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Non-Degeneracy of 2-Forms and Pfaffian

Jae-Hyouk Lee

Department of Mathematics, Ewha Womans University, 52 Ewhayeodae-gil, Seodaemun-gu, Seoul 03760, Korea; jaehyoukl@ewha.ac.kr

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Abstract: In this article, we study the non-degeneracy of 2-forms (skew symmetric $(0,2)$-tensor) α along the Pfaffian of α. We consider a symplectic vector space V with a non-degenerate skew symmetric $(0,2)$-tensor ω, and derive various properties of the Pfaffian of α. As an application we show the non-degenerate skew symmetric $(0,2)$-tensor ω has a property of rigidity that it is determined by its exterior power.

Keywords: Pfaffian; non-degeneracy; symplectic space

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1. Introduction

In tensor calculus, the non-degeneracy/degeneracy is one of the key criteria to characterize the tensors. When it comes to $(0,2)$-tensors, the non-degeneracy is determined by the determinant of matrix along the identification between $(0,2)$-tensors and square matrices. As each $(0,2)$-tensor is canonically decomposed into the sum of a symmetric $(0,2)$-tensor and a skew symmetric $(0,2)$-tensor, the non-degeneracy of the skew symmetric $(0,2)$-tensors is inherited to the Pfaffian of the corresponding matrix whose square is the determinant. In this article, we study the non-degeneracy of the skew symmetric $(0,2)$-tensors along the Pfaffian of the tensors and show certain rigidity of the non-degenerate skew symmetric $(0,2)$-tensors. The Pfaffian plays roles of the determinant for skew symmetric matrices, and the properties of Pfaffian are also similar [1–3]. Here we introduce the Pfaffian for the skew symmetric tensors and study its properties related to the determinant. A non-degenerate skew symmetric $(0,2)$-tensor ω of a $2n$-dimensional real vector space V is called a symplectic form. Here the even dimension of V is the necessary condition for the non-degeneracy of ω. For a symplectic vector space (V,ω) with a fixed symplectic form ω, we define the Pfaffian map $\mathfrak{pf}(\alpha) := \alpha^n/\omega^n$ of skew symmetric $(0,2)$-tensors α where α^n (respectively, ω^n) presents $\alpha \wedge \cdots \wedge \alpha$, n-th exterior power of α (respectively, $\omega \wedge \cdots \wedge \omega$, n-th exterior power of ω). As a matter of fact, the Pfaffian map $\mathfrak{pf}$ is the Pfaffian of matrix along the standard identification Φ (Section 2) between skew symmetric $(0,2)$-tensors and the skew symmetric matrices. Moreover, we define another map ϕ from skew symmetric $(0,2)$-tensors to skew symmetric matrices, and show Lemma 3

$$\Phi(\alpha)\,\phi(\alpha) = -\mathfrak{pf}(\alpha)\, I_{2n}.$$

As an application, we show that if $\omega^k = \alpha^k$ for $1 \leq k < n$, then $\omega = \pm\alpha$. This interesting property is observed while we study the geometry of symplectic knot spaces [4] and Grassmannians of symplectic subspaces [5]. Even though this result seems very natural, we could not find it in the literature and therefore we provide our proof here.

We note that there are various reasons to consider the exterior power ω^k in the research of geometry. The exterior power $\frac{\omega^k}{k!}$ satisfies Wirtinger's inequality which concludes that it is a calibration [6]. The calibration $\frac{\omega^k}{k!}$ calibrates complex k-dimensional sub-manifolds of a Kähler manifold. A generalization of Wirtinger's inequality for ω is studied in [7] to deal with systolic inequalities for projective spaces of real numbers, complex numbers, and quaternionic numbers. The non-degeneracy of skew symmetric $(0,2)$-forms α is also described as the orbit of α in $\Lambda^n V^*$ by $GL(V)$ is open. Therefrom, stable forms and related metrics are studies in [8].

2. Symplectic Vector Space and Pfaffian

A real vector space V with a non-degenerate 2-form $\omega \in \Lambda^2 V^*$ is called a *symplectic vector space* and the 2-form ω is called a *symplectic form* of V. Here the non-degeneracy means that for any non-zero $v \in V$, the linear functional $\iota_v \omega := \omega(v, \cdot) \in V^*$ is not zero. The non-degeneracy of ω is equivalent to the existence of a basis on V^*, say $e^1, f^1, \ldots, e^n, f^n$ such that ω is of the following standard form,

$$\omega = \sum_{j=1}^{n} e^j \wedge f^j$$

so that the dimension of V must be even, here $\dim(V) = 2n$. Equivalently, ω^n is a non-zero element in $\Lambda^{2n} V^*$([9] for further detail).

For any subspace P in a symplectic vector space (V, ω), we recall the symplectic complement P^ω in V as

$$P^\omega := \{u \in V \mid \omega(u, v) = 0 \text{ for all } v \in P\}.$$

We observe that $P^\omega \cap P$ is the set of vectors where the $\omega\mid_P$, the restriction of ω to the subspace P, is completely degenerate, and $P/(P^\omega \cap P)$, thereby, has an induced symplectic form $\tilde{\omega}$. In fact, the non-degeneracy of $\tilde{\omega}$ is equivalent to the fact that $\tilde{\omega}^m$, $m = \dim(P/P^\omega \cap P)/2$, is non-vanishing. Furthermore, by pulling back this wedge product to P, we can conclude the same wedge power of $\omega\mid_P$ is non-vanishing, but the bigger power vanishes by definition of $P^\omega \cap P$. This number $r(P)$ defines the symplectic rank of P

$$r(P) := \max\left\{r \in \mathbb{N} : (\omega|_P)^r \neq 0\right\}.$$

A linear subspace S of V is called *symplectic* if the restriction of ω to S is a symplectic from on S, or equivalently $\dim(S)/2 = r(S)$ ([5] for further detail).

A symplectic vector space (V, ω) is called a *Hermitian* vector space if it equipped with an inner product g which is compatible with the symplectic form ω, i.e., $\omega(u, v) = g(Ju, v)$ defines a Hermitian complex structure J on V. Indeed, these structures induce a Hermitian inner product of V defined as

$$h(u, v) := g(u, v) - \sqrt{-1} w(u, v)$$

regarding (V, ω, J) as a complex vector space.

Pfaffian and 2-Forms

Let (V, ω, g, J) be a real $2n$-dimensional Hermitian vector space. Fix an oriented unitary basis $\{e_1, \ldots, e_{2n}\}$ with $Je_{2k-1} = e_{2k}$ for $k = 1, \ldots, n$, and J is represented as a block diagonal matrix where each block is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using this orthonormal basis of V, the space $o(V, g)$ of infinitesimal isometries of V (i.e., the tangent space of the isometry group of V at the identity) can be identified with the set of skew symmetric $2n \times 2n$ real matrices, i.e., $o(V, g) \simeq o(2n)$ (the Lie algebra of the orthogonal group $O(2n)$).

For the space of 2-forms on V, we have an isomorphism

$$\begin{array}{rccc} \Phi: & \bigwedge^2 V^* & \to & o\left(2n\right) \\ & \alpha & \mapsto & A = \left(A_{ij}\right) \end{array}$$

where the (i,j)-entry A_{ij} of A is defined as $\alpha\left(e_i, e_j\right)$. Note that (i) $\Phi\left(\omega\right) = J$ and (ii) $\Phi\left(\alpha\right)$ is invertible if and only if α is a non-degenerate form, and the inverse of Φ is defined by

$$\sum_{1 \le i < j \le 2n} B_{ij} e_i^* \wedge e_j^*$$

for each B in $o\left(2n\right)$.

Since the nth wedge power of any element in $\bigwedge^2 V^*$ is a (possibly zero) top-degree form on V, we can define the *Pfaffian* map

$$\begin{array}{rccc} \mathfrak{pf}: & \bigwedge^2 V^* & \to & \mathbb{R} \\ & \alpha & \mapsto & \frac{\alpha^n}{\omega^n} \end{array}$$

where $\omega^n = n! e_1^* \wedge e_2^* \wedge \cdots \wedge e_{2n-1}^* \wedge e_{2n}^*$. Using the identification $\bigwedge^2 V^* \simeq o(2n)$, for a skew symmetric matrix A in $o\left(2n\right)$, we have

$$\mathfrak{pf}\ \left(\Phi^{-1}\right)\left(A\right) = \sum \left(-1\right)^{\sigma} A_{i_1 i_2} \ldots A_{i_{2n-1} i_{2n}}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & i_2 & \ldots & i_{2n-1} & i_{2n} \end{pmatrix}$$

and the summation is taken over all *partitions* of $\{1, 2, \ldots, 2n-1, 2n\}$ into pairs $\{i_{2k-1}, i_{2k}\}$ with $i_{2k-1} < i_{2k}$. Indeed this is the usual definition of the *Pfaffian* of matrices and we denote

$$\mathbf{pf}(A) := \mathfrak{pf}\ \left(\Phi^{-1}\right)\left(A\right).$$

Note that for any 2-form α, $\mathfrak{pf}(\alpha) = 0$ iff α is a degenerate form, and for any skew symmetric matrix A, $\mathbf{pf}(A) = (-1)^n \sqrt{\det A}$ and in particular $\mathbf{pf}(J) = (-1)^n$.

3. Rigidity of Symplectic Forms

In this section we show that symplectic form ω has a nice property that it is determined by its exterior power.

By means of the non-degeneracy ω, we define another map from $\bigwedge^2 V^*$ to $o\left(2n\right)$ as

$$\begin{array}{rccc} \phi: & \bigwedge^2 V^* & \to & o\left(2n\right) \\ & \alpha & \longmapsto & \Phi\left(* \frac{\alpha^{n-1}}{(n-1)!}\right) \end{array}$$

where $n > 1$ and $*$ is the Hodge star operator. Here we choose $\omega^n / n!$ as a volume form.

In the following three lemmas, we will see how this map ϕ is related to the Pfaffian.

Lemma 1. *For each α in $\bigwedge^2 V^*$ with $\Phi\left(\alpha\right) = A$,*

$$\phi\left(\alpha\right)_{ij} = \begin{cases} (-1)^{i+j-1} \mathbf{pf}\left(\hat{A}_{ij}\right) & \text{if } i > j \\ (-1)^{i+j} \mathbf{pf}\left(\hat{A}_{ij}\right) & \text{if } i < j \end{cases}$$

where $\hat{A}_{ij}$ is a $(2n-2) \times (2n-2)$ matrix obtained by removing the i and j rows and columns from A.

Proof. Since $\phi(\alpha)$ is skew symmetric, it is enough to check this when $i > j$. For a given oriented unitary basis $\{e_1, \ldots, e_{2n}\}$ on V,

$$\phi(\alpha)_{ij} = \left(* \frac{\alpha^{n-1}}{(n-1)!} \right) (e_i \wedge e_j) = \frac{\alpha^{n-1}}{(n-1)!} \left(* \left(e_i \wedge e_j \right) \right)$$
$$= (-1)^{i+j-1} \frac{\alpha^{n-1}}{(n-1)!} \left(e_1 \wedge \ldots \hat{e}_i \ldots \hat{e}_j \cdots \wedge e_{2n} \right)$$

where $e_1 \wedge \ldots \hat{e}_i \ldots \hat{e}_j \cdots \wedge e_{2n}$ is given by removing e_i and e_j from the dual of $\frac{\omega^n}{n!}$. Observe

$$\frac{\alpha^{n-1}}{(n-1)!} \left(e_1 \wedge \ldots \hat{e}_i \ldots \hat{e}_j \cdots \wedge e_{2n} \right) = \frac{\left(\left. \frac{\alpha^{n-1}}{(n-1)!} \right|_S \right)^{n-1}}{e_1^* \wedge \ldots \hat{e}_i^* \ldots \hat{e}_j^* \cdots \wedge e_{2n}^*}$$
$$= \mathbf{pf} \left(\Phi|_S \left(\alpha|_S \right) \right)$$

where $S = \left(span \left\{ e_i, e_j \right\} \right)^{\perp}$ and $\Phi|_S$ is the restriction of Φ to S with the oriented basis $\left\{ e_1 \ldots \hat{e}_i \ldots \hat{e}_j \ldots e_{2n} \right\}$. Therefore,

$$\phi(\alpha)_{ij} = (-1)^{i+j-1} \mathbf{pf} \left(\hat{A}_{ij} \right).$$

□

Remark 1. *Since $\phi(\alpha)$ is a skew symmetric, $\phi(\alpha)_{ii} = 0$, and $\boldsymbol{pf}(\hat{A}_{ii}) = 0$, because $\hat{A}_{ii}$ is a $(2n-1) \times (2n-1)$ skew symmetric matrix and its determinant vanishes.*

This lemma implies that $\phi(\alpha)$ can be understood as the adjugate matrix related to the Pfaffian which is an analog to the adjugate matrix related to the determinant. From this point of view, the next lemma is also an analog to cofactor expansion of the determinant. Note that for a fixed i, we choose entries A_{ki} $(1 \leq k \leq i)$ and A_{ik} $(i < k \leq 2n)$ so that A_{ik}'s are chosen from the upper part of the diagonal of A.

Lemma 2. *For each α in $\bigwedge^2 V^*$ with $\Phi(\alpha) = A$ and each i $(1 \leq i \leq 2n)$,*

$$\mathfrak{pf}(\alpha) = \sum_{k=1}^{i-1} A_{ki} \phi(\alpha)_{ki} + \sum_{k=i+1}^{2n} A_{ik} \phi(\alpha)_{ik}.$$

Proof. First, for a fixed i from the definition of $\mathbf{pf}(A)$ each term in

$$\mathfrak{pf}(\alpha) = \sum (-1)^{\sigma} A_{i_1 i_2} \ldots A_{i_{2n-1} i_{2n}}$$

contains exactly one of the A_{ki}'s or A_{ik}'s for k $(1 \leq k \leq 2n)$. Therefore, we may write

$$\mathfrak{pf}(\alpha) = \sum_{k=1}^{i-1} C_{ki} + \sum_{k=i+1}^{n} C_{ik}$$

where the C_{ki}'s are the sum of terms in $\mathbf{pf}(A)$ containing A_{ki}. In general, for $l < m$

$$C_{lm} = \sum (-1)^{\sigma} A_{lm} \ldots A_{i_{2n-1} i_{2n}}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n \\ l & m & i_3 & i_4 & \dots & i_{2n} \end{pmatrix}$$

and the summation is taken over all partitions of $\{1, 2, \dots, 2n-1, 2n\}$ into pairs $\{i_{2k-1}, i_{2k}\}$ with $i_{2k-1} < i_{2k}$ including $\{l, m\}$. But, σ can be decomposed into $\tau\tilde{\sigma}$ where

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n \\ l & m & 1 & 2 & \dots & 2n \end{pmatrix} \text{ and}$$
$$\tau = \begin{pmatrix} l & m & 1 & 2 & \dots & 2n \\ l & m & i_3 & i_4 & \dots & i_{2n} \end{pmatrix}.$$

Therefore, $(-1)^{\sigma} = (-1)^{l+m-1} (-1)^{\tau}$and

$$C_{lm} = (-1)^{l+m-1} A_{lm} \sum (-1)^{\tau} A_{i_3 i_4} \dots A_{i_{2n-1} i_{2n}}$$

where the summation is taken over all partitions of $\left\{1, 2, \dots \hat{l}, \dots \hat{m}, \dots, 2n-1, 2n\right\}$ into pairs $\{i_{2k-1}, i_{2k}\}$ with $i_{2k-1} < i_{2k}$. Since this sum is $\mathbf{pf}(\hat{A}_{lm})$, by the previous lemma

$$C_{lm} = (-1)^{l+m-1} A_{lm}\, \mathbf{pf}\left(\hat{A}_{lm}\right) = A_{lm} \phi\left(\alpha\right)_{lm}.$$

This gives the lemma. $\square$

Lemma 3. *For each α in $\bigwedge^2 V^*$,*

$$\Phi\left(\alpha\right) \phi\left(\alpha\right) = -\mathfrak{pf}\left(\alpha\right) I_{2n}$$

where I_{2n} is the $2n \times 2n$ identity matrix.

Proof. Let $A = \Phi(\alpha)$. For $1 \leq i, j \leq 2n$, the (i, i) entry of $\Phi(\alpha)\, \phi(\alpha)$ is

$$\begin{aligned} \left(\Phi\left(\alpha\right) \phi\left(\alpha\right)\right)_{ii} &= \sum_{k=1}^{2n} A_{ik} \phi\left(\alpha\right)_{ki} \\ &= \sum_{k=1}^{i-1} A_{ik} \phi\left(\alpha\right)_{ki} + \sum_{k=i+1}^{2n} A_{ik} \phi\left(\alpha\right)_{ki} \\ &= -\sum_{k=1}^{i-1} A_{ki} \phi\left(\alpha\right)_{ki} - \sum_{k=i+1}^{2n} A_{ik} \phi\left(\alpha\right)_{ik} \\ &= -\mathfrak{pf}\left(\alpha\right). \end{aligned}$$

Here, we used the previous lemma and the fact that A and $\phi(\alpha)$ are skew symmetric. The (i, j) entry with $i < j$ is

$$
\begin{aligned}
&\left(\Phi\left(\alpha\right)\phi\left(\alpha\right)\right)_{ij} \\
&= \sum_{k=1}^{2n} A_{ik}\phi\left(\alpha\right)_{kj} \\
&= \sum_{k=1}^{i-1} A_{ik}\phi\left(\alpha\right)_{kj} + \sum_{k=i+1}^{j-1} A_{ik}\phi\left(\alpha\right)_{kj} + \sum_{k=j+1}^{2n} A_{ik}\phi\left(\alpha\right)_{kj} \\
&= -\left(\sum_{k=1}^{i-1} A_{ki}\phi\left(\alpha\right)_{kj} + \sum_{k=i+1}^{j-1} A_{ki}\phi\left(\alpha\right)_{kj} + \sum_{k=j+1}^{2n} A_{ik}\phi\left(\alpha\right)_{jk}\right) \\
&= -\mathbf{pf}\left(\tilde{A}\right).
\end{aligned}
$$

where $\tilde{A}$ is obtained from A by replacing the j-th row (respectively, column) with the i-th row (respectively, column) and by setting the (j, j) entry to zero. Therefore, $\tilde{A}$ is singular and

$$\mathbf{pf}\left(\tilde{A}\right) = (-1)^n \sqrt{\det A} = 0.$$

Similarly, we can check that the (i, j) entry with $i > j$ is also zero. This proves the lemma. □

Corollary 1. *If α in $\bigwedge^2 V^*$ with $\Phi\left(\alpha\right) = A$ is non-degenerate*

$$A^{-1} = -\frac{1}{\mathbf{pf}\left(A\right)}\phi\left(\alpha\right).$$

From these three lemmas, we can obtain the following interesting result.

Theorem 1. *Let V be $2n$-dimensional vector space with symplectic structure ω. For a 2-form β, if $\beta^k = \omega^k$, $1 \leq k < n$, then $\beta = \pm\,\omega$.*

Proof. First, we will prove this for $k = n - 1$. Since the cases $n = 1$ or $k = 1$ are trivial, we only consider the case when $n > k > 1$.

Suppose there is a 2-form β such that $\beta^{n-1} = \omega^{n-1}$. Then we observe that

$$\phi\left(\beta\right) = \Phi\left(*\frac{\beta^{n-1}}{(n-1)!}\right) = \Phi\left(*\frac{\omega^{n-1}}{(n-1)!}\right) = \Phi\left(\omega\right) = J.$$

By the previous lemma,

$$\Phi\left(\beta\right) = -\mathfrak{pf}\left(\beta\right)\phi\left(\beta\right)^{-1} = \mathfrak{pf}\left(\beta\right)J,$$

and since Φ is injective, we conclude

$$\beta = \mathfrak{pf}\left(\beta\right)\omega.$$

From $\beta^{n-1} = \omega^{n-1}$, we induce

$$\left(\mathfrak{pf}\left(\beta\right)\right)^{n-1} = 1,$$

therefore $\mathfrak{pf}\left(\beta\right) = \pm 1$.

Now, we assume there is a 2-form β with $\beta^k = \omega^k$ where $1 < k < n$. To get the conclusion, it is enough to show

$$\beta\left(a, b\right) = \pm\omega\left(a, b\right)$$

for any linearly independent vectors a and b in V. It is easy to see that for those independent vectors a and b, there is $2\left(k+1\right)$-dimensional symplectic subspace S containing a and b. By using the above argument, $\left(\beta|_S\right)^k = \left(\omega|_S\right)^k$ implies $\beta|_S = \omega|_S$ for the induced orientation on S. Therefore, $\beta|_S\left(a, b\right) = \omega|_S\left(a, b\right)$ and this gives the above conclusion. □

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