symmetry

MDPI

Article

A Symmetry-Based Approach for First-Passage-Times of Gauss-Markov Processes through Daniels-Type Boundaries

Enrica Pirozzi

Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, 80126 Napoli, Italy; enrica.pirozzi@unina.it

Received: 14 January 2020; Accepted: 6 February 2020; Published: 13 February 2020

check for updates

Abstract: Symmetry properties of the Brownian motion and of some diffusion processes are useful to specify the probability density functions and the first passage time density through specific boundaries. Here, we consider the class of Gauss-Markov processes and their symmetry properties. In particular, we study probability densities of such processes in presence of a couple of Daniels-type boundaries, for which closed form results exit. The main results of this paper are the alternative proofs to characterize the transition probability density between the two boundaries and the first passage time density exploiting exclusively symmetry properties. Explicit expressions are provided for Wiener and Ornstein-Uhlenbeck processes.

Keywords: Symmetry functions; transition probability density function; first-exit-time; two-sided region; diffusion processes

---

1. Introduction

Due the wide range of applications, from those of the financial context to those in biology, computational neurosciences, genetics, and physics, the Gaussian processes have always been of great interest (see, for instance, [1–3]). The theory of diffusion processes has extensively developed and many interesting mathematical results, particularly useful for applications, were obtained (see, for instance, [4–7]). Indeed, even if accurate models were designed to describe and investigate real phenomena, often these models are too complicated to be treated mathematically; in most of these cases the diffusion approximation provided the solution that was the right compromise between the need to have more realistic models and a simple and effective mathematical description. Some instances can be found in the field of computational neuroscience, such as in [8–11], or in the field of queueing theory ([12,13]).

In the last two decades, also the Gauss-Markov (GM) processes ([14]) are often involved in a similar way to specialize existing models; they constitute a simplified mathematical tool respect to stochastic process that are Gaussian but do not have the Markov property. Under specific hypotheses, the Gauss-Markov processes are also diffusions, hence, they are called Gauss-Diffusion processes; in this case, their transition density function solves the Fokker-Planck partial differential equation (pde), typical of diffusion processes ([15–17]), but with specified coefficient functions. Here, we will focus our attention on these kind of processes.

The central interest is the determination of the probability density function (pdf) of the first passage time (FPT) of these kind of processes through boundaries (constant or time-dependent). Some results were already obtained for diffusion processes, even if most of them were obtained by transformation methods of the Bachelier-Lévy formula (see, for instance, [18–22]) for the Brownian motion and a linear boundary. Then, for GM processes, in [14,23,24] can be found contributions in the direction of determination of closed forms and for the approximations of the FPT pdf in presence of specified boundaries. Also in these case, some space-time transformation between the involved processes were used. Specifically, we can recall that in [25] Daniels, by using the method of images, provided an expression for the transition pdf of the standard Wiener process in the presence of a particular time-dependent absorbing boundary. On the other hand, in [26] (for diffusion processes) and [14] (for Gauss-Markov processes) the strategy to solve a Volterra integral equation for the FPT pdf was developed. Along this approach, in very few cases can the integral equation be solved, with a closed form result derived; in all other cases, numerical procedures have been specialized ad hoc to obtained very accurate approximations of the solution.

An alternative analytical approach particularly effective is that based on the exploiting of some special features of the densities, such as the symmetry of the transition pdf. Indeed, in [27], in order to derive closed forms for FPT pdf for diffusion processes, the authors studied special symmetry conditions on the transition pdf in presence of particular time-dependent boundaries. Then, in [28], a careful investigation was carried out for diffusion processes and their first exit time of these processes through two-sided region delimited by an upper (respect to the initial position of the process) boundary and a lower boundary.

Here, we are aimed to exploit the symmetry properties of the transition pdf specifically for GM processes. Indeed, the symmetry approach has the advantage to allow discovering particular expressions (characterizations) for the involved probability functions and consequentially a more specific investigation of the process itself. It is worth remarking that the investigation of such symmetry properties differ from the study of symmetry properties of diffusion processes (as done in [28]), because essentially our results are derived only by using specific functions involving mean and covariance of the Gauss-Markov processes. In particular, this paper is devoted to new characterizations of the transition pdf of GM processes taking into account its symmetry properties. It is shown, indeed, that some of these representations reveal to be particularly useful for the determination of the closed form of first-exit-time (FET) from an open set confined between two boundaries. A large number of papers is devoted to investigate first passage problem of diffusion processes restricted between two boundaries, in the past but also recently, (see, for instance, [29–38]), also for possible applications ([39,40]). Here, we focus on Gauss-Markov processes between Daniels-type boundaries ([25]) for which the closed form can be derived also by the proposed symmetry approach. The alternative proofs to determine specific closed forms for the transition pdf and for the FET pdf are the main results of this paper. Some specific examples for well-known processes are also provided.

The paper is organized in following way: in Section 2 we give some preliminary definitions and conditions under which it is possible to have FET pdf in closed form. In Section 3, the symmetry curves, the corresponding symmetry functions and the symmetry properties of GM processes are considered. Furthermore, the Daniels-type curves are defined in (21) and a two-sided region is defined as the open set (subset of the process state space $(\mathbb{R} \times T)$), with such curves as lower and upper boundaries. Hence, the transition pdf in the specified two-sided region is defined. In Theorem 1 and in Section 4 specific expressions for the transition pdf between the two boundaries are provided, highlighting relations with symmetry functions. In Proposition 1 it is proved that the transition pdf in the two-sided region solves a Fokker-Planck equation. Section 5 is devoted to the characterization of FET pdf and the proof of the second main theorem (Theorem 2) is provided. Specifically, in Proposition 2 the distribution function of FET is given, whereas in Lemma 3 a preliminary representation of the FET pdf is given in order it can be exploited in the proof of the Theorem 2. Finally, in Section 6, for useful comparisons, two GM processes, Wiener and Ornstein-Uhlenbeck processes, are considered and the corresponding results are specified for them.

2. Essentials on Gauss-Markov Processes and FET

From [14,23,24,37], we take the following definitions for GM processes and first passage times random variables.

Let $T \subseteq \mathbb{R}$ be a continuous parameter set and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \{X(t)\}_{t\in T}, \mathbb{P})$ be a stochastic process with state space $\mathcal{S} = \mathbb{R}$. The process $\{X(t), t \in T\}$ is a real continuous Gauss-Markov process if it is normal and it has a continuous mean function $m(t) := \mathbb{E}[X(t)]$ in T and a continuous covariance function $c(s,t) := \mathbb{E}\{[X(s) - m(s)][X(t) - m(t)]\}$ in $T \times T$. Moreover, $\{X(t)\}$ is non-singular process except at the end points of T, i.e., if $T = [a, b]$ $(a, b \in \mathbb{R})$, then $\{X(t)\}$ is non-singularly normal distributed except possibly in $t = a$ or $t = b$, where $X(t)$ could be $X(t) = m(t)$ with probability 1.

More specifically, the covariance function $c(s,t)$ of a GM process is typically such that

$$c(s,t) = h_1(s)h_2(t), \quad s \le t, \quad s, t, \in \mathring{T} \tag{1}$$

with $h_1(t)$, $h_2(t)$ we call the covariance factors. The ratio function of the covariance factors, i.e., $r(t) = \frac{h_1(t)}{h_2(t)}$ is a monotonically increasing function; note that $h_1(t)h_2(t) > 0, \quad \forall t \in T$ due the process is non-singular in the interior of T. The transition mean and variance of the process $X(t)$ are

$$\begin{aligned} \mathbb{E}[X(t)|X(\tau) = y] &= m(t) + \frac{h_2(t)}{h_2(\tau)}[y - m(\tau)] \\ \mathrm{Var}[X(t)|X(\tau) = y] &= h_2(t)\left[h_1(t) - \frac{h_2(t)}{h_2(\tau)}h_1(\tau)\right] \end{aligned} \tag{2}$$

for $t, \tau \in T$, $\tau < t$, and the normal transition pdf $f(x,t|y,\tau)$ remains completly specified by the above quantities.

Now, we consider two $C^1(T)$-class functions, i.e., $S_1(t)$ and $S_2(t)$ such that

(i) $S_1(t) < S_2(t), \forall t \in T$
(ii) $S_1(t_0) < X(t_0) \equiv x_0 < S_2(t_0)$, $t_0 \in T$.

We call $S_1(t)$ the lower and $S_2(t)$ the upper boundary, respectively.

We define the following random variables, $\forall t \ge t_0$, $t, t_0 \in T$,

$$\begin{aligned} T^{(1)}_{x_0} &= \inf_{t\ge t_0}\{t : X(t) < S_1(t); X(\theta) < S_2(\theta), \forall\theta \in (t_0, t)\}, \quad X(t_0) = x_0 \\ T^{(2)}_{x_0} &= \inf_{t\ge t_0}\{t : X(t) > S_2(t); X(\theta) > S_1(\theta), \forall\theta \in (t_0, t)\}, \quad X(t_0) = x_0 \\ T_{x_0} &= \inf_{t\ge t_0}\{t : X(t) \notin (S_1(t), S_2(t))\}, \quad X(t_0) = x_0. \end{aligned} \tag{3}$$

Specifically, $T^{(1)}_{x_0}$ is the lower FPT through the boundary $S_1(t)$, $T^{(2)}_{x_0}$ is the upper FPT through $S_2(t)$ and T_{x_0} is the FET from the $\mathbb{R} \times \mathbb{R}$ open subset $(S_1(t), S_2(t))$, respectively. Furthermore, the respective pdfs are the following

$$\begin{aligned} g_1(t|x_0, t_0) &= \frac{\partial}{\partial t}P(T^{(1)}_{x_0} < t), \quad g_2(t|x_0, t_0) = \frac{\partial}{\partial t}P(T^{(2)}_{x_0} < t), \\ g(t|x_0, t_0) &= \frac{\partial}{\partial t}P(T_{x_0} < t) \equiv g_1(t|x_0, t_0) + g_2(t|x_0, t_0). \end{aligned} \tag{4}$$

We note that, for $X(t_0) = x_0$, if we consider the events, ,

$$\mathscr{E}_1 = \{\exists t \in (t_0, +\infty) : X(t) < S_1(t); X(\theta) < S_2(\theta), \forall\theta \in (t_0, t)\}$$

and

$$\mathscr{E}_2 = \{\exists t \in (t_0, +\infty) : X(t) > S_2(t); X(\theta) > S_1(\theta), \forall\theta \in (t_0, t)\}$$

the pdf $g_i(t|x_0,t_0)$, for $i=1,2$, are such that

$$\int_{t_0}^{+\infty} g_1(t|x_0,t_0)dt = P\{\mathscr{E}_1\} = P\{\mathscr{E}_1 \cup \mathscr{E}_2\} - P\{\mathscr{E}_2\} = P\{\mathscr{E}_1 \cup \mathscr{E}_2\} - \int_{t_0}^{+\infty} g_2(t|x_0,t_0)dt.$$

Consequently, we can denote the probability that $X(t)$ firstly attains $S_1(t)$ $[S_2(t)]$ by t without crossing $S_2(t)$ $[S_1(t)]$ with $P(T_{x_0}^{(1)} < t)$ $[P(T_{x_0}^{(2)} < t)]$, and with $P(T_{x_0} < t)$ we denote the probability that $X(t)$ firstly attains either $S_1(t)$ or $S_2(t)$ by time t. We recall that the pdfs $g_1(t|x_0,t_0)$ and $g_2(t|x_0,t_0)$ are solutions of the two coupled non-singular second kind Volterra integral equations ([23]):

$$\begin{aligned} g_1(t|x_0,t_0) &= 2\Psi_1(t|x_0,t_0) \\ &- 2\int_{t_0}^{t} \{g_1(\tau|x_0,t_0)\Psi_1[t|S_1(\tau),\tau] + g_2(\tau|x_0,t_0)\Psi_1[t|S_2(\tau),\tau]\}\, d\tau, \\ g_2(t|x_0,t_0) &= -2\Psi_2(t|x_0,t_0) \\ &+ 2\int_{t_0}^{t} \{g_1(\tau|x_0,t_0)\Psi_2[t|S_1(\tau),\tau] + g_2(\tau|x_0,t_0)\Psi_2[t|S_2(\tau),\tau]\}\, d\tau, \end{aligned} \tag{5}$$

with

$$\begin{aligned} \Psi_j(t|y,\tau) &= \left\{ \frac{S_j'(t)-m'(t)}{2} - \frac{S_j(t)-m(t)}{2}\,\frac{h_1'(t)h_2(\tau)-h_2'(t)h_1(\tau)}{h_1(t)h_2(\tau)-h_2(t)h_1(\tau)} \right. \\ &- \left. \frac{y-m(\tau)}{2}\,\frac{h_2'(t)h_1(t)-h_2(t)h_1'(t)}{h_1(t)h_2(\tau)-h_2(t)h_1(\tau)} \right\} f[S_j(t),t|y,\tau], \quad (j=1,2). \end{aligned} \tag{6}$$

and

$$\lim_{\tau\to t} \Psi_i[S_i(t),t|S_j(\tau),\tau] = 0 \quad (i,j=1,2), \tag{7}$$

recalling that $f[x,t|y,\tau]$ is the transition pdf of $X(t)$. By solving the system (5) it is possible to evaluate g from (4). Closed form results for (5) are known in only a few cases (cf., [14,23]).

Closed-Forms Results

From [23] we recall that integral equations (5) can be reduced to a single equation under some conditions. Indeed, under specific assumptions on the process and the boundaries, the first-exit time pdf $g(t \mid x_0,t_0)$ solves a single non-singular Volterra integral equation in place of Equations (5).

In addition to all previous assumptions, if the following conditions are satisfied, i.e.,

$$\lim_{t\to+\infty} r(t) = +\infty, \quad P\big\{S_1(t) \le X(t) < S_2(t) \mid X(t_0)=x_0\big\} \neq 1 \quad \forall t \in T,$$

we have:

$$\int_{t_0}^{+\infty} g(t \mid x_0,t_0)\, dt = 1, \tag{8}$$

and if $\forall t \ge t_0 \in T$

$$S_1(t) + S_2(t) = 2\,m(t) + 2\,c\,h_2(t), \qquad (c \in \mathbb{R}), \tag{9}$$

then the system (5) reduces to the following integral equation

$$\begin{aligned} g(t \mid x_0,t_0) &= 2\Big[\Psi_1(t \mid x_0,t_0) - \Psi_2(t \mid x_0,t_0)\Big] \\ &\quad -2\int_{t_0}^{t} g(\tau \mid x_0,t_0)\, \Big\{\Psi_1[t \mid S_1(\tau),\tau] - \Psi_2[t \mid S_1(\tau),\tau]\Big\}\, d\tau. \end{aligned} \tag{10}$$

Finally, if (9) holds for all $t \geq t_0$ and the initial state x_0 satisfies the following relation

$$x_0 = m(t_0) + c\, h_2(t_0), \qquad (c \in \mathbb{R}), \tag{11}$$

then one has

$$g_1(t \mid x_0, t_0) = g_2(t \mid x_0, t_0). \tag{12}$$

Hence, in this case, the solution $g(t \mid x_0, t_0)$ of the integral Equation (10) is given by

$$g(t \mid x_0, t_0) \equiv 2\, g_1(t \mid x_0, t_0) \equiv 2\, g_2(t \mid x_0, t_0). \tag{13}$$

In [23], solutions of (10) are given as series of functions when the two boundaries are specific functions of the mean and covariance of the GM process. Successively, such solutions are specialized for some GM processes and boundaries in [39]. In any other case, the system (5) can be solved by numerical procedures providing reliable approximations of the solutions.

3. Symmetry Properties

In the state space of the process $\{X(t), t \in T\}$, consider the following curves:

$$\begin{aligned} &y(t) = m(t) + d_1 h_1(t) + d_2 h_2(t), && \textit{(the symmetry curve or mirror)} \\ &u(t) = m(t) + d_1^* h_1(t) + d_2^* h_2(t), && \textit{(an assigned curve)} \\ &v(t) = 2y(t) - u(t), && \textit{(the symmetric curve of } u(t) \textit{ respect to the mirror } y(t)\textit{)} \end{aligned} \tag{14}$$

such that $v(t) < y(t) < u(t)\ \forall t \geq t_0$ with $t, t_0 \in T$, $d_1, d_2, d_1^*, d_2^* \in \mathbb{R}$, and the corresponding symmetry functions denoted by

$$\begin{aligned} &\psi_0(x,t), \quad \phi_0(x,t), \quad \textit{(associated with } y(t)\textit{)} \\ &\psi_1(x,t), \quad \phi_1(x,t), \quad \textit{(associated with } u(t)\textit{)} \\ &\psi_2(x,t), \quad \phi_2(x,t), \quad \textit{(associated with } v(t)\textit{)} \end{aligned} \tag{15}$$

with $v(t) = \psi_0(u(t), t)$.

The symmetry properties of GM processes ([28]) are such that for a general curve $z(t) = m(t) + ah_1(t) + bh_2(t)$, $(a, b \in \mathbb{R})$, with the associated symmetry functions

$$\psi(x,t) = 2z(t) - x, \qquad \phi(x,t) = \exp\left\{ -\frac{2a[x - z(t)]}{h_2(t)} \right\}, \tag{16}$$

the following relations hold

$$f(x,t|x_0,t_0) = \frac{\phi(x,t)}{\phi(x_0,t_0)} f[\psi(x,t), t|\psi(x_0,t_0), t_0] \tag{17}$$

$$\phi(x,t) f[\psi(x,t), t|x_0,t_0] = f(x,t|x_0,t_0) \exp\left\{ -\frac{2[x - z(t)](x_0 - z(t_0))}{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)} \right\}. \tag{18}$$

We point that the above relations written for the couple of functions (ψ, ϕ) hold for the symmetry functions (ψ_i, ϕ_i) , for $i = 0, 1, 2$, for symmetry curves $y(t)$, $u(t)$, $v(t)$ of (14), respectively.

3.1. Transition Distribution Function in a Two-Sided Region

Assuming that

$$\mathrm{P}\{X(\tau) = y(\tau)\} = 1$$

for a fixed $\tau \in T$, we denote by $S_1(t,\tau)$ and $S_2(t,\tau)$ the $C^1(T)$ functions such that, for $t \in T$ and $t \geq \tau$, verify the following conditions

(i) $S_1(t,\tau) < S_2(t,\tau) \qquad \forall t \geq \tau;$
(ii) $\lim_{t\downarrow\tau} S_1(t,\tau) < y(\tau) < \lim_{t\downarrow\tau} S_2(t,\tau).$

Furthermore, $\forall t \geq \tau$ and $x \in (S_1(t,\tau), S_2(t,\tau))$, we define

$$\mathrm{B}(x,t|y(\tau),\tau) = \mathrm{P}\{X(t) < x; S_1(\theta,\tau) < X(\theta) < S_2(\theta,\tau), \forall\theta \in (\tau,t)|X(\tau) = y(\tau)\} \tag{19}$$

and

$$\beta(x,t|y(\tau),\tau) = \frac{\partial}{\partial x}\mathrm{B}(x,t|y(\tau),\tau) \tag{20}$$

the transition probability distribution function of $X(t)$ between the two boundaries $S_1(t,\tau)$, $S_2(t,\tau)$, and its density, respectively.

Now, in the next theorem, we give our first main result in which we give a closed form expression for $\beta(x,t|y(\tau),\tau)$ when the boundaries $S_1(t,\tau)$, $S_2(t,\tau)$ are of Daniels type ([20,25]). (Note that we can also consider these boundaries as absorbing boundaries.)

Theorem 1. *For a fixed $t_0 \in T$, let the lower and upper boundaries be*

$$\begin{aligned} S_1(t,t_0) &= v(t) - \frac{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}{2(u(t_0) - y(t_0))} \ln\left[\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_2}\right] \\ S_2(t,t_0) &= u(t) + \frac{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}{2(u(t_0) - y(t_0))} \ln\left[\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right] \end{aligned} \tag{21}$$

with $u(t)$, $v(t)$ as in (14), $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\lim_{t\to supT} \Delta(t,t_0) > 0$
and

$$\Delta(t;t_0) = 1 - 4\alpha_1\alpha_2 \exp\left\{-\frac{4[u(t_0) - y(t_0)][u(t) - y(t)]}{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}\right\}. \tag{22}$$

Then, the transition probability density function of the process $X(t)$ between the two boundaries $S_1(t,t_0)$ and $S_2(t,t_0)$, for $t \geq t_0$, has the form

$$\begin{aligned} \beta(x,t|y(t_0),t_0) = f(x,t|y(t_0),t_0) &- \alpha_1\phi_1(x,t) f[\psi_1(x,t),t|y(t_0),t_0] \\ &- \alpha_2\phi_2(x,t) f[\psi_2(x,t),t|y(t_0),t_0] \end{aligned} \tag{23}$$

for $S_1(t,t_0) < x < S_2(t,t_0)$ and $v(t_0) < y(t_0) < u(t_0)$.

We will give the proof of the above theorem in Section 4.1, but before we need some preliminary results about some representations and properties of the transition probability density $\beta(x,t|y(t_0),t_0)$.

4. Characterization of the Transition Density in a Two-Sided Region

For $t_0 \in T$, let us define $\mathcal{D} = (S_1(t,t_0), S_2(t,t_0)) \subset (\mathbb{R} \times T)$ as the two-sided region. Consider $\tilde{\beta}(x,t|y(t_0),t_0)$ the right-hand-side of (23) defined on $(\mathcal{D} \times T)^2$.

Lemma 1. *The function $\tilde{\beta}(x,t|y(t_0),t_0)$ is such that*

$$\begin{aligned} \tilde{\beta}(x,t|y(t_0),t_0) = f(x,t|y(t_0),t_0) &- \alpha_1\phi_1[2u(t_0) - y(t_0),t_0] f[x,t|2u(t_0) - y(t_0),t_0] \\ &- \alpha_2\phi_2[2v(t_0) - y(t_0),t_0] f[x,t|2v(t_0) - y(t_0),t_0], \end{aligned} \tag{24}$$

with

$$\phi_1[2u(t_0) - y(t_0),t_0] = \exp\left\{-\frac{2d_1^*[u(t_0) - y(t_0)]}{h_2(t_0)}\right\} \tag{25}$$

and

$$\phi_2[2v(t_0) - y(t_0), t_0] \quad = \quad \exp\left\{-\frac{4[d_1^* - 2d_1][u(t_0) - y(t_0)]}{h_2(t_0)}\right\}. \tag{26}$$

Proof. Note that, referring to (15), $y(t_0)$ can also be written as follows

$$y(t_0) = 2u(t_0) - [2u(t_0) - y(t_0)] = \psi_1[2u(t_0) - y(t_0), t_0] = \psi_2[2v(t_0) - y(t_0), t_0]. \tag{27}$$

Hence, using (27) in the right-hand-side of (23), we have

$$\tilde{\beta}(x, t|y(t_0), t_0) = f(x, t|y(t_0), t_0) - \alpha_1\phi_1(x, t) f[\psi_1(x, t), t|\psi_1(2u(t_0) - y(t_0), t_0), t_0] \\ - \alpha_2\phi_2(x, t) f[\psi_2(x, t), t|\psi_2(2v(t_0) - y(t_0), t_0), t_0]. \tag{28}$$

Then, referring to the general symmetry relation (17) and setting $x_0 = y(t_0)$, one obtains

$$\phi(x, t) f[\psi(x, t), t|\psi(y(t_0), t_0), t_0] = \phi(y(t_0), t_0) f(x, t|y(t_0), t_0). \tag{29}$$

Finally, (24) is obtained by using (27) and (29) in (28). Furthermore, (25) is the second of (16) for the symmetry function $u(t) = m(t) + d_1^* h_1(t) + d_2^* h_2(t)$. Again by using the second of (16) for $v(t)$ and recalling that $v(t) = 2y(t) - u(t) = m(t) + (2d_1 - d_1^*)h_1(t) + (2d_2 - d_2^*)h_2(t)$, the (26) is obtained as follows

$$\phi_2[2v(t_0) - y(t_0), t_0] \quad = \quad \exp\left\{-\frac{2[2d_1 - d_1^*][(2v(t_0) - y(t_0)) - v(t_0)]}{h_2(t_0)}\right\} \tag{30}$$

$$= \quad \exp\left\{-\frac{4[2d_1 - d_1^*][\frac{v(t_0) - y(t_0)}{2}]}{h_2(t_0)}\right\} \tag{31}$$

$$= \quad \exp\left\{-\frac{4[d_1^* - 2d_1][u(t_0) - y(t_0)]}{h_2(t_0)}\right\}. \tag{32}$$

□

Remark 1. *We can also note that the function $\tilde{\beta}(x, t|y(t_0), t_0)$ can also be rewritten as*

$$\tilde{\beta}(x, t|y(t_0), t_0) = f(x, t|y(t_0), t_0) - \alpha_1\phi_1[2u(t_0) - y(t_0), t_0] f[x, t|2u(t_0) - y(t_0), t_0]$$

$$-\alpha_2\phi_2[2v(t_0) - y(t_0), t_0] f[x, t|3y(t_0) - 2u(t_0), t_0],$$

being $2v(t_0) - y(t_0) = 4y(t_0) - 2u(t_0) - y(t_0) = 3y(t_0) - 2u(t_0)$.

The last form will be useful in the next section.

Proposition 1. *The function $\tilde{\beta}(x, t|y(t_0), t_0)$ solves the Fokker-Planck partial differential equation:*

$$\frac{\partial\tilde{\beta}(x, t|y(t_0), t_0)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x, t)\tilde{\beta}(x, t|y(t_0), t_0)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(t)\tilde{\beta}(x, t|y(t_0), t_0)] \tag{33}$$

where

$$A_1(x, t) = m'(t) + [x - m(t)]\frac{h_2'(t)}{h_2(t)}, \quad A_2(t) = h_2^2(t) r'(t) \tag{34}$$

with the initial delta-type condition, i.e.,

$$\lim_{t_0 \uparrow t} \tilde{\beta}(x, t|y(t_0), t_0) \quad = \quad \delta(x - y(t_0)).$$

Proof. We note that from (24) of Lemma 1

$$
\begin{aligned}
\frac{\partial\tilde{\beta}(x,t|y(t_0),t_0)}{\partial t} = \frac{\partial f(x,t|y(t_0),t_0)}{\partial t} \\
- \alpha_1\phi_1[2u(t_0)-y(t_0),t_0)]\frac{\partial}{\partial t}f[x,t|2u(t_0)-y(t_0),t_0] \\
- \alpha_2\phi_2[2v(t_0)-y(t_0),t_0]\frac{\partial}{\partial t}f[x,t|2v(t_0)-y(t_0),t_0]. \qquad (35)
\end{aligned}
$$

Hence, $\tilde{\beta}$ is a linear combination of $f(x,t|\cdot,t_0)$. We recall that the transition pdf $f(x,t|\cdot,t_0)$ of the GM process $X(t)$ solves the following Fokker-Planck pde

$$
\frac{\partial f(x,t|\cdot,t_0)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x,t)]f(x,t|\cdot,t_0)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(t)f(x,t|\cdot,t_0)] \qquad (36)
$$

with corresponding initial delta-type conditions and $A_1(x,t)$, $A_2(t)$ as in (34). Then, taking into account (36), Equation (35) can be explicitly written as

$$
\begin{aligned}
\frac{\partial\tilde{\beta}(x,t|y(t_0),t_0)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x,t)]f(x,t|y(t_0),t_0)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(t)f(x,t|y(t_0),t_0)] \\
- \alpha_1\phi_1[2u(t_0-y(t_0),t_0)] \\
\times\left\{-\frac{\partial}{\partial x}\Big[A_1(x,t)f[x,t|2u(t_0)-y(t_0),t_0]\Big]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[A_2(t)f[x,t|2u(t_0)-y(t_0),t_0]\Big]\right\} \\
- \alpha_2\phi_2[2v(t_0)-y(t_0),t_0)] \\
\times\left\{-\frac{\partial}{\partial x}\Big[A_1(x,t)f[x,t|2v(t_0)-y(t_0),t_0]\Big]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[A_2(t)f[x,t|2v(t_0)-y(t_0),t_0]\Big]\right\}.
\end{aligned}
$$

Finally, rearranging the last equation, we obtain Equation (33) with the corresponding initial condition. Hence, the thesis holds. □

Lemma 2. *For $S_1(t,t_0) < x < S_2(t,t_0)$ and $\forall t \geq t_0 \in T$, setting*

$$
U(x,t,t_0) = \left\{\frac{[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}
$$

and

$$
R(t,t_0) = \left\{\frac{[u(t)-y(t)][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}
$$

the function $\tilde{\beta}(x,t|y(t_0),t_0)$ has the following expression:

$$
\tilde{\beta}(x,t|y(t_0),t_0) = -f[x,t|y(t_0),t_0]e^{2U(x,t,t_0)}[\alpha_1 e^{-4U(x,t,t_0)} - e^{-2U(x,t,t_0)} + \alpha_2 e^{-4R(t,t_0)}]. \qquad (37)
$$

Proof. Coming back to the expression (23) of $\tilde{\beta}(x,t|y(t_0),t_0)$ and by using the symmetry relation (18) we have

$$
\begin{aligned}
\tilde{\beta}(x,t|y(t_0),t_0) = f(x,t|y(t_0),t_0) \\
- \alpha_1 f(x,t|y(t_0),t_0)\exp\left\{-\frac{2[x-u(t)][y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\} \\
- \alpha_2 f(x,t|y(t_0),t_0)\exp\left\{-\frac{2[x-v(t)][y(t_0)-v(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}. \qquad (38)
\end{aligned}
$$

Recalling that $y(t_0) - v(t_0) = y(t_0) - 2y(t_0) + u(t_0) = u(t_0) - y(t_0)$ and $v(t) = \psi[u(t), t]$, i.e., $2y(t) - u(t) = v(t)$, and by using the symmetry property of the symmetry curve $z(t)$ such that $z(t) - x = \psi(x,t) - z(t)$ and $\psi(x,t) - x = 2[z(t) - x]$, we can write that $x - v(t) = x - 2y(t) + u(t) = x - u(t) - 2[y(t) - u(t)]$. Hence, the last term in (38) becomes:

$$\exp\left\{-\frac{2[x-v(t)][y(t_0)-v(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\} = \exp\left\{\frac{2\big[[x-u(t)]-2[y(t)-u(t)]\big][y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$
$$= \exp\left\{\frac{2[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\} \times \exp\left\{-\frac{4[u(t)-y(t)][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}. \quad (39)$$

Therefore, from (38) and (39), we have

$$\tilde{\beta}(x,t|y(t_0),t_0) = f[x,t|y(t_0),t_0]\Bigg\{1 - \alpha_1 \exp\Big\{-\frac{2[x-u(t)][y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\}$$
$$- \alpha_2 \exp\Big\{\frac{2[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\} \times \exp\Big\{-\frac{4[u(t)-y(t)][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\}\Bigg\}$$

that is the (37). □

4.1. Proof of Theorem 1

Proof. To prove the thesis, i.e., $\tilde{\beta}(x,t|y(t_0),t_0) \equiv \beta(x,t|y(t_0),t_0)$, we have to prove that the function $\tilde{\beta}(x,t|y(t_0),t_0)$ is the effective transition probability density of $X(t)$ in the two-sided region $\mathcal{D}$. To do this we have to verify that $\tilde{\beta}(x,t|y(t_0),t_0)$ is such that

(i) $\tilde{\beta}(x,t|y(t_0),t_0) = 0$ for $x = S_i(t,t_0)$ $(i = 1,2)$, $\forall t > t_0$;

(ii) $\tilde{\beta}(x,t|y(t_0),t_0) \geq 0$, $\forall x \in (S_1(t,t_0), S_2(t,t_0))$, $\forall t > t_0$;

(iii) $\tilde{\beta}(x,t|y(t_0),t_0)$ satisfies the delta condition $\forall x \in (S_1(t,t_0), S_2(t,t_0))$, i.e.,

$$\lim_{t \downarrow t_0} \tilde{\beta}(x,t|y(t_0),t_0) = \delta(x - y(t_0))$$

with $y(t_0) \in (\lim_{t\downarrow t_0} S_1(t,t_0), \lim_{t\downarrow t_0} S_2(t,t_0))$.

We firstly prove i) and ii). From (37), we note that $\tilde{\beta}(x,t|y(t_0),t_0) = 0 \Leftrightarrow \alpha_1\alpha_2 < 0$ or $\alpha_1\alpha_2 > 0$ and $\lim_{t\to supT} \Delta(t,t_0) > 0$ with $\Delta(t,t_0) = 1 - 4\alpha_1\alpha_2 e^{-4R(t,t_0)}$. Indeed, under these conditions, the zeros of

$$[\alpha_1 e^{-4U(x,t,t_0)} - e^{-2U(x,t,t_0)} + \alpha_2 e^{-4R(t,t_0)}]$$

with U and R as in Lemma 2, are:

$$\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1} \quad \text{and} \quad \frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}.$$

Hence, we can write

$$\tilde{\beta}(x,t|y(t_0),t_0) = -\alpha_1 f[x,t|y(t_0),t_0] e^{2U(x,t,t_0)}$$
$$\left[e^{-2U(x,t,t_0)} - \frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]\left[e^{-2U(x,t,t_0)} - \frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]. \quad (40)$$

More specifically (in the case $\alpha_1 > 0$ that implies $\alpha_2 > 0$ and $\lim_{t\to supT} \Delta(t,t_0) > 0$), $\tilde{\beta}(x,t|y(t_0),t_0)$ is such that

$$\tilde{\beta}(x,t|y(t_0),t_0) = -\alpha_1 f[x,t|y(t_0),t_0] \exp\left\{\frac{2[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$
$$\times\left[\exp\left\{-\frac{2[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}-\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]$$
$$\times\left[\exp\left\{-\frac{2[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}-\frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]$$

and

$$\tilde{\beta}(x,t|y(t_0),t_0)=0 \Leftrightarrow e^{-2U(x,t,t_0)}=\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1} \text{ or } e^{-2U(x,t,t_0)}=\frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}.$$

We note that

$$-2U(x,t,t_0)=\ln\left[\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]$$
$$\Leftrightarrow -2\left\{\frac{[u(t_0)-y(t_0)][u(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}=\ln\left[\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]$$
$$\Leftrightarrow u(t)-x=-\frac{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}{2[u(t_0)-y(t_0)]}\ln\left[\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right]$$

that gives a first solution:

$$x(t,t_0)=u(t)+\frac{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}{2[u(t_0)-y(t_0)]}\ln\left[\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right] \tag{41}$$

And similarly, we also note that from the condition $e^{-2U(x,t,t_0)}=\frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}$ another solution is

$$x(t,t_0)=u(t)+\frac{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}{2[u(t_0)-y(t_0)]}\ln\left[\frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}\right],$$

that we discard because it is a not continuous solution.

Furthermore, an equivalent representation of (37) of $\tilde{\beta}(x,t|y(t_0),t_0)$ holds:

$$\tilde{\beta}(x,t|y(t_0),t_0)=-f[x,t|y(t_0),t_0]e^{-2V(x,t,t_0)}[\alpha_1 e^{-4R(t,t_0)}-e^{2V(x,t,t_0)}+\alpha_2 e^{4V(x,t,t_0)}], \tag{42}$$

with

$$V(x,t,t_0)=\frac{[u(t_0)-y(t_0)][v(t)-x]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}.$$

Indeed, in (38), being $x-u(t)=x-u(t)+v(t)-v(t)=[x-v(t)]+[v(t)-u(t)]$, one has:

$$\exp\left\{-\frac{2[y(t_0)-u(t_0)][x-u(t)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$
$$=\exp\left\{-\frac{2[[x-v(t)]+[v(t)-u(t)]][y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$
$$=\exp\left\{-\frac{2[v(t)-x][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}\exp\left\{\frac{2[v(t)-u(t)]][y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$
$$\exp\left\{-\frac{2[v(t)-x][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}\exp\left\{\frac{4[y(t)-u(t)][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$
$$=e^{-2V(x,t,t_0)}e^{-4R(t,t_0)}.$$

From (42), in the case $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\lim_{t \to supT} \Delta(t, t_0) > 0$, we can again write

$$\tilde{\beta}(x, t|y(t_0), t_0) = -\alpha_2 f[x, t|y(t_0), t_0] e^{-2V(x,t,t_0)} \left[e^{2V(x,t,t_0)} - \frac{1 + \sqrt{\Delta(t, t_0)}}{2\alpha_2}\right] \left[e^{2V(x,t,t_0)} - \frac{1 - \sqrt{\Delta(t, t_0)}}{2\alpha_2}\right]$$

and

$$\tilde{\beta}(x, t|y(t_0), t_0) = 0 \Leftrightarrow$$
$$2V(x, t, t_0) = \ln\left[\frac{1 + \sqrt{\Delta(t, t_0)}}{2\alpha_2}\right] \quad \text{or} \quad 2V(x, t, t_0) = \ln\left[\frac{1 - \sqrt{\Delta(t, t_0)}}{2\alpha_2}\right].$$

Hence, in this case, the solutions of equation $\tilde{\beta}(x, t|y(t_0), t_0) = 0$ are

$$x(t, t_0) = v(t) - \frac{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}{2[u(t_0) - y(t_0)]} \ln\left[\frac{1 + \sqrt{\Delta(t, t_0)}}{2\alpha_2}\right] \tag{43}$$

and

$$x(t, t_0) = v(t) - \frac{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}{2[u(t_0) - y(t_0)]} \ln\left[\frac{1 - \sqrt{\Delta(t, t_0)}}{2\alpha_2}\right],$$

the last one has to be discarded being a not continuous solution.

Hence, for $\alpha_1 > 0$ and $\alpha_2 > 0$ and $\lim_{t \to supT} \Delta(t, t_0) > 0$, the function $\tilde{\beta}(x, t|y(t_0), t_0)$ verifies the conditions *i*) and *ii*); specifically, we have that, from (41) and (43), it is equal to zero on the continuous functions

$$S_1(t, t_0) = v(t) - \frac{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}{2(u(t_0) - y(t_0))} \ln\left[\frac{1 + \sqrt{\Delta(t, t_0}}{2\alpha_2}\right]$$

and

$$S_2(t, t_0) = u(t) + \frac{h_1(t)h_2(t_0) - h_1(t_0)h_2(t)}{2(u(t_0) - y(t_0))} \ln\left[\frac{1 + \sqrt{\Delta(t, t_0}}{2\alpha_1}\right]$$

and it has positive values for $x \in (S_1(t, t_0), S_2(t, t_0))$.

Furthermore, *iii*) also holds. Indeed, $\tilde{\beta}(x, t|y(t_0), t_0)$ satisfies the delta condition due the result of Proposition 1 with $y(t_0) \in (\lim_{t \downarrow t_0} S_1(t, t_0), \lim_{t \downarrow t_0} S_2(t, t_0))$.

We finally claim that $\tilde{\beta} = \beta$, i.e., $\tilde{\beta}$ is the transition pdf of the process $X(t)$ in presence of the two boundaries (21). □

5. Pdf of FET

Here, exploiting the form and the properties of the transition pdf $\beta(x, t|y(t_0), t_0)$ in the presence of the boundaries (21), we can obtain the first-exit-time probability density from the above two-sided open set in closed form. First, we provide an expression of the distribution function of the FET density.

Proposition 2. *Under the hypotheses of Theorem 1, for the two-sided region delimited by the boundaries (21), for $t > t_0$, we have the following result for the distribution function of FET as the integral of $g(t|y(t_0), t_0)$:*

$$\int_{t_0}^{t} g[\theta|y(t_0),t_0]d\theta = 1 - [F[S_2(t,t_0),t|y(t_0),t_0] - F[S_1(t,t_0),t|y(t_0),t_0]]$$
$$+\alpha_1 \exp\left\{-\frac{2d_1^*[u(t_0)-y(t_0)]}{h_2(t_0)}\right\}\Big[F[S_2(t,t_0),t|2u(t_0)-y(t_0),t_0] - [F[S_1(t,t_0),t|2u(t_0)-y(t_0),t_0]\Big]$$
$$+\alpha_2 \exp\left\{-\frac{4[d_1^*-2d_1][u(t_0)-y(t_0)]}{h_2(t_0)}\right\}$$
$$\times\Big[F[S_2(t,t_0),t|3y(t_0)-2u(t_0),t_0] - [F[S_1(t,t_0),t|3y(t_0)-2u(t_0),t_0]\Big].$$

Proof. Recalling that the following relation between the transition density $\beta(x,t|y(t_0),t_0)$ and the FET density $g[t|y(t_0),t_0]$ holds

$$\int_{S_1(t,t_0)}^{S_2(t,t_0)} dz\beta(z,t|y(t_0),t_0) + \int_{t_0}^{t} d\theta g[\theta|y(t_0),t_0] = 1. \tag{44}$$

Using here Remark 1, and proceeding to integrate $\beta(x,t|y(t_0),t_0)$ between the two boundaries (21), the thesis holds, with $F[x,t|y,\tau]$ the transition probability distribution function of the process. □

We now need to prove a preliminary lemma about an integral representation of FET density.

Lemma 3. *Under the hypotheses of Theorem 1, for the two-sided region delimited by the boundaries (21), taking into account the functions U and R of Lemma 2 , for the GM process $X(t)$, for $t > t_0$, we first obtain the FET pdf such as*

$$g[t|y(t_0),t_0] = \int_{S_1(t,t_0)}^{S_2(t,t_0)} dz\frac{\partial}{\partial z}\left\{A_1(z,t)f[z,t|y(t_0),t_0] - \frac{1}{2}\frac{\partial}{\partial z}\Big[A_2(t)f[z,t|y(t_0),t_0]\Big]\right\}$$
$$-\alpha_1\int_{S_1(t,t_0)}^{S_2(t,t_0)} dz\frac{\partial}{\partial z}\left\{A_1(z,t)f[z,t|y(t_0),t_0]e^{-2U(z,t,t_0)} - \frac{1}{2}\frac{\partial}{\partial z}A_2(t)f[z,t|y(t_0),t_0]e^{-2U(z,t,t_0)}\right\} \tag{45}$$
$$-\alpha_2\int_{S_1(t,t_0)}^{S_2(t,t_0)} dz\frac{\partial}{\partial z}\left\{A_1(z,t)f[z,t|y(t_0),t_0]e^{2V(z,t,t_0)} - \frac{1}{2}\frac{\partial}{\partial z}A_2(t)f[z,t|y(t_0),t_0]e^{2V(z,t,t_0)}\right\}.$$

Proof. From (44), by differentiating, we obtain

$$g[t|y(t_0),t_0] = -\frac{\partial}{\partial t}\int_{S_1(t,t_0)}^{S_2(t,t_0)} \beta(z,t|y(t_0),t_0)dz$$

in which we will insert $\beta(x,t|y(t_0),t_0)$ such as from Remark 1. Furthermore, from Proposition 1, we know that $\beta(x,t|y(t_0),t_0)$ solves the Fokker-Planck pde (33). Hence, we can write:

$$g[t|y(t_0),t_0] = \int_{S_1(t,t_0)}^{S_2(t,t_0)}\Big[-\frac{\partial}{\partial t}\beta(z,t|y(t_0),t_0)\Big]dz$$
$$= -\int_{S_1(t,t_0)}^{S_2(t,t_0)}\frac{\partial}{\partial t}\Big[f[z,t|y(t_0),t_0] - \alpha_1\phi_1[2u(t_0)-y(t_0),t_0]f[z,t|2u(t_0)-y(t_0),t_0]$$
$$-\alpha_2\phi_2[2v(t_0)-y(t_0),t_0]f[z,t|3y(t_0)-2u(t_0),t_0]\Big]dz$$

$$
\begin{aligned}
&= -\int_{S_1(t,t_0)}^{S_2(t,t_0)} \Big(-\frac{\partial}{\partial z}\Big[A_1(z,t)f[z,t|y(t_0),t_0] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[A_2(t)f[z,t|y(t_0),t_0]\Big]\Big) \\
&\quad -\alpha_1\phi_1[\psi_1(y(t_0),t_0),t_0]\Big(-\frac{\partial}{\partial z}\Big[A_1(z,t)f[z,t|\psi_1(y(t_0)),t_0]\Big] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\Big[A_2(t)f[z,t|\psi_1(y(t_0)),t_0]\Big]\Big) \\
&\quad -\alpha_2\phi_2[\psi_2(y(t_0),t_0),t_0]\Big(-\frac{\partial}{\partial z}\Big[A_1(z,t)f[z,t|\psi_2(y(t_0)),t_0]\Big] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\Big[A_2(t)f[z,t|\psi_2(y(t_0)),t_0]\Big]\Big)dz \\
&= \int_{S_1(t,t_0)}^{S_2(t,t_0)} dz \frac{\partial}{\partial z}\Big\{\Big[A_1(z,t)f(z,t|y(t_0),t_0)\Big] - \frac{1}{2}\frac{\partial}{\partial z}\Big[A_2(t)f(z,t|y(t_0),t_0)\Big]\Big\} \\
&\quad -\alpha_1\phi_1[\psi_1(y(t_0),t_0),t_0]\int_{S_1(t,t_0)}^{S_2(t,t_0)} dz \frac{\partial}{\partial z}\Big\{\Big[A_1(z,t)f[z,t|\psi_1(y(t_0)),t_0]\Big] - \frac{1}{2}\frac{\partial}{\partial z}\Big[A_2(t)f[z,t|\psi_1(y(t_0),t_0),t_0]\Big]\Big\} \\
&\quad -\alpha_2\phi_2[\psi_2(y(t_0),t_0),t_0]\int_{S_1(t,t_0)}^{S_2(t,t_0)} dz \frac{\partial}{\partial z}\Big\{\Big[A_1(z,t)f[z,t|\psi_2(y(t_0)),t_0]\Big] - \frac{1}{2}\frac{\partial}{\partial z}\Big[A_2(t)f[z,t|\psi_2(y(t_0),t_0),t_0]\Big]\Big\}.
\end{aligned} \tag{46}
$$

Now, taking into account

$$
f(x,t|y,\tau) = \frac{\phi(x,t)}{\phi(y,\tau)} f[\psi(x,t),t|\psi(y,\tau),\tau]
$$

and

$$
\psi[\psi(x,t),t] = x,
$$

we obtain

$$
\begin{aligned}
\phi_1[\psi_1(y(t_0),t_0),t_0]f[z,t|\psi_1(y(t_0),t_0),t_0] &= \phi_1(z,t)f[\psi_1(z,t),t|\psi_1[\psi_1(y(t_0),t_0),t_0],t_0] \\
&= \phi_1(z,t)f[\psi_1(z,t),t|y(t_0),t_0] \\
&= f[z,t|(y(t_0),t_0]\exp\Big\{-\frac{2[u(t)-z][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\} \\
&= f[z,t|(y(t_0),t_0]e^{-2U(z,t,t_0)}.
\end{aligned} \tag{47}
$$

Similarly, it can be proved

$$
\phi_2[\psi_2(y(t_0),t_0),t_0]f[z,t|\psi_2(y(t_0),t_0),t_0] = f[z,t|(y(t_0),t_0]e^{2V(z,t,t_0)}.
$$

Indeed, taking into account that $v(t_0) = 2y(t_0) - u(t_0) - y(t_0) = y(t_0) - u(t_0)$, one has:

$$
\begin{aligned}
\phi_2[\psi_2(y(t_0),t_0),t_0]f[z,t|\psi_2(y(t_0),t_0),t_0] &= \phi_2(z,t)f[\psi_2(z,t),t|y(t_0,t_0] \\
&= f[z,t|(y(t_0),t_0]\exp\Big\{-\frac{2[v(t)-x][v(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\} \\
&= f[z,t|(y(t_0),t_0]\exp\Big\{\frac{2[v(t)-x][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\} \\
&= f[z,t|(y(t_0),t_0]e^{2V(z,t,t_0)}.
\end{aligned} \tag{48}
$$

Finally, by inserting (47) and (48) in (46), the lemma is proved. □

Theorem 2. *Under the hypoteses of Theorem 1, for the two-sided region delimited by the boundaries (21), taking into account the result and the functions U and R of Lemma 2 , for the GM process $X(t)$, for $t > t_0$, we have the following closed form for the FET pdf*

$$
\begin{aligned}
g[t|y(t_0),t_0] = &\frac{h_2^2(t)[u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\frac{dr(t)}{dt}\sqrt{\Delta(t,t_0)} \\
&\times\Big[f[S_1(t,t_0),t|y(t_0),t_0] + f[S_2(t,t_0),t|y(t_0),t_0]\Big].
\end{aligned} \tag{49}
$$

Proof. Taking into account Lemma 3, in order to evaluate the right-hand-side of (45), we first calculate the following derivatives:

$$\frac{\partial}{\partial z}U(z,t,t_0) = \frac{\partial}{\partial z}\left\{\frac{[u(t)-z][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\} = \left\{\frac{y(t_0)-u(t_0)}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}$$

and

$$\frac{\partial}{\partial z}V(z,t,t_0) = \frac{\partial}{\partial z}\left\{\frac{[v(t)-z][u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\} = \left\{\frac{y(t_0)-u(t_0)}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\right\}.$$

Hence, by integrating the right-hand-side of (45), by using Lemma 2, Equation (38), Theorem 1, and the above expressions, by exploiting the symmetry properties of the process in the presence of the boundaries (21), we have

$$\begin{aligned}
&g[t|y(t_0),t_0] = \\
&f[S_2(t,t_0),t|y(t_0),t_0)]\Big\{A_1[S_2(t,t_0),t] - \alpha_1 A_1[S_2(t,t_0),t]e^{-2U[S_2(t,t_0),t,t_0]} \\
&\qquad - \alpha_1 A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}e^{-2U[S_2(t,t_0),t,t_0]} \\
&\qquad - \alpha_2 A_1[S_2(t,t_0),t]e^{2V[S_2(t,t_0),t,t_0]} \\
&\qquad + \alpha_2 A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}e^{2V[S_2(t,t_0),t,t_0]}\Big\} \\
&-f[S_1(t,t_0),t|y(t_0),t_0)]\Big\{A_1[S_1(t,t_0),t] - \alpha_1 A_1[S_1(t,t_0),t]e^{-2U[S_1(t,t_0),t,t_0]} \\
&\qquad - \alpha_1 A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}e^{-2U[S_1(t,t_0),t,t_0]} \\
&\qquad - \alpha_2 A_1[S_1(t,t_0),t]e^{2V[S_1(t,t_0),t,t_0]} \\
&\qquad + \alpha_2 A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}e^{2V[S_1(t,t_0),t,t_0]}\Big\} \\
&-\frac{1}{2}A_2(t)\frac{\partial}{\partial z}f[z,t|y(t_0),t_0\Big]_{z=S_2(t,t_0)}\Big\{1-\alpha_1 e^{-2U[S_2(t,t_0),t,t_0]} - \alpha_2 e^{2V[S_2(t,t_0),t,t_0]}\Big\} \\
&+\frac{1}{2}A_2(t)\frac{\partial}{\partial z}f[z,t|y(t_0),t_0\Big]_{z=S_1(t,t_0)}\Big\{1-\alpha_1 e^{-2U[S_1(t,t_0),t,t_0]} - \alpha_2 e^{2V[S_1(t,t_0),t,t_0]}\Big\} \\
&\quad = f[S_2(t,t_0),t|y(t_0),t_0)]\Big\{A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)} \\
&\qquad\qquad \Big(\alpha_2 e^{2V[S_2(t,t_0),t,t_0]} - \alpha_1 e^{-2U[S_2(t,t_0),t,t_0]}\Big)\Big\} \\
&\quad -f[S_1(t,t_0),t|y(t_0),t_0)]\Big\{A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)} \\
&\qquad\qquad \Big(\alpha_2 e^{2V[S_1(t,t_0),t,t_0]} - \alpha_1 e^{-2U[S_1(t,t_0),t,t_0]}\Big)\Big\} \\
&\quad = A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)} \\
&\quad\ \Big\{f[S_2(t,t_0),t|y(t_0),t_0)]\Big(\alpha_2 e^{2V[S_2(t,t_0),t,t_0]} - \alpha_1 e^{-2U[S_2(t,t_0),t,t_0]}\Big) \\
&\quad\ - f[S_1(t,t_0),t|y(t_0),t_0)](\alpha_2 e^{2V[S_1(t,t_0),t,t_0]} - \alpha_1 e^{-2U[S_1(t,t_0),t,t_0]})\Big\}.
\end{aligned}$$

Recalling that:

$$e^{2V[S_2(t,t_0),t,t_0]} = \frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_2}, \quad e^{2V[S_1(t,t_0),t,t_0]} = \frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_2},$$

$$e^{-2U[S_2(t,t_0),t,t_0]} = \frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}, \quad e^{-2U[S_1(t,t_0),t,t_0]} = \frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1},$$

one has

$$\begin{aligned} g[t|y(t_0),t_0] &= A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)} \\ &\quad \Big\{ f[S_2(t,t_0),t|y(t_0),t_0)]\Big[\alpha_2\Big(\frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_2}\Big)-\alpha_1\Big(\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_1}\Big)\Big] \\ &\quad - f[S_1(t,t_0),t|y(t_0),t_0)]\Big[\alpha_2\Big(\frac{1+\sqrt{\Delta(t,t_0)}}{2\alpha_2}\Big)-\alpha_1\Big(\frac{1-\sqrt{\Delta(t,t_0)}}{2\alpha_1}\Big)\Big]\Big\} \\ &= A_2(t)\frac{[y(t_0)-u(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\Big\{ f[S_2(t,t_0),t|y(t_0),t_0)](-\sqrt{\Delta(t,t_0)}) \\ &\quad - f[S_1(t,t_0),t|y(t_0),t_0)]\sqrt{\Delta(t,t_0)}\Big\} \\ &= A_2(t)\frac{[u(t_0)-y(t_0)]}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)}\sqrt{\Delta(t,t_0)}\Big\{ f[S_2(t,t_0),t|y(t_0),t_0] \\ &\quad + f[S_1(t,t_0),t|y(t_0),t_0)]\Big\}. \end{aligned} \tag{50}$$

By using the following relation

$$A_2(t) = h_2^2(t)\frac{dr(t)}{dt} \tag{51}$$

and inserting this in (50) the proof is completed. □

Remark 2. *We note that*

$$h_1(t)h_2(t_0)-h_1(t_0)h_2(t) = h_2(t)h_2(t_0)\Big[\frac{h_1(t)}{h_2(t)}-\frac{h_1(t_0)}{h_2(t_0)}\Big] = h_2(t)h_2(t_0)[r(t)-r(t_0)].$$

Hence, by using also (51), we can write

$$\frac{A_2(t)}{h_1(t)h_2(t_0)-h_1(t_0)h_2(t)} = \frac{dr(t)}{dt}\frac{h_2^2(t)}{h_2(t)h_2(t_0)[r(t)-r(t_0)]}$$

and finally, by substituting the last relation in (50), we obtain the following expression (comparable with that of [24])

$$g[t|y(t_0),t_0] = \frac{u(t_0)-y(t_0)}{r(t)-r(t_0)}\frac{h_2(t)}{h_2(t_0)}\frac{dr(t)}{dt}\sqrt{\Delta(t,t_0)} \tag{52}$$

$$\Big\{ f[S_2(t,t_0),t|y(t_0),t_0] + f[S_1(t,t_0),t|y(t_0),t_0)]\Big\}. \tag{53}$$

6. Two Specific Examples

Only for explanatory purposes and comparisons with known results (see, for instance, [28,34]), we give some explicit expressions for the above functions of two GM processes such as Wiener and Ornstein-Uhlenbeck processes, due their central rule in the class of GM processes ([14]).

6.1. The Wiener Process

Consider the Wiener process $\{X(t), t \geq 0\}$ with mean function $m(t) = \mu t, (\mu \in \mathbb{R}^+)$, covariance function $c(s,t) = s$, $(s, t \in \mathbb{R}^+, s < t)$, covariance factors $h_1(t) = t$, $h_2(t) = 1$, and $r(t) = t, \forall t \geq 0$.

According to (14), we set the following symmetry curves:

$$u(t) = \mu t + \eta$$

$$y(t) = \mu t$$

$$v(t) = \mu t - \eta$$

with $\eta \in \mathbb{R}^+$. The symmetry functions are

$$\begin{aligned} \psi_0(x,t) &= 2\mu t - x, \ \ \phi_0(x,t) = 1, \quad \textit{(associated with } y(t)\textit{)} \\ \psi_1(x,t) &= 2\mu t + 2\eta - x, \ \ \phi_1(x,t) = 1, \quad \textit{(associated with } u(t)\textit{)} \\ \psi_2(x,t) &= 2\mu t - 2\eta - x, \ \ \phi_2(x,t) = 1 \quad \textit{(associated with } v(t)\textit{)}. \end{aligned} \tag{54}$$

Please note that from (2) the transition mean and variance functions are

$$\begin{aligned} \mathbb{E}[X(t)|X(\tau) = y] &= \mu t + [y - \mu \tau] \\ \mathrm{Var}[X(t)|X(\tau) = y] &= t - \tau \end{aligned}$$

for $y \in \mathbb{R}$, $t, \tau \in \mathbb{R}^+$, $\tau < t$, and the free transition pdf $f(x,t|y,\tau)$ remains specified as the corresponding normal density with the above mean and variance. Setting $t_0 = 0$, $y(0) = 0$, we have, from (22)

$$\Delta(t;0) = 1 - 4\alpha_1\alpha_2 \exp\{-4\eta^2/t\},$$

with $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and such that $\lim_{t\to+\infty} \Delta(t,0) > 0$, whereas the Daniels-type boundaries from (21) are

$$\begin{aligned} S_1(t,0) &= \mu t - \eta - \frac{t}{2\eta} \ln \left[\frac{1 + \sqrt{1 - 4\alpha_1\alpha_2 \exp\{-4\eta^2/t\}}}{2\alpha_2} \right] \\ S_2(t,0) &= \mu t + \eta + \frac{t}{2\eta} \ln \left[\frac{1 + \sqrt{1 - 4\alpha_1\alpha_2 \exp\{-4\eta^2/t\}}}{2\alpha_1} \right]. \end{aligned}$$

Please note that for instance, choosing $\eta > 0$, one has $S_1(0,0) < y(0) < S_2(0,0)$, with $y(0) = 0$, $S_1(0,0) = -\eta$ and $S_2(0,0) = \eta$. From (23), the transition pdf between the two Daniels-type boundaries is, for $S_1(t,0) < x < S_2(t,0)$,

$$\beta(x,t|0,0) = \frac{1}{\sqrt{2\pi t}} \left[\exp\left\{ -\frac{(x-\mu t)^2}{2t} \right\} - \alpha_1 \exp\left\{ -\frac{(x-\mu t - 2\eta)^2}{2t} \right\} - \alpha_2 \exp\left\{ -\frac{(x-\mu t + 2\eta)^2}{2t} \right\} \right]. \tag{55}$$

Finally, from (52), the FET pdf is $\forall t > 0$

$$g[t|0,0] = \frac{\eta}{t}\sqrt{1 - 4\alpha_1\alpha_2 \exp\{-4\eta^2/t\}} \tag{56}$$

$$\times \frac{1}{\sqrt{2\pi t}}\left\{ \exp\left[-\frac{[S_1(t,0)-\mu t]^2}{2t}\right] + \exp\left[-\frac{[S_2(t,0)-\mu t]^2}{2t}\right]\right\}. \tag{57}$$

6.2. The Ornstein-Uhlenbeck Process

Consider the well-known Ornstein-Uhlenbeck process $\{X(t), t \geq 0\}$ with $X(0) = x_0$, that is the GM process with mean

$$m(t) = x_0 e^{-t/\vartheta} + \mu\vartheta\left(1 - e^{-t/\vartheta}\right)$$

and covariance, for $s \leq t$,

$$c(s,t) = \frac{\sigma^2\vartheta}{2}\left(e^{-(t-s)/\vartheta} - e^{-(t+s)/\vartheta}\right)$$

where $\sigma, \vartheta > 0$, and with covariance factors

$$h_1(t) = \frac{\sigma\vartheta}{2}\left(e^{t/\vartheta} - e^{-t/\vartheta}\right), \; h_2(t) = \sigma e^{-t/\vartheta}.$$

Moreover, we note that the variance is $Var(X(t)) = \frac{\sigma^2\vartheta}{2}\left(1 - e^{-2t/\vartheta}\right)$, and the ratio function $r(t)$ is such that $r(t) = \frac{\vartheta}{2}\left(e^{2t/\vartheta} - 1\right)$ with $r(0) = 0$. Then, according to (14), and with $x_0 = 0$, we set the following symmetry curves:

$$u(t) = \mu\vartheta\left(1 - e^{-t/\vartheta}\right) + \eta\frac{\sigma\vartheta}{2}e^{t/\vartheta}$$

$$y(t) = \mu\vartheta\left(1 - e^{-t/\vartheta}\right)$$

$$v(t) = \mu\vartheta\left(1 - e^{-t/\vartheta}\right) - \eta\frac{\sigma\vartheta}{2}e^{t/\vartheta}$$

with $\eta \in \mathbb{R}^+$. From (15) and (16), the symmetry functions are

$$\psi_0(x,t) = 2\mu\vartheta\left(1 - e^{-t/\vartheta}\right) - x, \;\; \phi_0(x,t) = 1,$$

$$\psi_1(x,t) = 2\mu\vartheta\left(1 - e^{-t/\vartheta}\right) + \eta\sigma\vartheta e^{t/\vartheta} - x, \;\; \phi_1(x,t) = \exp\{-\frac{2\eta}{\sigma}e^{t/\vartheta}[x - u(t)]\}, \tag{58}$$

$$\psi_2(x,t) = 2\mu\vartheta\left(1 - e^{-t/\vartheta}\right) - \eta\sigma\vartheta e^{t/\vartheta} - x, \;\; \phi_2(x,t) = \exp\{\frac{2\eta}{\sigma}e^{t/\vartheta}[x - v(t)]\}.$$

Please note that from (2), with $y(\tau) = 0$ and $\tau = 0$, the transition mean and variance functions are

$$\mathbb{E}[X(t)|X(0) = 0] = m(t) = \mu\vartheta\left(1 - e^{-t/\vartheta}\right)$$

$$\text{Var}[X(t)|X(0) = 0] = Var(X(t)) = \frac{\sigma^2\vartheta}{2}\left(1 - e^{-2t/\vartheta}\right)$$

for $t \in \mathbb{R}^+$, and the free transition pdf $f(x,t|0,0)$ remains specified as the corresponding normal density with the above mean and variance. Setting again $t_0 = 0$, $y(0) = 0$, we have, from (22)

$$\Delta(t;0) = 1 - 4\alpha_1\alpha_2 \exp\left\{-\frac{2\eta^2\vartheta}{1 - e^{-2t/\vartheta}}\right\},$$

with $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and such that $\lim_{t\to+\infty}\Delta(t,0) > 0$, whereas the Daniels-type boundaries from (21) are

$$S_1(t,0) = \mu\vartheta\left(1 - e^{-t/\vartheta}\right) - \eta\frac{\sigma\vartheta}{2}e^{t/\vartheta} - \frac{\sigma\left(e^{t/\vartheta} - e^{-t/\vartheta}\right)}{2\eta}\ln\left[\frac{1 + \sqrt{1 - 4\alpha_1\alpha_2\exp\left\{-\frac{2\eta^2\vartheta}{1-e^{-2t/\vartheta}}\right\}}}{2\alpha_2}\right]$$

$$S_2(t,0) = \mu\vartheta\left(1 - e^{-t/\vartheta}\right) + \eta\frac{\sigma\vartheta}{2}e^{t/\vartheta} + \frac{\sigma\left(e^{t/\vartheta} - e^{-t/\vartheta}\right)}{2\eta}\ln\left[\frac{1 + \sqrt{1 - 4\alpha_1\alpha_2\exp\left\{-\frac{2\eta^2\vartheta}{1-e^{-2t/\vartheta}}\right\}}}{2\alpha_1}\right].$$

Please note that for instance, choosing $\eta > 0$, one has $S_1(0,0) < y(0) < S_2(0,0)$, with $y(0) = 0$, $S_1(0,0) = -\eta\sigma\vartheta/2$ and $S_2(0,0) = \eta\sigma\vartheta/2$.

Finally, by using all above specified functions, from (23) and from (52), the transition pdf $\beta(x,t|0,0)$ between the two Daniels-type boundaries is, for $S_1(t,0) < x < S_2(t,0)$, and the FET pdf $g[t|0,0]$ can be explicitly obtained, respectively.

As last remark, we note that the Ornstein-Uhlenbeck process $X(t)$ here considered is also solution of the following stochastic differential equation (SDE):

$$dX(t) = \left[-\frac{X(t)}{\vartheta} + \mu\right]dt + \sigma^2 dW(t), \quad X(0) = 0,$$

with $W(t)$ is a standard Wiener process. The determination of the first passage time density of $X(t)$ from a region is the central problem for very large number of models based on the above SDE. The symmetry strategy and the obtained expressions in presence of Daniels-type boundaries can be useful also in such modeling contexts, because, under specific assumptions, some (piecewise) approximations can be derived.

Funding: This research is partially supported by MIUR—PRIN 2017, project "Stochastic Models for Complex Systems", no. 2017JFFHSH.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Rasmussen, C.E.; Williams, C.K.I. *Gaussian Processes for Machine Learning*; MIT Press: Cambridge, MA, USA, 2006; ISBN 978-0-262-18253-9.
2. Mehr, C.B.; McFadden, J.A. Certain properties of gaussian processes and their first passage time. *J. R. Statist. Soc.* **1965**, *27*, 505–522. [CrossRef]
3. Adler, R.J. An introduction to continuity, extrema, and related topics for general Gaussian processes. Lecture Notes-Monograph Series. *Inst. Math. Stat.* **1990**, *12*, 1–155.
4. Janssen, J.; Manca, O.; Manca, R. *Applied Diffusion Processes from Engineering to Finance*; Wiley: Hoboken, NJ, USA, 2013; 416p.
5. Ricciardi, L.M.; Di Crescenzo, A.; Giorno, V.; Nobile, A.G. An outline of theoretical and algorithmic approaches to first passage time problems with applications to biological modeling. *Math. Jpn.* **1999**, *50*, 247–322
6. Tuckwell, H.C. On the First-Exit Time Problem for Temporally Homogeneous Markov Processes. *J. Appl. Probab.* **1976**, *13*, 39–48. [CrossRef]
7. Giorno, V.; Nobile, A.G. On the Construction of a Special Class of Time-Inhomogeneous Diffusion Processes. *J. Stat. Phys.* **2019**, *177*, 299–323. [CrossRef]
8. Pirozzi, E. Colored noise and a stochastic fractional model for correlated inputs and adaptation in neuronal firing. *Biol. Cybern.* **2018**, *112*, 25–39. [CrossRef] [PubMed]
9. Ascione, G.; Pirozzi, E. On a stochastic neuronal model integrating correlated inputs. *Math. Biosci. Eng.* **2019**, *16*, 5206–5225. [CrossRef] [PubMed]

10. D'Onofrio, G.; Lansky, P.; Pirozzi, E. On two diffusion neuronal models with multiplicative noise: The mean first-passage time properties. *Chaos* **2018**, *28*, 043103. [CrossRef] [PubMed]
11. D'Onofrio, G.; Macci, C.; Pirozzi, E. Asymptotic Results for First-Passage Times of Some Exponential Processes. *Methodol. Comput. Appl. Probab.* **2018**, *20*, 1453–1476. [CrossRef]
12. Di Crescenzo, A.; Giorno, V.; Krishna Kumar, B.; Nobile, A.G. A double-ended queue with catastrophes and repairs, and a jump-diffusion approximation. *Methodol. Comput. Appl. Probab.* **2012**, *14*, 937–954. [CrossRef]
13. Di Crescenzo, A.; Giorno, V.; Krishna Kumar, B.; Nobile, A.G. A Time-Non-Homogeneous Double-Ended Queue with Failures and Repairs and Its Continuous Approximation. *Mathematics* **2018**, *6*, 81. [CrossRef]
14. Di Nardo, E.; Nobile, A.; Pirozzi, E.; Ricciardi, L. A computational approach to first-passage-time problems for Gauss—Markov processes. *Adv. Appl. Probab.* **2001**, *33*, 453–482. [CrossRef]
15. Buonocore, A.; Caputo, L.; Pirozzi, E.; Carfora, M.F. Gauss-diffusion processes for modeling the dynamics of a couple of interacting neurons. *Math. Biosci. Eng.* **2014**, *11*, 189–201. [CrossRef] [PubMed]
16. Buonocore, A.; Caputo, L.; D'Onofrio, G.; Pirozzi, E. Closed-form solutions for the first-passage-time problem and neuronal modeling. *Ricerche di Matematica* **2015**, *64*, 421–439. [CrossRef]
17. Buonocore, A.; Caputo, L.; Pirozzi, E.; Ricciardi, L.M. The First Passage Time Problem for Gauss-Diffusion Processes: Algorithmic Approaches and Applications to LIF Neuronal Model. *Methodol. Comput. Appl. Probab.* **2011**, *13*, 29–57. [CrossRef]
18. Lerche, H.R. *Boundary Crossing of Brownian Motion: Its Relation to the Law of the Iterated Logarithm and to Sequential Analysis*; Springer: Berlin, Germany, 1986; Volume 40.
19. Durbin, J. The first-passage density of a continuous Gaussian process to a general boundary. *J. Appl. Probab.* **1985**, *22*, 99–122. [CrossRef]
20. Daniels, H.E. The first crossing-time density for Brownian motion with a perturbed linear boundary. *Bernoulli* **2000**, *6*, 571–580. [CrossRef]
21. Giorno, V.; Nobile, A.G.; Ricciardi, L.M. On the asymptotic behaviour of first-passage-time densities for one-dimensional diffusion processes and varying boundaries. *Adv. Appl. Probab.* **1990**, *22*, 883–914. [CrossRef]
22. Alili, L.; Patie, P. On the first crossing times of a Brownian motion and a family of continuous curves. *Comptes Rendus Math.* **2005**, *340*, 225–228. [CrossRef]
23. Nobile, A.G.; Pirozzi, E.; Ricciardi, L.M. On The Two-Boundary First-Passage Time For a Class of Markov Processes. *Sci. Math. Jpn.* **2006**, *64*, 421–442.
24. Nobile, A.G.; Pirozzi, E.; Ricciardi, L.M. Asymptotics and Evaluations of FPT Densities Through Varying Boundaries For Gauss-Markov Processes. *Sci. Math. Jpn.* **2008**, *67*, 241–266.
25. Daniels, H.E. Approximating the first crossing time density for a curved boundary. *Bernoulli* **1996**, *2*, 133–143. [CrossRef]
26. Buonocore, A.; Nobile, A.G.; Ricciardi, L.M. A New Integral Equation for the Evaluation of First-Passage-Time Probability Densities. *Adv. Appl. Probab.* **1987**, *19*, 784–800. [CrossRef]
27. Giorno, V.; Nobile, A.G.; Ricciardi, L.M. A symmetry-based constructive approach to probability densities for one-dimensional diffusion processes. *J. Appl. Prob.* **1989**, *27*, 707–721. [CrossRef]
28. Di Crescenzo, A.; Giorno, V.; Nobile, A.G.; Ricciardi, L.M. On first-passage-time and transition densities for strongly symmetric diffusion processes. *Nagoya Math. J.* **1997**, *145*, 143–161. [CrossRef]
29. Abundo, M. One-dimensional reflected diffusions with two boundaries and an inverse first-hitting problem. *Stoch. Anal. Appl.* **2014**, *32*, 975–991. [CrossRef]
30. Fernandez, L.; Hieber, P.; Scherer, M. Double-barrier first-passage times of jump-diffusion processes. *Monte Carlo Methods Appl.* **2013**, *19*, 107–141.
31. Fleming, W.H.; James, M.R. Asymptotic Series and Exit Time Probabilities. *Ann. Probab.* **1992**, *20*, 1369–1384. [CrossRef]
32. Hieber, P.; Scherer, M. A note on first-passage times of continuously time-changed Brownian motion. *Stat. Probab. Lett.* **2012**, *82*, 165–172. [CrossRef]
33. Patie, P. Two-sided exit problem for a Spectrally Negative α-Stable Ornstein-Uhlenbeck Process and the Wright's generalized hypergeometric functions. *Electron. Commun. Probab.* **2007**, *12*, 146–160. [CrossRef]
34. Buonocore, A.; Giorno, V.; Nobile, A.G.; Ricciardi, L.M. On two-boundary first-crossing-time problem for diffusion processes. *J. Appl. Prob.* **1990**, *27*, 102–114. [CrossRef]

35. Sacerdote, L.; Telve, O.; Zucca, C. Joint densities of first passage times of a diffusion process through two constant boundaries. *J. Adv. Appl. Probab.* **2013**, *4*, 186–202.
36. Sacerdote, L.; Tamborrino, M.; Zucca, C. First passage times for two-dimensional correlated diffusion processes: Analytical and numerical methods. *J. Comput. Appl. Math.* **2016**, *296*, 275–292. [CrossRef]
37. D'Onofrio, G.; Pirozzi, E. Asymptotics of Two-boundary First-exit-time Densities for Gauss-Markov Processes. *Methodol. Comput. Appl. Probab.* **2019**, *21*, 735–752. [CrossRef]
38. Sweet, A.L.; Hardin, J.C. Solutions for Some Diffusion Processes with Two Barriers. *J. Appl. Probab.* **1970**, *7*, 423–431. [CrossRef]
39. D'Onofrio, G.; Pirozzi, E. On two-boundary first exit time of Gauss-diffusion processes: Closed-form results and biological modeling. *Seminario Interdisciplinare di Matematica* **2015**, *12*, 111–124
40. D'Onofrio, G.; Pirozzi, E. Two-boundary first exit time of Gauss-Markov processes for stochastic modeling of acto-myosin dynamics. *J. Math. Biol.* **2017**, *74*, 1511–1531. [CrossRef]