k \equiv 0 \ (mod\ 3)$, then we say $G = \frac{k-6}{3}P_5 \cup G_3'$.

Let $r \geq 4$ and $n \geq r+1$. For graph G with $\gamma_t(G) = k$ and $\gamma_{t\{R3\}}(G) = 2k+r$ where $r+1 \leq k \leq n$, there consider two cases.

Case 1. Let r be an even integer. Then there exists a graph G' for which $\gamma_t(G') = k-(r-2)$ and $\gamma_{t\{R3\}}(G') = 2k - 2(r-2) + 2$. Now let $G = \frac{r-2}{2}P_4 \cup G'$. Then $\gamma_t(G) = r - 2 + \gamma_t(G') = k$ and $\gamma_{t\{R3\}}(G) = 3(r-2) + \gamma_{t\{R3\}}(G') = 3(r-2) + 2k - 2(r-2) + 2 = 2k + r$.

Case 2. Let r be an odd integer. Then there exists a graph G'' for which $\gamma_t(G'') = k-(r-3)$ and $\gamma_{t\{R3\}}(G'') = 2k - 2(r-3) + 3$. If we consider $G = \frac{r-3}{2}P_4 \cup G''$. Then $\gamma_t(G) = r - 3 + \gamma_t(G'') = k$ and $\gamma_{t\{R3\}}(G) = 3(r-3) + \gamma_{t\{R3\}}(G'') = 2k + r$.

Finally, we want discuss the case of $r = n$, that is we want to find graphs G with $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 3n$. For $n = 2$ and $3n = 6$, let $G = P_4$. For G with $\gamma_t(G) = 3$ and $\gamma_{t\{R3\}}(G) = 9$, let $G = H_1$ be a graph constructed from P_5 with vertices v_1, v_2, v_3, v_4, v_5 with three additional vertices w_1, w_2, w_3 and three pendant edges v_2w_2, v_3w_3, v_4w_4. Then it can be seen that $\gamma_t(H_1) = 3$ and $\gamma_{t\{R3\}}(H_1) = 9$.

Let $n \geq 4$. If n is an even, then let $G = \frac{n}{2}P_4$ and if n is an odd, then let $G = \frac{n-3}{2}P_4 \cup H_1$. In both cases $\gamma_t(G) = n$ and $\gamma_{t\{R3\}}(G) = 3n$. □

4. Total Roman $\{3\}$ and Total Roman $\{2\}$-domination

In [13] it has been shown that, for a connected graph G with a $\gamma_{\{R3\}}$-function $f = (V_0, V_2, V_3)$, $\gamma_{\{R3\}}(G) \geq \gamma(G) + \gamma_{\{R2\}}(G)$.

In this section we investigate the relation between total Roman $\{3\}$ and total Roman $\{2\}$-domination. First we have the following.

Observation 7. *Let G be a graph and (V_0, V_1, V_2) be a $\gamma_{t\{R2\}}$ function of G. Then $(V_0' = V_0, V_2' = V_1, V_3' = V_2)$ is a TR$\{3\}$-DF function. Conversely, if (V_0, V_1, V_2, V_3) is a $\gamma_{t\{R3\}}$ of G, then $(U_0 = V_0, U_1 = V_1 \cup V_2, U_2 = V_3)$ is a TR$\{2\}$-DF of G.*

Proof. The proof is straightforward. □

The following results state the relation between $\gamma_{t\{R3\}}$ and $\gamma_{t\{R2\}}$ of graphs G when $\gamma_{t\{R3\}}(G)$ is small.

Proposition 8. *Let G be a graph. Then:*
1. *$\gamma_{t\{R3\}}(G) = 3$ if and only if $\gamma_{t\{R2\}}(G) = 2$.*
2. *If $\gamma_{t\{R3\}}(G) = 4$, then $\gamma_{t\{R2\}}(G) = 3$.*
3. *If $\gamma_{t\{R2\}}(G) = 3$, then $4 \leq \gamma_{t\{R3\}}(G) \leq 5$.*

Proof. 1. Let $\gamma_{t\{R3\}}(G) = 3$. Then there exist two adjacent vertices v, u with label $2, 1$ respectively so that each vertex with label 0 is adjacent to them or there exist three mutually adjacent vertices v, u, w with label 1 so that each vertex with label 0 is adjacent to them. In the first case, we change the vertex with label 2 to the label 1 and in the second case we change one of the vertices with label 1 to the label 0. These changing labels give us a $\gamma_{t\{R2\}}(G)$-function with weight 2. Conversely, let $\gamma_{t\{R2\}}(G) = 2$. Then there exist two vertices with label 1 for which every vertex is adjacent to them. We change one of the labels to 2, and therefore the result holds.

2. Let $\gamma_{t\{R3\}}(G) = 4$. There are three cases.

2.1. There exist 4 vertices v, u, w, z with label 1 for which the induced subgraph by them is the cycle C_4, the graph $K = K_4 - e$ or the complete graph K_4. In any induced subgraph, there are no two vertices of them for which any vertex with label 0 is adjacent to them. Thus in the case of a TR{2}-DF we change one of the labels 1 to the label 0. Therefore $\gamma_{t\{R2\}}(G) = 3$.

2.2. There exist 2 vertices v, u with label 1 and one vertex w with label 2, for which the induced subgraph by them is the cycle C_3, or the path $P_3 = v - w - u$. In any of the two cases each vertex with label 0 is adjacent to v, w or u, w or three of them. Now we change the label of w to 1, and we obtain a $\gamma_{t\{R2\}}$-function for G with weight 3.

2.3. There exist 2 vertices v, u with label 3 and label 1, respectively, for which the induced subgraph by v, u is K_2. By this assumption each vertex with label 0 is adjacent to v, but there maybe exist some vertices (none of them) which are adjacent to u. Now we change the label v to 2, and we obtain a $\gamma_{t\{R2\}}$-function for G with weight 3.

3. Let $\gamma_{t\{R2\}}(G) = 3$. There are two cases.

3.1. There exist 3 vertices v, u, w with label 1 for which the induced subgraph by v, u, w is the cycle C_3 or a path P_3. If each vertex with label 0 is adjacent to v, w or u, w, then by changing the label w to 2, we obtain a $\gamma_{t\{R3\}}$-function for G with weight 4.
If some vertices with label 0 are adjacent to v, u, some of them are adjacent to v, w and the other are adjacent u, w, then by changing two vertices of v, u, w to label 2, we obtain a $\gamma_{t\{R3\}}$-function for G with weight 5.

3.2. There exist 2 vertices v, u with label 2 and label 1, respectively, for which the induced subgraph by v, u is K_2. By this assumption each vertex with label 0 is adjacent to v, but there maybe exist some vertices (none of them) which are adjacent to u. Now we change the label v to 3, and we obtain a $\gamma_{t\{R3\}}$-function for G with weight 4. Therefore $4 \leq \gamma_{t\{R3\}}(G) \leq 5$. □

In the following we want to find the relation between total Roman {3}-domination, total domination and total Roman {2}-domination of graphs.

Observation 8. *Let G be a connected graph with a $\gamma_{t\{R3\}}$-function $f = (V_0, V_2, V_3)$. Then $\gamma_{t\{R3\}}(G) \geq \gamma_t(G) + \gamma_{t\{R2\}}(G)$.*

Proof. Let (V_0, V_2, V_3) be a $\gamma_{t\{R3\}}$-function of G. Then $\gamma_t(G) \leq |V_2| + |V_3|$. If we define $g = (V_0' = V_0, V_1' = V_2, V_2' = V_3)$, then g is a total Roman {2}-dominating function on G. Therefore $\gamma_t(G) + \gamma_{t\{R2\}}(G) \leq |V_2| + |V_3| + |V_1'| + 2|V_2'| \leq 2|V_2| + 3|V_3| = \gamma_{t\{R3\}}(G)$. □

In Observation 8 the condition of $\gamma_{t\{R3\}}$-function $f = (V_0, V_2, V_3)$ is necessary. Because there are many graphs for which the result of Observation 8 does not hold. For example, for the complete graphs K_n $(n \geq 2)$, cycles C_n and paths P_n for $n \geq 5$, we observe that $\gamma_{t\{R3\}}(G) < \gamma_t(G) + \gamma_{t\{R2\}}(G)$. However, in the following we establish, for any integer $n \geq 5$, there is a graph G such that $\gamma_{t\{R3\}}(G) = \gamma_{t\{R2\}}(G) + \gamma_t(G)$.

Proposition 9. *For any positive integer $n \geq 5$, there is a graph G for which $\gamma_{t\{R3\}}(G) = \gamma_{t\{R2\}}(G) + \gamma_t(G)$.*

Proof. For $n = 5$ let $G = C_5^+$. Then $\gamma_{t\{R2\}}(G) = 3$, $\gamma_t(G) = 2$ and $\gamma_{t\{R3\}}(G) = 5$. For $n = 6$, let G be a bistar of order 6. Then $\gamma_{t\{R3\}}(G) = 6 = 4 + 2 = \gamma_{t\{R2\}}(G) + \gamma_t(G)$. For $n = 7$, let $G = G_1$ in Figure 3. For $n = 8$, let $G = G_2$ in Figure 3. For $n = 9$, let $G = G_2$ in Figure 3. For $n \geq 10$, by induction we consider the graph $G = C_5^+ \cup H$ where the graph H (H may be connected or disconnected) for which $\gamma_{t\{R3\}}(H) = n - 5 = \gamma_{t\{R2\}}(H) + \gamma_t(H)$.

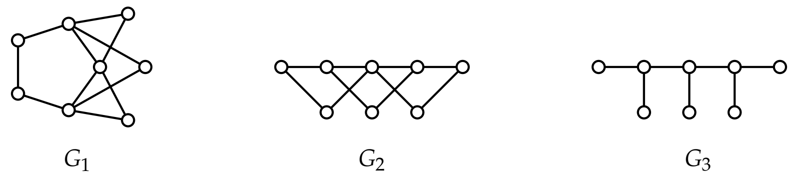

Figure 3. Examples.

□

Finally, we show that for any positive integer $n \geq 5$, there is a graph G such that $\gamma_{t\{R3\}}(G) = n$, $\gamma_{t\{R2\}}(G) = n - 1$ and $\gamma_t(G) = n - 2$.
For this, let G be the graph constructed in Proposition 3 as graph H for $n \geq 5$. Then $\gamma_{t\{R3\}}(H) = n$, $\gamma_{t\{R2\}}(H) = n - 1$ and $\gamma_t(H) = n - 2$.

5. Large Total Roman $\{3\}$-domination Number

In this section, we characterize connected graphs G of order n with $\gamma_{t\{R3\}}(G) = 2n - k$ for $1 \leq k \leq 4$. For this we use the following result.

Theorem 9. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{t\{R3\}}(G) \leq (3n)/2$, with equality if and only if G is the corona $H \circ K_1$ where H is a connected graph.*

Proof. If $n = 2$, then the statement is valid. Let now $n \geq 3$. If $|L(G)| \leq n/2$, then define $f : V(G) \to \{0, 1, 2, 3\}$ by $f(x) = 2$ for $x \in L(G)$ and $f(x) = 1$ for $x \in V(G) \setminus L(G)$. Then f is a total Roman $\{3\}$-dominating function on G of weight

$$2|L(G)| + n - |L(G)| = n + |L(G)| \leq \frac{3n}{2}.$$

If $|L(G)| > n/2$, then define $f : V(G) \to \{0, 1, 2, 3\}$ by $f(x) = 1$ for $x \in L(G)$ and $f(x) = 2$ for $x \in V(G) \setminus L(G)$. Then f is a total Roman $\{3\}$-dominating function on G of weight

$$|L(G)| + 2(n - |L(G)|) = 2n - |L(G)| < \frac{3n}{2}.$$

If $G = H \circ K_1$ for a is a connected graph H, then $\gamma_{t\{R3\}}(G) = (3n)/2$.

Conversely, let $\gamma_{t\{R3\}}(G) = (3n)/2$. Then the proof above shows that $|L(G)| = n/2$. Assume that there exists a vertex $v \in V(G)$ which is neither a leaf nor a support vertex. Define $f : V(G) \to \{0, 1, 2, 3\}$ by $f(x) = 1$ for $x \in L(G) \cup \{v\}$ and $f(x) = 2$ for $x \in V(G) \setminus (L(G) \cup \{v\})$. Then f is a total Roman $\{3\}$-dominating function on G of weight

$$|L(G)| + 2(n - |L(G)| - 1) + 1 = 2n - |L(G)| - 1 = \frac{3n}{2} - 1,$$

a contradiction. Thus every vertex is a leaf or a support vertex. Since $|L(G)| = n/2$, we deduce that $G = H \circ K_1$ with a connected graph H. □

Corollary 4. *For any connected graph G of order $n \geq 2$, $\gamma_{t\{R3\}}(G) = 2n - 1$ if and only if $G = P_2$.*

Proof. Let $\gamma_{t\{R3\}}(G) = 2n - 1$. Then Theorem 9 implies $2n - 1 \leq (3n)/2$ and thus $n = 2$. Clearly, the statement is valid for P_2. □

Corollary 5. *For any connected graph G of order $n \geq 3$, $\gamma_{t\{R3\}}(G) = 2n - 2$ if and only if $G \in \{P_3, P_4\}$.*

Proof. If $G \in \{P_3, P_4\}$, then the statement is valid. Conversely, let $\gamma_{t\{R3\}}(G) = 2n - 2$. Then Theorem 9 implies $2n - 2 \leq (3n)/2$ and thus $n \leq 4$, with equality if and only if $G = P_4$. In the remaining case $n = 3$, we observe that $G \in \{P_3, C_3\}$ with $\gamma_{t\{R3\}}(P_3) = 4$ and $\gamma_{t\{R3\}}(C_3) = 3$, and therefore $G = P_3$. □

Next we characterize the graphs G with the property that $\gamma_{t\{R3\}}(G) = 2|V(G)| - 3$.

Theorem 10. *For any connected graph G of order $n \geq 3$, $\gamma_{t\{R3\}}(G) = 2n - 3$ if and only if $G \in \{C_3, P_3 \circ K_1, C_3 \circ K_1\}$.*

Proof. If $G \in \{C_3, P_3 \circ K_1, C_3 \circ K_1\}$, then the statement is valid. Conversely, let $\gamma_{t\{R3\}}(G) = 2n - 3$. If $\Delta(G) = 2$, then $G \in \{P_n, C_n\}$ and we conclude by Observations 1, 2 that $G = C_3$. If $\Delta(G) = 3$, then $\gamma_{t\{R3\}}(G) = 2n - 3 = 2n - \Delta(G)$ and so by Theorem 6, $G \in \mathcal{G}$. Therefore $G \in \{P_3 \circ K_1, C_3 \circ K_1\}$. Let $\Delta(G) \geq 4$. Then by Theorem, 6 $\gamma_{t\{R3\}}(G) \leq 2n - \Delta(G) = 2n - 4 < 2n - 3$. Thus $G \in \{C_3, P_3 \circ K_1, C_3 \circ K_1\}$, and the proof is complete. □

Let $\mathcal{H}$ be the family of connected graphs order 5 with $\Delta(G) = 3$ which have exactly one leaf or the tree T_5 consisting of the path $v_1v_2v_3v_4$ such that v_2 is adjacent to a further vertex w.

Let $\mathcal{F}$ be the family of graphs $G = Q \circ K_1$ with a connected graph Q of order 4.

Observation 11. *If $G \in \{\mathcal{F}, \mathcal{H}\}$, then $\gamma_{t\{R3\}}(G) = 2n - 4$.*

Proof. Clearly, $\gamma_{t\{R3\}}(T_5) = 2n - 4 = 6$. Let $G \in \mathcal{H}$ be of order 5 with exactly one leaf u. If v is the support vertex of u, then $f : V(G) \to \{0, 1, 2, 3\}$ with $f(v) = 2$ and $f(x) = 1$ for $x \in V(G) \setminus \{v\}$ is a TR{3}-DF on G and therefore $\gamma_{t\{R3\}}(G) = 6 = 2n - 4$.

If $G = Q \circ K_1$ with a connected graph Q of order 4, then we have seen in proof of Theorem 9 that $\gamma_{t\{R3\}}(G) = (3n)/2 = 2n - 4 = 12$.

Theorem 12. *For any connected graph G of order $n \geq 4$, we have $\gamma_{t\{R3\}}(G) = 2n - 4$ if and only if $G \in \{C_4, P_5\} \cup \{claw, paw\} \cup \mathcal{H} \cup \mathcal{F}$ where claw is $K_{1,3}$ and paw is obtained from $K_{1,3}$ by adding one edge between two arbitrary distinct vertices.*

Proof. Let $G \in \{C_4, P_5\} \cup \{claw, paw\} \cup \mathcal{H} \cup \mathcal{F}$. By Observations 1, 2 and 11, we have $\gamma_{t\{R3\}}(G) = 2n - 4$.

Conversely, let $\gamma_{t\{R3\}}(G) = 2n - 4$. According to Theorem 9, we have $2n - 4 = \gamma_{t\{R3\}}(G) \leq (3n)/2$ and thus $n \leq 8$ with equality if and only if G is the corona $H \circ K_1$ with a connected graph H of order 4. Therefore $G \in \mathcal{F}$ if $n = 8$. Let now $n \leq 7$.

If $\Delta(G) = 2$, then $G \in \{P_n, C_n\}$ and by Observations 1, 2, we have $n = 2n - 4$ which implies $n = 4$ and $G = C_4$, or $n + 1 = 2n - 4$ which implies $n = 5$ and $G = P_5$ or $n + 2 = 2n - 4$ which implies $G = P_6$. Since $\gamma_{t\{R3\}}(C_4) = 4 = 2n - 4$ and $\gamma_{t\{R3\}}(P_5) = 6 = 2n - 4$ but $\gamma_{t\{R3\}}(P_6) = 7 \neq 2n - 4$, we deduce that $G \in \{C_4, P_5\}$.

Let now $\Delta(G) = 3$. Next we discuss the cases $n = 4, 5, 6$ or $n = 7$.

If $n = 4$, then for only two graphs G, the claw and the paw, we have $\gamma_{t\{R3\}}(G) = 4 = 2n - 4$.

If $n = 5$, it is simply verified that $\gamma_{t\{R3\}}(G) = 6 = 2n - 4$ if an only if $G \in \mathcal{H}$.

If $n = 6$, then let v be a vertex of degree 3 with the neighbors u_1, u_2, u_3, and let w_1 and w_2 be the remaining vertices. Assume, without loss of generality, that w_1 is adjacent to u_1.

Case 1: Assume that w_2 is adjacent to u_1. Then $f : V(G) \to \{0, 1, 2, 3\}$ with $f(v) = f(u_1) = 3$ and $f(x) = 0$ for $x \neq v, u_1$ is a TR{3}-DF on G and therefore $\gamma_{t\{R3\}}(G) \leq 6$.

Case 2: Assume that w_2 is adjacent to w_1. Then $f : V(G) \to \{0,1,2,3\}$ with $f(v) = f(w_1) = 3$, $f(u_1) = 1$ and $f(x) = 0$ for $x \neq v, u_1, w_1$ is a TR{3}-DF on G and therefore $\gamma_{t\{R3\}}(G) \leq 7$.

Case 3: Assume that w_2 is adjacent to u_2 or u_3, say u_2. If there are no further edges, then $\gamma_{t\{R3\}}(G) = 9 \neq 2n-4$.

Now assume that there are further edges. If w_2 is adjacent to u_3, then $f : V(G) \to \{0,1,2,3\}$ with $f(u_1) = 2$ and $f(x) = 1$ for $x \neq u_1$ is a TR{3}-DF on G and therefore $\gamma_{t\{R3\}}(G) \leq 7$. If w_1 is adjacent to u_2, then $f : V(G) \to \{0,1,2,3\}$ with $f(v) = f(u_2) = 3$ and $f(x) = 0$ for $x \neq v, u_2$ is a TR{3}-DF on G and therefore $\gamma_{t\{R3\}}(G) \leq 6$. If u_1 is adjacent to u_2 and there are no further edges, then $\gamma_{t\{R3\}}(G) = 9 \neq 2n-4$. If finally, u_3 is adjacent to u_2 or u_1, say u_2, then $f : V(G) \to \{0,1,2,3\}$ with $f(u_1) = f(u_2) = 3$, $f(v) = 1$ and $f(x) = 0$ for $x \neq v, u_1, u_2$ is a TR{3}-DF on G and therefore $\gamma_{t\{R3\}}(G) \leq 7$. Thus we see that there is no graph G of order 6 with $\gamma_{t\{R3\}}(G) = 8 = 2n-4$.

Let now $n = 7$. If $|L(G)| \leq 2$, then define $f : V(G) \to \{0,1,2,3\}$ by $f(x) = 2$ for $x \in L(G)$ and $f(x) = 1$ for $x \in V(G) \setminus L(G)$. Then f is a total Roman {3}-dominating function on G of weight $9 < 10 = 2n-4$. If $|L(G)| \geq 4$, then define $f : V(G) \to \{0,1,2,3\}$ by $f(x) = 0$ for $x \in L(G)$ and $f(x) = 3$ for $x \in V(G) \setminus L(G)$. Then f is a total Roman {3}-dominating function on G of weight $9 < 10 = 2n-4$.
Finally, assume that $|L(G)| = 3$. If G has exactly 3 support vertices, then define $f : V(G) \to \{0,1,2,3\}$ by $f(x) = 1$ for $x \in L(G)$, $f(x) = 2$ for $x \in S(G)$ and $f(x) = 0$ for the remaining vertex. Then f is a total Roman {3}-dominating function on G of weight $9 < 10 = 2n-4$. If G has exactly 2 support vertices, then define $f : V(G) \to \{0,1,2,3\}$ by $f(x) = 0$ for $x \in L(G)$, $f(x) = 3$ for $x \in S(G)$ and $f(x) = 1$ for the remaining two vertices. Then f is a total Roman {3}-dominating function on G of weight $8 < 10 = 2n-4$.
Let $\Delta(G) = 4$. By Theorem 6, $\gamma_{t\{R3\}}(G) = 2n-4$ if and only if $G \in \mathcal{G} \subseteq \mathcal{F}$.
Let $\Delta(G) \geq 5$. Then by Theorem 6 $\gamma_{t\{R3\}}(G) \leq 2n-5 < 2n-4$. Therefore the proof is complete. □

6. Complexity

In this section, we study the complexity of total Roman {3}-domination of graphs. We show that the total Roman {3}-domination problem is NP-complete for bipartite graphs. Consider the following decision problem.

Total Roman {3}-domination problem TR3DP.
Instance: Graph $G = (V, E)$, and a positive integer $k \leq |V|$.
Question: Does G have a total Roman {3}-domination of weight at most k?

It is well-known that the Exact-3-Cover (X3C) problem is NP-complete. We show that the NP-completeness of TR3D problem by reducing the Exact-3-Cover (X3C), to TR3D.

EXACT 3-COVER ($X3C$)

Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 13. *TR3D is NP-Complete for bipartite graphs.*

Proof. It is clear that TR3DP belongs to $\mathcal{NP}$. Now we show that, how to transform any instance of X3C into an instance G of TR3D so that, the solution one of them is equivalent to the solution of the other one. Let $X = \{x_1, x_2, \cdots, x_{3r}\}$ and $C = \{C_1, C_2, \cdots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we form a graph G_i obtained from a path $P_5 : y_{i_1}$-y_{i_2}-y_{i_3}-y_{i_4}-y_{i_5} by adding the edge $y_{i_2}y_{i_5}$. For each $C_j \in C$, we form a star $K_{1,5}$ with center c_j for which one leaf is labeled l_j. Let $L = \{l_1, l_2, \cdots, l_t\}$. Now to obtain a graph G, we add edges $l_jy_{i_1}$ if $y_{i_1} \in C_j$. Set $k =$

$4t+13r$. Let $H=\langle\bigcup_{i=1}^{3r}V(G_i)\rangle$ be the subgraph of G induced by the $\bigcup_{i=1}^{3r}V(G_i)$. Observe that for every total Roman $\{3\}$-dominating function f on G with $f(V(G_i))\geq 4$, all vertices on each cycle $C_4=y_{i_2}$-y_{i_3}-y_{i_4}-y_{i_5}-y_{i_2} are total Roman $\{3\}$-dominated. Moreover, since G_i has a total Roman $\{3\}$-domination number equal to 6, we can assume that $f(V(G_i))\leq 6$. More precisely, if $f(V(G_i))=6$, then, without loss of generality, we may assume that $f(y_{i_2})=f(y_{i_3})=f(y_{i_2})=f(y_{i_3})=1$ and $f(y_{i_1})=2$. If also, $f(V(G_i))\in\{4,5\}$, then obviously at least one vertex of G_i (including y_{i_1}) is not total Roman $\{3\}$-dominated. In this case, we can assume that vertices of G_i are assigned as $f(y_{i_2})=f(y_{i_3})=f(y_{i_4})=f(y_{i_5})=1$ so that, only y_{i_1} is not total Roman $\{3\}$-dominated and $f(y_{i_1})\in\{0,1\}$.

Suppose that the instance X, C of $X3C$ has a solution C'. We build a total Roman $\{3\}$-dominating function f on G of weight k. For every C_j, assign the value 2 to l_j if $C_j\in C'$ and 1 to the other l_j if $C_j\notin C'$. Assign value 3 to every c_j and value 0 to each leaf adjacent to c_j. Finally, for every i, assign 1 to $y_{i_2},y_{i_3},y_{i_4},y_{i_5}$, and 0 to y_{i_1} of G_i. Since C' exists, $|C'|=r$, the number of l_js with weight 2 is r, having disjoint neighborhoods in $\{y_{1_1},y_{2_1},\cdots,y_{3r_1}\}$, where every y_{i_1} has one neighbors assigned 1 and one neighbor assigned 2. Also since the number of l_js with weight 1 is $t-r$. Hence, it can be easily seen that f is a TR3-D function with weight $f(V)=3t+2r+t-r+12r=k$.

Conversely, let $g=(V_0,V_1,V_2,V_3)$ be a total Roman $\{3\}$-dominating function of G with weight at most k. Obviously, every star needs a weight of at least 4, and so without loss of generality, we may assume that $g(c_j)=3$ and all the leaves neighbor of c_j are assigned 0. Since $l_jc_j\in E(G)$, it implies that each vertex l_j can be assigned by 1. Moreover, for each i, $g(V(G_i))\in\{4,6\}$, as mentioned above. We can let the vertices of G_i are assigned the values given in the above paragraph depending on whether $g(V(G_i))=4$ or $g(V(G_i))=6$. Let p be the number of G_is having weight 6. Then $g(V(H))=6p+4(3r-p)=12r+2p$. Now, if $g(l_j)>1$ for some j, then l_j total Roman $\{3\}$-dominates some vertex y_{s_1}, and, in that case, $g(l_j)=2$ (since $g(y_{i_2})=1$). Let z be the number of l_js assigned 2 and $t-z$ of others be assigned 1. Then $3t+2z+t-z+12r+2p\leq k=4t+13r$, implies that $z+2p\leq q$. On the other hand, since each l_j has exactly three neighbors in $\{x_{1_1},x_{2_1},\cdots,x_{3r_1}\}$, we must have $3z\geq 3r-p$. From these two inequalities, we achieve at $p=0$ and then $z=q$. Consequently, $C'=\{C_j:g(l_j)=2\}$ is an exact cover for C. □

7. Open Problems

In the preceding sections a new model of total Roman domination, total Roman $\{R3\}$-domination has been introduced. There are the relationships between the total domination, total Roman $\{R2\}$-domination and total Roman $\{R3\}$-domination numbers as follows:

If G is a graph without isolated vertices, then $\gamma_t(G)+1\leq\gamma_{t\{R3\}}(G)\leq 3\gamma_t(G)$, (Proposition 5).

If G is a graph without isolated vertices, then $\gamma_{t\{R3\}}(G)=\gamma_t(G)+1$ if and only if G has at least two vertices of degree $\Delta=|V(G)|-1$, in the other words $\gamma_{t\{R3\}}(G)=3$ and $\gamma_t(G)=2$. (Proposition 6).

For any positive integer $n\geq 5$, there is a graph G of order n in which $\gamma_{t\{R3\}}(G)=\gamma_{t\{R2\}}(G)+\gamma_t(G)$, (Proposition 9).

For a family of graphs we have shown that $\gamma_{t\{R3\}}(G)=|V(G)|$, (Observation 3).

We have already characterized graphs G in which $\gamma_{t\{R3\}}(G)=2|V(G)|-r$, where $1\leq r\leq 4$.

Problems

1. Characterize the graphs G for which $\gamma_{t\{R3\}}(G)=3\gamma_t(G)$.
2. Does there exist any characterization of graphs G for which $\gamma_{t\{R3\}}(G)=\gamma_t(G)+r$, where $2\leq r\leq\gamma_t(G)-2$?
3. For positive integers $n\geq 5$, characterize the graphs G for which $\gamma_{t\{R3\}}(G)=\gamma_{t\{R2\}}(G)+\gamma_t(G)$.
4. Does there exist any characterization of graphs G for which $\gamma_{t\{R3\}}(G)=|V(G)|$?
5. Can one characterize graphs G in which $\gamma_{t\{R3\}}(G)=2|V(G)|-r$ for $5\leq r\leq|V(G)|-1$?

6. Is it possible to construct a polynomial algorithm for computing of $\gamma_{t\{R3\}}(T)$ for any tree T?

Author Contributions: Conceptualization, D.A.M., Z.S., and L.V.; writing, D.A.M.; revising, L.V., Z.S., and D.A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The first author's research work was supported by the National Key R & D Program of China (Grant No. 2019YFA0706402) and the Natural Science Foundation of Guangdong Province under Grant 2018A0303130115, and the second author's research work has been supported by a research grant from the University of Mazandaran. The authors thank the three referees of this paper for valuable suggestions and useful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mojdeh, D.A.; Firoozi, P.; Hasni, R. On connected (γ, k)-critical graphs. *Australas. J. Comb.* **2010**, *46*, 25–36.
2. Mojdeh, D.A.; Rad, N.J. On the total domination critical graphs. *Electron. Notes Discret. Math.* **2006**, *24*, 89–92. [CrossRef]
3. Alhevaz, A.; Darkooti, M.; Rahbani, H.; Shang, Y. Strong Equality of Perfect Roman and Weak Roman Domination in Trees. *Mathematics* **2019**, *7*, 997, doi:10.3390/math7100997. [CrossRef]
4. Stewart, I. Defend the Roman empire! *Sci. Am.* **1999**, *281*, 136–138. [CrossRef]
5. ReVelle, C.S.; Rosing, K.E. Defendens imperium romanum: A classical problem in military strategy. *Am. Math. Mon.* **2000**, *107*, 585–594. [CrossRef]
6. Chellali, M.; Haynes, T.W.; Hedetniemi, S.T.; McRae, A.A. Roman 2-domination. *Discret. Appl. Math.* **2016**, *204*, 22 – 28. [CrossRef]
7. Gao, H.; Xi, C.; Li, K.; Zhang, Q.; Yang, Y. The Italian Domination Numbers of Generalized Petersen Graphs P(n,3). *Mathematics* **2019**, *7*, 714. [CrossRef]
8. Henning, M.A.; Klostermeyer, W.F. Italian domination in trees. *Discret. Appl. Math.* **2017**, *217*, 557–564. [CrossRef]
9. García, S.C.; Martínez, A.C.; Mira, F.A.H.; Yero, I.G. Total Roman 2-domination in graphs. *Quaest. Math.* **2019**, 1–24. [CrossRef]
10. Beeler, R.A.; Haynes, T.W.; Hedetniemi, S.T. Double Roman domination. *Discret. Appl. Math.* **2016**, *211*, 23–29. [CrossRef]
11. Zhang, X.; Li, Z.; Jiang, H.; Shao, Z. Double Roman domination in trees. *Inf. Process. Lett.* **2018**, *134*, 31–34. [CrossRef]
12. Hao, G.; Volkmann, L.; Mojdeh, D.A. Total double Roman domination in graphs. *Commun. Comb. Optim.* **2020**, *5*, 27–39.
13. Mojdeh, D.A.; Volkmann, L. Roman 3-domination (Double Italian Domination). article in press.[CrossRef]