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Discrete Symmetry Group Approach for Numerical Solution of the Heat Equation

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Abstract: In this article, an invariantized finite difference scheme to find the solution of the heat equation, is developed. The scheme is based on a discrete symmetry transformation. A comparison of the results obtained by the proposed scheme and the Crank Nicolson method is carried out with reference to the exact solutions. It is found that the proposed invariantized scheme for the heat equation improves the efficiency and accuracy of the existing Crank Nicolson method.

Keywords: lie symmetries; invariantized difference scheme; numerical solutions

1. Introduction

Lie's theory of symmetry groups for differential equations was initiated and utilized for obtaining solutions or reductions of the differential equations [1]. Lie constructed a highly algorithmic technique for the solutions of differential equations. Lie established that several techniques to find the solutions of differential equations can be described and deduced by considering the Lie group analysis of differential equations [2]. Thus, Lie symmetry methodology has a great significance in the theory and applications of differential equations. It is broadly applied by several researchers to solve difficult nonlinear problems. Lie studied the groups of continuous transformations. These transformations (symmetries) can be described by the infinitesimal generators.

The symmetries, which are not continuous are called discrete symmetries. Discrete symmetries have several applications in differential equations, e.g., they are used to simplify the numerical scheme and to find the new exact solutions from the known solutions [3]. The nature of bifurcations in nonlinear dynamical systems are also obtained by using discrete symmetry groups [4]. The procedure to find the discrete symmetries of the differential equations is discussed in [5–8].

Nowadays, it is a challenging area of research to solve the dynamical equations, as various phenomena in nature are modeled in dynamical systems. Many researchers have considered dynamical systems, e.g., Martinez, Yu Zhang, and Timothy Gordon studied the uses of the control scheme in the classical dynamical systems theory to predict driver behavior and vehicle trajectories [9]. Martinez and Timothy also discussed the uses of machine learning for the systematic understanding of human control behaviors in driving [10]. Lie group theory has became a universal tool for the analysis of dynamical equations. Lie symmetry analysis provides an effective way to solve the partial differential equations.

But recently, interest is rising on the applications of Lie group analysis in the partial differential equations for their numerical solutions. Some work has been dedicated to building those numerical schemes that preserve the symmetries of the given differential equations. Invariantized finite difference schemes by using the idea of moving frame were constructed by Kim [11] and Olver [12]. The technique of discretization that preserves some continuous symmetries of the original differential equation was also studied in [13–17]. In [18], the exact solutions of Fisher's type equation with the help of Lie symmetries, which are continuous symmetries in nature, are studied. Since most of the partial differential equations have some geometrical properties and some discrete symmetry groups correspond to these geometrical features of partial differential equations [8], the new invaiantized finite difference methods constructed by using these discrete symmetry groups may show better performances than the other finite difference methods.

In this article, first, it is shown that the Crank Nicolson scheme of a diffusion (heat) equation is invariant under a group of the discrete symmetry transformation. Here, a modification is proposed for the invariantization of the Crank Nicolson method given by Kim et al. [19]. This modification is proposed with the help of the composition of discrete and continuous symmetry transformations of the heat equation. It is also shown that the proposed invariantized scheme gives better results, as compared to the other classical finite difference methods.

2. Heat Equation

The homogeneous heat (diffusion) [20] equation

$$\frac{\partial w}{\partial t} - D\frac{\partial^2 w}{\partial x^2} = 0, \tag{1}$$

plays a vital role in the study of heat conduction and other diffusion processes, in which a thin metal rod of length L, whose sides are insulated, is considered. The temperature of the bar at the point x and at the time t is represented by $w(x,t)$. The parameter D is called the thermal conductivity and it depends only upon the material from which the rod is made. For simplicity, we take the parameter $D = 1$ then the Equation (1) becomes

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0. \tag{2}$$

2.1. Continuous Symmetry Groups

Equation (2) is a linear and homogenous partial differential equation with one dependent variable w and two independent variables x and t. The point transformation of Equation (2) defines the local diffeomorphism

$$\Gamma : (x,t,w) \mapsto (\hat{x}(x,t,w), \hat{t}(x,t,w), \hat{w}(x,t,w)).$$

This transformation maps any surface $w = f(x,t)$ to the following surface

$$\hat{x} = \hat{x}(x,t,f(x,t)),$$

$$\hat{t} = \hat{t}(x,t,f(x,t)),$$

$$\hat{w} = \hat{w}(x,t,f(x,t)).$$

The heat Equation (2) has infinite dimensional Lie algebra. The infinitesimal generators and the continuous symmetry groups of the Equation (2) [21] are presented in Table 1.

Table 1. Continuous symmetry transformations of (2).

	Generators	Symmetry Transformations
1	$\frac{\partial}{\partial x}$	Space translation: $(x,t,w) \mapsto (x+\epsilon, t, w)$
2	$\frac{\partial}{\partial t}$	Time translation: $(x,t,w) \mapsto (x, t+\epsilon, w)$
3	$w\frac{\partial}{\partial w}$	Scale Transformation: $(x,t,w) \mapsto (x,t,e^{\epsilon}w)$
4	$x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$	Scale Transformation: $(x,t,w) \mapsto (e^{\epsilon}x, e^{2\epsilon}t, w)$
5	$2t\frac{\partial}{\partial x} - xw\frac{\partial}{\partial w}$	Galilean boost: $(x,t,w) \mapsto (x+2\epsilon t, t, -xw\epsilon + w)$
6	$4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)w\frac{\partial}{\partial w}$	Projection: $(x,t,w) \mapsto \left(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, w\sqrt{1-4\epsilon t}\exp(\frac{-\epsilon x^2}{1-4\epsilon t})\right)$
7	$q(x,t)\frac{\partial}{\partial w}$	

Due to the arbitrariness of the function appearing in the last generator of Table 1, Lie algebra of Equation (2) is infinite dimensional. Each of the groups given in Table 1 has the property of mapping solutions of heat Equation (2) to the other solutions. For example, consider the explicit form of the projection

$$(x,t,w) \mapsto \Big(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, w\sqrt{1-4\epsilon t}\exp(\frac{-\epsilon x^2}{1-4\epsilon t})\Big),$$

where ϵ is the group parameter. Computing the induced action on graphs of functions, we conclude that if $w = f(x,t)$ is any solution to heat Equation (2), so is

$$w = \frac{1}{\sqrt{1-4\epsilon t}}\exp\Big(\frac{\epsilon x^2}{1-4\epsilon t)}\Big) f\Big(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}\Big).$$

2.2. Discrete Symmetry Group

Hydon introduced a technique [7] by which all discrete point symmetries of the partial differential equations can be found on basis of the results [6] that each continuous symmetry generator of a Lie algebra ℓ of a differential equation brings an automorphism that preserve the following commutator relation

$$[X_l, X_k] = c_{lk}^m X_m.$$

Hydon's method categorizes and factor out all those automorphisms of a Lie algebra ℓ that are equivalent under the action of a Lie symmetry in a Lie group that is generated by the Lie algebra ℓ and provide the most general realization of these automorphisms as point transformations. Finally, by using these point transformations, an entire list of discrete point symmetries of a partial differential equation, is obtained.

The discrete symmetry group of Equation (2) has already been obtained in [8], which is isomorphic to Z_4 = {group of residues modulo 4} and is generated by

$$(x,t,w) \to \Big(\frac{x}{2t}, \frac{-1}{4t}, \sqrt{2\iota t}\exp\Big(\frac{x^2}{4t}\Big)w\Big), \tag{3}$$

where $\iota = \sqrt{-1}$.

3. Finite Difference Schemes for the Heat Equation

Some finite difference schemes are available in the literature, which help us to find the numerical solutions of the partial differential equations. In this section, the backward difference scheme, forward difference scheme, and the Crank Nicolson method for the heat equation along their stability conditions, are given [22].

3.1. Forward Difference Scheme

The forward difference scheme for heat Equation (2) is given by

$$w_{n,j+1} = \alpha w_{n+1,j} + (1-2\alpha)w_{n,j} + \alpha w_{n-1,j},$$

where $\alpha = \frac{k}{h^2}$. Here h is the $x-$axis step size and k is the time step size for the grid points (x_i, t_j), where $x_i = ih$, $t_j = jk$ for non-negative integers i, j. This scheme is Forward in Time and Centered in Space (FTCS). This method is explicit and converges to the solution for $0 < \alpha \le \frac{1}{2}$, so is conditionally stable [22].

3.2. Backward Difference Scheme

The backward difference scheme for Equation (2) is given by

$$w_{n,j-1} = -\alpha w_{n+1,j} + (1+2\alpha) w_{n,j} - \alpha w_{n-1,j}.$$

This scheme is Backward in Time and Centered in Space (BTCS). This is an implicit and unconditionally stable scheme [22].

3.3. Crank Nicolson Method

The Crank Nicolson method (CNM) for Equation (2) is

$$\alpha w_{n-1,j} + 2(1-\alpha) w_{n,j} + \alpha w_{n+1,j} = -\alpha w_{n-1,j+1}$$
$$+2(1+\alpha) w_{n,j+1} - \alpha w_{n+1,j+1}. \tag{4}$$

The Crank Nicolson method is also implicit and unconditionally stable [22]. It has significant advantages for the time-accurate solutions. The temporal truncation error of CNM is $O(\triangle t^2)$, whereas the truncation error of FTCS and BTCS is $O(\triangle t)$.

3.4. Invariantization of the Crank Nicolson Method under the Discrete Symmetry Transformation

It is observed that among the mentioned finite difference schemes for the heat equation, the Crank Nicolson method gives the more accurate results [23], so we are interested in constructing an invariantization of the Crank Nicolson method. In the present subsection, we show that the Crank Nicolson method is invariant under the discrete symmetry transformation (3).

Let $w_{n,j} = w(n\triangle x, j\triangle t)$ be an approximation of $w(x,t)$ at the mesh point (x_n, t_j). Now, by using the discrete symmetry group of heat equation given in (3), we have the following transformation

$$w = \sqrt{2it} \exp\left(\frac{x^2}{4t}\right) w. \tag{5}$$

By using (5), the Crank Nicolson method transformed to

$$\alpha w_{n-1,j} \sqrt{2ij\triangle t} \exp\left\{(n-1)^2 \frac{(\triangle x)^2}{4j\triangle t}\right\} + 2(1-\alpha) w_{n,j} \sqrt{2ij\triangle t} \exp\left\{\frac{(n\triangle x)^2}{4j\triangle t}\right\} +$$
$$\alpha w_{n+1,j} \sqrt{2ij\triangle t} \exp\left\{(n+1)^2 \frac{(\triangle x)^2}{4j\triangle t}\right\} = -\alpha w_{n-1,j+1} \sqrt{2i(j+1)\triangle t} \exp\left\{(n-1)^2 \frac{(\triangle x)^2}{4(j+1)\triangle t}\right\}$$
$$+2(1+\alpha) w_{n,j+1} \sqrt{2i(j+1)\triangle t} \exp\left\{\frac{(n\triangle x)^2}{4(j+1)\triangle t}\right\} -$$
$$\alpha w_{n+1,j+1} \sqrt{2i(j+1)\triangle t} \exp\left\{(n+1)^2 \frac{(\triangle x)^2}{4(j+1)\triangle t}\right\}. \tag{6}$$

By considering the following transformation

$$w_{N,J} = \sqrt{J} \exp\left(\frac{N^2}{4J}\right) w_{n,j},$$

with

$$J = j\triangle t \text{ and } N = n\triangle x,$$

(6) can be written as

$$\alpha w_{N-1,J} + 2(1-\alpha) w_{N,J} + \alpha w_{N+1,J} =$$
$$-\alpha w_{N-1,J+1} + 2(1+\alpha) w_{N,J+1} - \alpha w_{N+1,J+1}. \tag{7}$$

The finite difference approximation to heat Equation (2) obtained in (7) is similar to the Crank Nicolson method for heat Equation (4). Thus, the Crank Nicolson method remains invariant under the discrete symmetry transformation (3). The consequence of the result obtained in Section 3 can be written in the form of the following theorem.

Theorem 1. *The Crank Nicolson method for heat equation given in (4) is invariant under the only discrete symmetry transformation (3) of the heat Equation (2).*

4. Discrete Symmetry Numerical Scheme For the Heat Equation

Most of the finite difference methods including the Crank Nicolson method are invariant under time, space translations, and scale transformation [24]. It is also proved in the previous literature that the Crank Nicolson method for the heat Equation (2) is invariant under the transformation of the discrete symmetry. Notice that the Galilean boost and projection group are the transformations under which the Crank Nicolson method is not invariant. However, the Crank Nicolson method invariantized by Galilean boost, projection transformations or composition of these transformations, becomes unstable (i.e., does not converge to the exact solution). Nevertheless, by taking the composition of the discrete symmetry group and the projective symmetry group, which is a continuous symmetry group of the heat equation, invariantized Crank Nicolson method converges to the exact solution and gives the more adequate results as compare to other existing finite difference schemes of Equation (2).

In the present section, we construct an invariantization of the Crank Nicolson method for Equation (2) by using the composition of discrete and continuous symmetry groups. The construction of this method is based on the composition of the variable w of these two groups. We deal with the following transformation to construct the new scheme:

$$w = \sqrt{2\iota t}\exp\left\{\frac{x^2}{4(1-4\epsilon t)}\left(\frac{1}{t}-4\epsilon\right)\right\}w,$$

where $\iota = \sqrt{-1}$. By transforming the variable w in the Crank Nicolson method (4) with the above transformation for Equation (2), we get:

$$\begin{aligned}
&\alpha\sqrt{t_j}\, w_{n-1,j}\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4\epsilon\right)\right\}\\
&+(2-2\alpha)\sqrt{t_j}\, w_{n,j}\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4\epsilon\right)\right\}\\
&+\alpha\sqrt{t_j}\, w_{n+1,j}\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4\epsilon\right)\right\}\\
&=-\alpha\sqrt{t_{j+1}}\, w_{n-1,j+1}\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4\epsilon\right)\right\}\\
&+(2+2\alpha)\sqrt{t_{j+1}}\, w_{n,j+1}\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4\epsilon\right)\right\}\\
&-\alpha\sqrt{t_{j+1}}\, w_{n+1,j+1}\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4\epsilon\right)\right\},
\end{aligned}$$

which can be further simplified as

$$\sqrt{t_j}\Big[\alpha w_{n-1,j}\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4\epsilon\right)\right\}$$

$$+(2-2\alpha)w_{n,j}\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4\epsilon\right)\right\}+$$

$$\alpha w_{n+1,j}\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4\epsilon\right)\right\}\Big]=$$

$$\sqrt{t_{j+1}}\Big[-\alpha w_{n-1,j+1}\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4\epsilon\right)\right\}$$

$$+(2+2\alpha)w_{n,j+1}\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4\epsilon\right)\right\}$$

$$-\alpha w_{n+1,j+1}\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4\epsilon\right)\right\}\Big].$$

Since ϵ is a continuous parameter and we can choose the optimal value of ϵ for which the proposed method gives better performance than the Crank Nicolson method. We choose $\epsilon = c_i$, where $0 \le c_i \le 1$ and i is a positive integer, so the above equation takes the form:

$$\sqrt{t_j}\Big[\alpha w_{n-1,j}\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4c_i\right)\right\}$$

$$+(2-2\alpha)w_{n,j}\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4c_i\right)\right\}+$$

$$\alpha w_{n+1,j}\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4c_i\right)\right\}\Big]=$$

$$\sqrt{t_{j+1}}\Big[-\alpha w_{n-1,j+1}\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4c_i\right)\right\}$$

$$+(2+2\alpha)w_{n,j+1}\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}}-4c_i\right)\right\}$$

$$-\alpha w_{n+1,j+1}\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_{j+1})}(\frac{1}{t_{j+1}}-4c_i\right\}\Big].$$

The final form of the method is

$$A\Big(\alpha w_{n-1,j}B_1+(2-2\alpha)w_{n,j}B_2+\alpha w_{n+1,j}B_3\Big)$$

$$=C\Big(-\alpha w_{n-1,j+1}D_1+(2+2\alpha)w_{n,j+1}D_2-\alpha w_{n+1,j+1}D_3\Big), \tag{8}$$

where

$$A=\sqrt{t_j}\,,$$

$$B_1=\exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4c_i\right)\right\},$$

$$B_2=\exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4c_i\right)\right\},$$

$$B_3=\exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_j)}\left(\frac{1}{t_j}-4c_i\right)\right\},$$

and

$$C=\sqrt{t_{j+1}}\,,$$

$$D_1 = \exp\left\{\frac{(x_{n-1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}} - 4c_i\right)\right\},$$

$$D_2 = \exp\left\{\frac{(x_n)^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}} - 4c_i\right)\right\},$$

$$D_3 = \exp\left\{\frac{(x_{n+1})^2}{4(1-4\epsilon t_{j+1})}\left(\frac{1}{t_{j+1}} - 4c_i\right)\right\}.$$

Since the Crank Nicolson method for the heat equation is unconditionally stable [22] and the *discritized Crank Nicolson method (DCNM)* for heat Equation (2) provided in (8) is the invariantization of the Crank Nicolson method, so this method also preserves the unconditional stability condition and therefore converges to the exact solution without having any condition on α.

5. Solutions of the Heat Equation

In this section, we find the analytic solution of heat Equation (2) and compare it with the numerical solutions calculated by using the CNM and the proposed method DCNM.

5.1. Analytic Solution

We consider one dimensional homogeneous heat Equation (2) with the following boundary conditions

$$w(x,0) = g(x), \;\; 0 \le x \le L, \tag{9}$$

$$w(0,t) = f_1(t), \;\; 0 \le t \le T,$$

$$w(1,t) = f_2(t),$$

where $g(x)$, $f_1(t)$ and $f_2(t)$ are two times continuously differentiable functions on $x \in [0, L]$, L is the length of the rod and T is the maximum time.

Heat Equation (2) is reduced to an ordinary differential equation by using the following similarity variable

$$\xi = t, \; V = \frac{w}{g(x)},$$

where $g(x)$ is obtained from the initial condition (5.1) with $g(x) \neq 0$. An exact solution of the system given by the Equations (2) and (9) for particular values of arbitrary functions $g(x)$, $f_1(t)$, and $f_2(t)$ is given in the following example.

Example 1. *Our aim is to find the solution of (2) with the following initial and boundary conditions*

$$w(x,0) = sin\pi x, \;\; 0 \le x \le 1,$$

$$w(0,t) = 0, \;\; 0 \le t \le 1,$$

$$w(1,t) = 0.$$

Using the similarity transformation

$$\xi = t, \; V = \frac{w}{sin\pi x},$$

the above problem is reduced to the following ODE

$$V_\xi + \pi^2 V = 0,$$

with the solution $V = e^{-\pi^2 t}$.

Hence $w(x,t) = e^{-\pi^2 t} sin(\pi x)$ is the exact solution of the above boundary value problem.

5.2. Numerical Solutions Using CNM and the Proposed Method DCNM

In this subsection, the performance of the proposed method DCNM is investigated by applying it to Example 1. The efficiency of the present method DCNM is shown by calculating the absolute errors, root mean square errors L_2 and maximum errors L_∞. These errors are computed by the following formulas [18]:

$$\text{Absolute error} = |e_i|,$$

$$L_2 = \left(\sum_{i=1}^{n} e_i^2\right)^{1/2},$$

$$L_\infty = \max_{1\le i\le n} |e_i|,$$

with $e_i = (w_i - W_i)$, where w_i are numerical and W_i are the exact solutions.

To check the computational accuracy of the proposed method (DCNM), we reconsider Example 1 given in the previous subsection, which has the analytic solution $w(x,t) = \exp(-\pi^2 t)\sin(\pi x)$ [22]. We compare our results of the numerical solutions (DCNM) with the exact solutions and the solutions that are obtained by the Crank Nicolson method for the x-axis step size $h = 0.1$, a time step size $k = 0.01$ and for the time $T = 0.5$. The comparisons of the numerical solutions obtained by DCNM with the solutions calculated by FTCS, CNM, and the exact solutions are presented in Table 2, where Table 3 is showing the absolute errors of the solutions calculated by FTCS, CNM and the DCNM given in (8) for the different x-axis step sizes. Table 4 shows the values of $w(x,t)$ for fixed $h = 0.1$ and for different values of k, where Tables 5 shows the values of $w(x,t)$ for fixed $k = 0.02$ and for different values of h.

Table 2. The values of $w(x,t)$ for different x.

x_i	FTCS	CNM	DCNM	Exact Solutions
0.0	0.000000	0.000000	0.0000000000	0.000000
0.2	0.000892	0.004517	0.0042956284	0.004227
0.4	0.001444	0.007309	0.00695047277	0.006840
0.6	0.001444	0.007309	0.00695047277	0.006840
0.8	0.000892	0.004517	0.0042956284	0.004227
1.0	0.000000	0.000000	0.0000000000	0.000000

Table 3. Absolute errors for different values of x.

x_i	FTCS	CNM	DCNM
0.0	0.000000	0.000000	0.0000000000
0.2	0.003335	2.9×10^{-4}	6.8345438×10^{-5}
0.4	0.005396	4.69×10^{-4}	$1.10585242 \times 10^{-4}$
0.6	0.005396	4.69×10^{-4}	$1.10585242 \times 10^{-4}$
0.8	0.003335	2.9×10^{-4}	6.8345438×10^{-5}
1.0	0.000000	0.000000	0.000000000000

Table 4. $w(x,t)$ for fixed $h = 0.1$ and for different values of k.

x_i	$k = 0.03$	$k = 0.02$	$k = 0.01$
0.0	0.0000	0.0000	0.0000
0.2	0.0053	0.0045	0.0042
0.4	0.0085	0.0076	0.0069
0.6	0.0085	0.0076	0.0069
0.8	0.0053	0.0045	0.0042
1.0	0.0000	0.0000	0.0000

Table 5. $w(x, t)$ for fixed $k = 0.02$ and for different values of h.

x_i	$h = 0.3$	$h = 0.2$	$h = 0.1$
0.0	0.0000	0.0000	0.0000
0.2	0.0050	0.0047	0.0045
0.4	0.0081	0.0078	0.0076
0.6	0.0081	0.0078	0.0076
0.8	0.0050	0.0047	0.0045
1.0	0.0000	0.0000	0.0000

Table 6 presents root mean square errors L_2 and the maximum errors L_∞ for DCNM and CNM of Example 1 for the different values of t.

Table 6. L_2 and L_∞ errors for different values of t.

t	L_2 DCNM	L_∞ DCNM	L_2 CNM	L_∞ CNM
0.1	9.6×10^{-3}	4.3×10^{-3}	4.86×10^{-3}	6.79×10^{-3}
0.3	4.0×10^{-4}	1.8×10^{-4}	8.87×10^{-5}	3.76×10^{-4}
0.5	9.0830×10^{-4}	4.0812×10^{-4}	1.73×10^{-3}	2.44×10^{-4}
0.7	1.8024×10^{-4}	1.0097×10^{-4}	2.04×10^{-4}	3.17×10^{-4}
0.9	1.0224×10^{-4}	6.1223×10^{-5}	2.14×10^{-3}	3.14×10^{-3}
1	1.1556×10^{-4}	5.0808×10^{-5}	2.15×10^{-3}	3.32×10^{-3}

Figure 1 shows the comparisons between the numerical solutions, obtained by the DCNM given in (8), the Crank Nicolson method, and the exact solutions of Example 1 in 2D for fixed $T = 0.5$. For $h = 0.1$, $k = 0.05$ and $T = 1$, the graphical representations of the space-time graph of the numerical solutions calculated by the DCNM and exact solutions for the above boundary value problem for $x \in [0, 1]$ are presented in Figures 2 and 3, respectively, and it can be observed that both solutions are very similar.

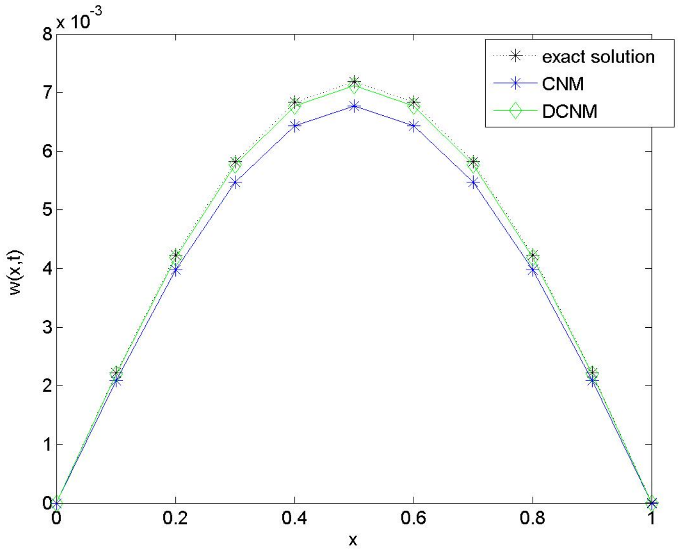

Figure 1. Comparison of the exact solution with solutions obtained by CNM and DCNM.

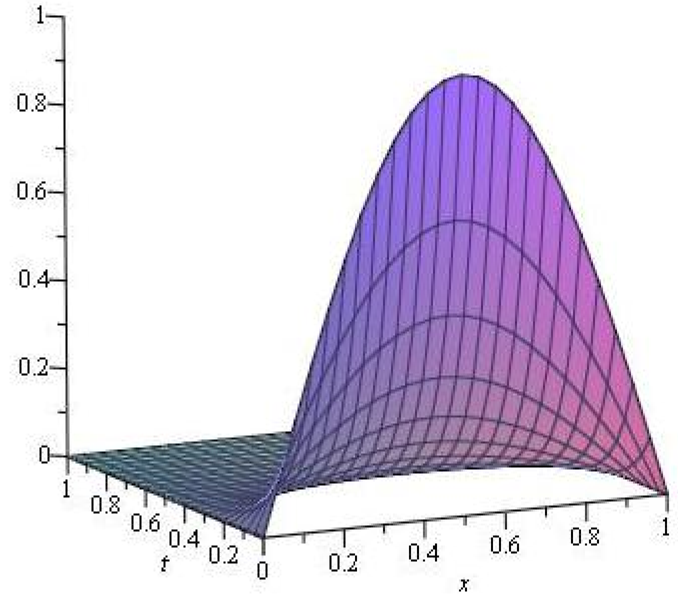

Figure 2. Space-time graph of DCNM solution for Example 1.

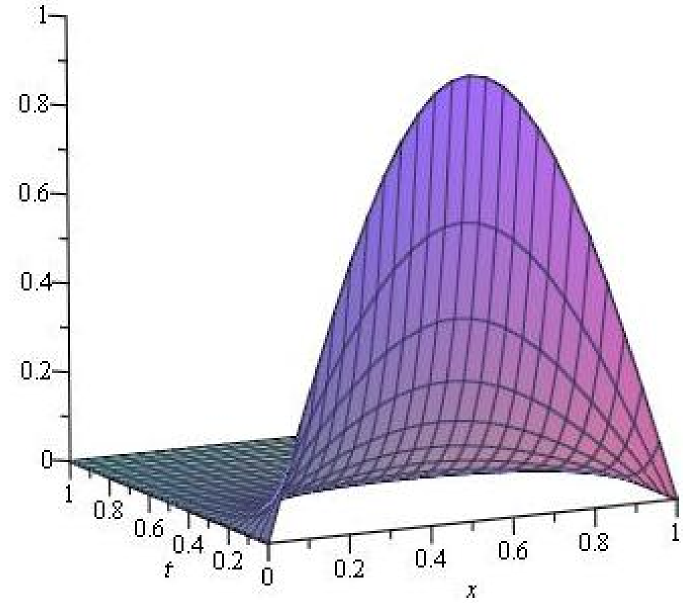

Figure 3. Space-time graph of exact solution for Example 1.

6. Conclusions

This paper contains the application of the discrete symmetry transformation for the boundary value problem of the diffusion equation. An invariantized finite difference method to find the solution of the heat equation using the composition of discrete symmetry group and projection group of the heat equation is developed. Tables 2–4 show that the proposed invariantized method DCNM improves the efficiency and accuracy of the existing Crank Nicolson method.

Similarly, with the help of discrete symmetry groups of partial differential equations, different invariantized finite difference schemes can be constructed to improve the efficiency and performance of the existing finite difference methods.

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Abbreviations

The following abbreviations are used in this manuscript:

FTCS	Forward in Time and Centered in Space
CNM	Crank Nicolson Method
DCNM	Discritized Crank Nicolson Method

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