symmetry

MDPI

*Article*

# Exceptional Set for Sums of Symmetric Mixed Powers of Primes

**Jinjiang Li [1], Chao Liu [1], Zhuo Zhang [1] and Min Zhang [2,*]**

[1] Department of Mathematics, China University of Mining and Technology, Beijing 100083, China; jinjiang.li.math@gmail.com (J.L.); chao.liu@student.cumtb.edu.cn (C.L.); zhuo.zhang.math@foxmail.com (Z.Z.)
[2] School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China
* Correspondence: min.zhang.math@gmail.com

Received: 17 January 2020; Accepted: 25 February 2020; Published: 2 March 2020

check for updates

**Abstract:** The main purpose of this paper is to use the Hardy–Littlewood method to study the solvability of mixed powers of primes. To be specific, we consider the even integers represented as the sum of one prime, one square of prime, one cube of prime, and one biquadrate of prime. However, this representation can not be realized for all even integers. In this paper, we establish the exceptional set of this kind of representation and give an upper bound estimate.

**Keywords:** Waring–Goldbach problem; circle method; exceptional set; symmetric form

**MSC:** 11P05; 11P32; 11P55

## 1. Introduction and Main Result

Let $N, k_1, k_2, \ldots, k_s$ be natural numbers which satisfy $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$, $N > s$. Waring's problem of unlike powers concerns the possibility of representation of $N$ in the form

$$N = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}. \tag{1}$$

For previous literature, the reader could refer to section P12 of LeVeque's *Reviews in number theory* and the bibliography of Vaughan [1]. For the special case, $k_1 = k_2 = \cdots = k_s$, an interesting problem is to determine the value for $k \geqslant 2$, called Waring's problem, of the function $G(k)$, the least positive number $s$ such that every *sufficiently large* number can be represented the sum of at most $s$ $k$-th powers of natural numbers. For this problem, there are only two values of the function $G(k)$ determined exactly. To be specific, $G(2) = 4$, by Lagrange in 1770, and $G(4) = 16$, by Davenport [2]. The majority of information for $G(k)$ has been derived from the Hardy–Littlewood method. This method has arised from a celebrated paper of Hardy and Ramanujan [3], which focused on the partition function.

There are many authors who devoted to establish many kinds of generalisations of this classical version of Waring's problem. Among these results, it is necessary to illustrate some of the majority variants. We begin with the most famous Waring–Goldbach problem, for which one devotes to investigate the possibility of the representation of integers as sums of $k$-th powers of prime numbers. In order to explain the associated congruence conditions, we denote by $k$ a natural number and $p$ a prime number. We write $\theta = \theta(k; p)$ as the integer with the properties $p^\theta | k$ and $p^\theta \nmid k$, and then define $\gamma = \gamma(k, p)$ by

$$\gamma(k,p) = \begin{cases} \theta + 2, & \text{when } p = 2 \text{ and } \theta > 0, \\ \theta + 1, & \text{otherwise.} \end{cases}$$

Also, we set

$$K(k) = \prod_{(p-1)|k} p^{\gamma}.$$

Denote by $H(k)$ the smallest integer $s$, which satisfies every sufficiently large integer congruent to $s$ modulo $K(k)$ can be represented as the sum of $s$ $k$-th powers of primes . By noting the fact that for $(p-1)|k$, we have $p^{\theta}(p-1)|k$, provided that $a^k \equiv 1 \pmod{p^{\gamma}}$ and $(p,a) = 1$. This states the seemingly awkward definition of $H(k)$, because if $n$ is the sum of $s$ $k$-th powers of primes exceeding $k+1$, then it must satisfy $n \equiv s \pmod{K(k)}$. Trivially, further congruence conditions could arise from the primes $p$ which satisfy $(p-1) \nmid k$. Following the previous investigations of Vinogradov [4,5], Hua systematically considered and investigated the additive problems involving prime variables in his famous book (see Hua [6,7]).

For the nonhomogeneous case, the most optimistic conjecture suggests that, for each prime $p$, if the Equation (1) has $p$-adic solutions and satisfies

$$k_1^{-1} + k_2^{-1} + \cdots + k_s^{-1} > 1, \tag{2}$$

then $n$ can be written as the sum of unlike powers of positive integers (1) provided that $n$ is sufficiently large in terms of $k$. For $s = 3$, such an claim maybe not true in certain situations (see Jagy and Kaplansky [8], or Exercise 5 of Chapter 8 of Vaughan [1]). However, a guide of application for the Hardy–Littlewood method suggests that the condition (2) should ensure at least that *almost all* integers satisfying the expected congruence conditions can be represented. Moreover, once subject to the following condition

$$k_1^{-1} + k_2^{-1} + \cdots + k_s^{-1} > 2, \tag{3}$$

a standard application of the Hardy–Littlewood method suggests that all the integers, which satisfy necessary congruence conditions, could be written in the form (1). Meanwhile, a conventional argument of the circle method shows that in situations in which the condition (2) does not hold, then every sufficiently large integer can not be represented in the expected form.

Since the Hardy–Littlewood method, the investigation of Waring's problem for unlike powers has produced splendid progress in circle method, especially for the classical version of Waring's problem. Additive Waring's problems of unlike powers involving squares, cubes or biquadrates offen attract greater interest of many mathematicians than those cases with higher mixed powers, and the current circumstance is quite satisfactory. For example, the reader can refer to references [9–19].

The Waring–Goldbach problem of mixed powers concerns the representation of $N$ which satisfying some necessary congruence conditions as the form

$$N = p_1^{k_1} + p_2^{k_2} + \cdots + p_s^{k_s},$$

where $p_1, p_2, \ldots, p_s$ are prime variables.

In 2002, Brüdern and Kawada [20] proved that for every sufficiently large even integer $N$, the equation

$$N = x + p_2^2 + p_3^3 + p_4^4$$

is solvable with $x$ being an almost–prime $\mathcal{P}_2$ and the $p_j$ $(j = 2, 3, 4)$ primes. As usual, $\mathcal{P}_r$ denotes an almost–prime with at most $r$ prime factors, counted according to multiplicity. On the other hand, in 2015, Zhao [21] established that, for $k = 3$ or 4, every sufficiently large even integer $N$ can be represented as the form

$$N = p_1 + p_2^2 + p_3^3 + p_4^k + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{t(k)}},$$

where $p_1, \ldots, p_4$ are primes, $\nu_1, \nu_2, \ldots, \nu_{t(k)}$ are natural numbers, and $t(3) = 16$, $t(4) = 18$, which is an improvement result of Liu and Lü [22]. Afterwards, Lü [23] improved the result of Zhao [21] and showed that every sufficiently large even integer $N$ can be represented as a sum of one prime, one square of prime, one cube of prime, one biquadrate of prime and 16 powers of 2.

In view of the results of Brüdern and Kawada [20], Zhao [21], Liu and Lü [22] and Lü [23], it is reasonable to conjecture that, for sufficiently large integer $N$ satisfying $N \equiv 0 \,(\bmod 2)$, the following Diophantine equation

$$N = p_1 + p_2^2 + p_3^3 + p_4^4$$

is solvable, here and below the letter $p$, with or without subscript, always denotes a prime number. However, this conjecture may be out of reach at present with the known methods and techniques.

In this paper, we shall consider the exceptional set of the problem (4) and establish the following result.

**Theorem 1.** *Let $E(N)$ denote the number of positive integers $n$, which satisfy $n \equiv 0 \,(\bmod 2)$, up to $N$, which can not be represented as*

$$n = p_1 + p_2^2 + p_3^3 + p_4^4. \tag{4}$$

*Then, for any $\varepsilon > 0$, we have*

$$E(N) \ll N^{\frac{61}{144}+\varepsilon}.$$

We will establish Theorem 1 by using a pruning process into the Hardy–Littlewood circle method. For the treatment on minor arcs, we will employ the argument developed by Wooley in [24] combined with the new estimates for exponential sum over primes developed by Zhao [25]. For the treatment on major arcs, we shall prune the major arcs further and deal with them respectively. The explicit details will be given in the related sections.

**Notation.** In this paper, let $p$, with or without subscripts, always denote a prime number; $\varepsilon$ always denotes a sufficiently small positive constant, which may not be the same at different occurrences. The letter $c$ always denotes a positive constant. As usual, we use $\chi \bmod q$ to denote a Dirichlet character modulo $q$, and $\chi^0 \bmod q$ the principal character. Moreover, we use $\varphi(n)$ and $d(n)$ to denote the Euler's function and Dirichlet's divisor function, respectively. $e(x) = e^{2\pi i x}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. $N$ is a sufficiently large integer and $n \in (N/2, N]$, and hence $\log N \asymp \log n$.

## 2. Outline of the Proof of Theorem 1

Let $N$ be a sufficiently large positive integer. By a splitting argument, it is sufficient to consider the even integers $n \in (N/2, N]$. For the application of the Hardy–Littlewood method, it is necessary to define the Farey dissection. For this purpose, we set the parameters as follows

$$A = 100^{100}, \quad Q_0 = \log^A N, \quad Q_1 = N^{\frac{1}{6}}, \quad Q_2 = N^{\frac{5}{6}}, \quad \mathfrak{I}_0 = \left[-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\right].$$

By Dirichlet's rational approximation lemma (for instance, see Lemma 12 on p.104 of [26], or Lemma 2.1 of [1]), each $\alpha \in (-1/Q_2, 1 - 1/Q_2]$ can be represented in the form

$$\alpha = \frac{a}{q} + \lambda, \qquad |\lambda| \leqslant \frac{1}{qQ_2},$$

for some integers $a$, $q$ with $1 \leqslant a \leqslant q \leqslant Q_2$ and $(a,q)=1$. Define

$$\mathfrak{M}(q,a)=\left[\frac{a}{q}-\frac{1}{qQ_2},\frac{a}{q}+\frac{1}{qQ_2}\right],\qquad \mathfrak{M}=\bigcup_{1\leqslant q\leqslant Q_1}\bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\mathfrak{M}(q,a),$$

$$\mathfrak{M}_0(q,a)=\left[\frac{a}{q}-\frac{Q_0^{100}}{qN},\frac{a}{q}+\frac{Q_0^{100}}{qN}\right],\qquad \mathfrak{M}_0=\bigcup_{1\leqslant q\leqslant Q_0^{100}}\bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\mathfrak{M}_0(q,a),$$

$$\mathfrak{m}_1=\mathfrak{I}_0\setminus\mathfrak{M},\qquad \mathfrak{m}_2=\mathfrak{M}\setminus\mathfrak{M}_0.$$

Then we obtain the Farey dissection

$$\mathfrak{I}_0=\mathfrak{M}_0\cup\mathfrak{m}_1\cup\mathfrak{m}_2. \tag{5}$$

For $k=1,2,3,4$, we define

$$f_k(\alpha)=\sum_{X_k<p\leqslant 2X_k}e(p^k\alpha),$$

where $X_k=(N/16)^{\frac{1}{k}}$. Let

$$\mathscr{R}(n)=\sum_{\substack{n=p_1+p_2^2+p_3^3+p_4^4\\X_i<p_i\leqslant 2X_i\\i=1,2,3,4}}1.$$

From (5), one has

$$\begin{aligned}\mathscr{R}(n)&=\int_0^1\left(\prod_{k=1}^4 f_k(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha=\int_{-\frac{1}{Q_2}}^{1-\frac{1}{Q_2}}\left(\prod_{k=1}^4 f_k(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\\&=\left\{\int_{\mathfrak{M}_0}+\int_{\mathfrak{m}_1}+\int_{\mathfrak{m}_2}\right\}\left(\prod_{k=1}^4 f_k(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha.\end{aligned}$$

In order to prove Theroem 1, we need the two following propositions:

**Proposition 1.** *For $n\in(N/2,N]$, there holds*

$$\int_{\mathfrak{M}_0}\left(\prod_{k=1}^4 f_k(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha=\frac{\Gamma(2)\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})\Gamma(\frac{5}{4})}{\Gamma(\frac{25}{12})}\mathfrak{S}(n)\frac{n^{\frac{13}{12}}}{\log^4 n}+O\left(\frac{n^{\frac{13}{12}}}{\log^5 n}\right), \tag{6}$$

*where $\mathfrak{S}(n)$ is the singular series defined in (10), which is absolutely convergent and satisfies*

$$(\log\log n)^{-c^*}\ll\mathfrak{S}(n)\ll d(n) \tag{7}$$

*for any integer $n$ satisfying $n\equiv 0\,(\bmod 2)$ and some fixed constant $c^*>0$.*

The proof of (6) in Proposition 1 follows from the well–know standard technique in the Hardy–Littlewood method. For more information, one can see pp. 90–99 of Hua [7], so we omit the details herein. For the properties (7) of singular series, we shall give the proof in Section 4.

**Proposition 2.** *Let $\mathcal{Z}(N)$ denote the number of integers $n\in(N/2,N]$ satisfying $n\equiv 0\,(\bmod 2)$ such that*

$$\sum_{j=1}^2\left|\int_{\mathfrak{m}_j}\left(\prod_{k=1}^4 f_k(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\right|\gg\frac{n^{\frac{13}{12}}}{\log^5 n}.$$

*Then we have*

$$\mathcal{Z}(N) \ll N^{\frac{61}{144}+\varepsilon}.$$

The proof of Proposition 2 will be given in Section 5. The remaining part of this section is devoted to establishing Theorem 1 by using Proposition 1 and Proposition 2.

**Proof of Theorem 1.** From Proposition 2, we deduce that, with at most $O\big(N^{\frac{61}{144}+\varepsilon}\big)$ exceptions, all even integers $n \in (N/2, N]$ satisfy

$$\sum_{j=1}^{2}\left|\int_{\mathfrak{m}_j}\left(\prod_{k=1}^{4} f_k(\alpha)\right)e(-n\alpha)\mathrm{d}\alpha\right| \ll \frac{n^{\frac{13}{12}}}{\log^5 n},$$

from which and Proposition 1, we conclude that, with at most $O\big(N^{\frac{61}{144}+\varepsilon}\big)$ exceptions, for all even integers $n \in (N/2, N]$, $\mathscr{R}(n)$ holds the asymptotic formula

$$\mathscr{R}(n) = \frac{\Gamma(2)\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})\Gamma(\frac{5}{4})}{\Gamma(\frac{25}{12})}\mathfrak{S}(n)\frac{n^{\frac{13}{12}}}{\log^4 n} + O\left(\frac{n^{\frac{13}{12}}}{\log^5 n}\right).$$

In other words, all even integers $n \in (N/2, N]$ can be represented in the form $p_1 + p_2^2 + p_3^3 + p_4^4$ with at most $O\big(N^{\frac{61}{144}+\varepsilon}\big)$ exceptions, where $p_1, p_2, p_3, p_4$ are prime numbers. By a splitting argument, we get

$$E(N) \ll \sum_{0\leqslant \ell \ll \log N} \mathcal{Z}\left(\frac{N}{2^\ell}\right) \ll \sum_{0\leqslant \ell \ll \log N}\left(\frac{N}{2^\ell}\right)^{\frac{61}{144}+\varepsilon} \ll N^{\frac{61}{144}+\varepsilon}.$$

This completes the proof of Theorem 1.

## 3. Some Auxiliary Lemmas

In this section, we shall list some necessary lemmas which will be used in proving Proposition 2.

**Lemma 1.** *Suppose that $\alpha$ is a real number, and that $|\alpha - a/q| \leqslant q^{-2}$ with $(a, q) = 1$. Let $\beta = \alpha - a/q$. Then we have*

$$f_k(\alpha) \ll d^{\delta_k}(q)(\log x)^c\left(X_k^{1/2}\sqrt{q(1+N|\beta|)} + X_k^{4/5} + \frac{X_k}{\sqrt{q(1+N|\beta|)}}\right),$$

*where $\delta_k = \frac{1}{2} + \frac{\log k}{\log 2}$ and $c$ is a constant.*

**Proof.** See Theorem 1.1 of Ren [27]. □

**Lemma 2.** *Suppose that $\alpha$ is a real number, and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a, q) = 1, \qquad 1 \leqslant q \leqslant X \qquad \textit{and} \qquad |q\alpha - a| \leqslant X^{-1}.$$

*If $P^{2\delta 2^{1-k}} \leqslant X \leqslant P^{k-2\delta 2^{1-k}}$, then one has*

$$\sum_{P<p\leqslant 2P} e\big(p^k\alpha\big) \ll P^{1-\delta 2^{1-k}+\varepsilon} + \frac{P^{1+\varepsilon}}{q^{1/2}\big(1+P^k|\alpha - a/q|\big)^{1/2}},$$

*where $\delta = 1/3$ for $k \geqslant 4$.*

**Proof.** See Lemma 2.4 of Zhao [25]. □

**Lemma 3.** *Suppose that $\alpha$ is a real number, and that there are $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a,q)=1, \qquad 1 \leqslant q \leqslant Q \qquad \textit{and} \qquad |q\alpha - a| \leqslant Q^{-1}.$$

*If $P^{\frac{1}{2}} \leqslant Q \leqslant P^{\frac{5}{2}}$, then one has*

$$\sum_{P<p\leqslant 2P} e(p^3\alpha) \ll P^{1-\frac{1}{12}+\varepsilon} + \frac{q^{-\frac{1}{6}}P^{1+\varepsilon}}{\big(1+P^3|\alpha - a/q|\big)^{1/2}}.$$

**Proof.** See Lemma 8.5 of Zhao [25]. □

**Lemma 4.** *For $\alpha \in \mathfrak{m}_1$, we have*

$$f_3(\alpha) \ll N^{\frac{11}{36}+\varepsilon} \qquad \textit{and} \qquad f_4(\alpha) \ll N^{\frac{23}{96}+\varepsilon}.$$

**Proof.** For $\alpha \in \mathfrak{m}_1$, we have $Q_1 \leqslant q \leqslant Q_2$. By Lemma 3, we get

$$f_3(\alpha) \ll X_3^{\frac{11}{12}+\varepsilon} + X_3^{1+\varepsilon}Q_1^{-\frac{1}{6}} \ll N^{\frac{11}{36}+\varepsilon}.$$

From Lemma 2, we obtain

$$f_4(\alpha) \ll X_4^{\frac{23}{24}+\varepsilon} + X_4^{1+\varepsilon}Q_1^{-\frac{1}{2}} \ll N^{\frac{23}{96}+\varepsilon}.$$

This completes the proof of Lemma 4. □

For $1 \leqslant a \leqslant q$ with $(a,q)=1$, set

$$\mathcal{I}(q,a) = \left[\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0}\right], \qquad \mathcal{I} = \bigcup_{1\leqslant q\leqslant Q_0} \bigcup_{\substack{a=-q\\(a,q)=1}}^{2q} \mathcal{I}(q,a). \tag{8}$$

For $\alpha \in \mathfrak{m}_2$, by Lemma 1, we have

$$f_3(\alpha) \ll \frac{N^{\frac{1}{3}}\log^c N}{q^{\frac{1}{2}-\varepsilon}\big(1+N|\lambda|\big)^{1/2}} + N^{\frac{4}{15}+\varepsilon} = V_3(\alpha) + N^{\frac{4}{15}+\varepsilon}, \tag{9}$$

say. Then we obtain the following Lemma.

**Lemma 5.** *We have*

$$\int_{\mathcal{I}} |V_3(\alpha)|^4 \mathrm{d}\alpha = \sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathcal{I}(q,a)} |V_3(\alpha)|^4 \mathrm{d}\alpha \ll N^{\frac{1}{3}}\log^c N.$$

**Proof.** We have

$$\begin{aligned}
&\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|V_3(\alpha)|^4\mathrm{d}\alpha\\
\ll&\sum_{1\leqslant q\leqslant Q_0}q^{-2+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{|\lambda|\leqslant\frac{1}{Q_0}}\frac{N^{\frac{4}{3}}\log^c N}{(1+N|\lambda|)^2}\mathrm{d}\lambda\\
\ll&\sum_{1\leqslant q\leqslant Q_0}q^{-2+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\left(\int_{|\lambda|\leqslant\frac{1}{N}}N^{\frac{4}{3}}\log^c N\mathrm{d}\lambda+\int_{\frac{1}{N}\leqslant|\lambda|\leqslant\frac{1}{Q_0}}\frac{N^{\frac{4}{3}}\log^c N}{N^2\lambda^2}\mathrm{d}\lambda\right)\\
\ll&N^{\frac{1}{3}}\log^c N\sum_{1\leqslant q\leqslant Q_0}q^{-2+\varepsilon}\varphi(q)\ll N^{\frac{1}{3}}Q_0^{\varepsilon}\log^c N\ll N^{\frac{1}{3}}\log^c N.
\end{aligned}$$

This completes the proof of Lemma 5. □

## 4. The Singular Series

In this section, we shall concentrate on investigating the properties of the singular series which appear in Proposition 1. First, we illustrate some notations. For $k\in\{1,2,3,4\}$ and a Dirichlet character $\chi$ mod $q$, we define

$$C_k(\chi,a)=\sum_{h=1}^{q}\overline{\chi(h)}e\left(\frac{ah^k}{q}\right),\qquad C_k(q,a)=C_k(\chi^0,a),$$

where $\chi^0$ is the principal character modulo $q$. Let $\chi_1,\chi_2,\chi_3,\chi_4$ be Dirichlet characters modulo $q$. Set

$$B(n,q,\chi_1,\chi_2,\chi_3,\chi_4)=\sum_{\substack{a=1\\(a,q)=1}}^{q}C_1(\chi_1,a)C_2(\chi_2,a)C_3(\chi_3,a)C_4(\chi_4,a)e\left(-\frac{an}{q}\right),$$

$$B(n,q)=B(n,q,\chi^0,\chi^0,\chi^0,\chi^0),$$

and write

$$A(n,q)=\frac{B(n,q)}{\varphi^4(q)},\qquad\mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q).\tag{10}$$

**Lemma 6.** *For $(a,q)=1$ and any Dirichlet character $\chi$ mod $q$, there holds*

$$|C_k(\chi,a)|\leqslant 2q^{1/2}d^{\beta_k}(q)$$

*with $\beta_k=(\log k)/\log 2$.*

**Proof.** See the Problem 14 of Chapter VI of Vinogradov [28]. □

**Lemma 7.** *Let $p$ be a prime and $p^\alpha\|k$. For $(a,p)=1$, if $\ell\geqslant\gamma(p)$, we have $C_k(p^\ell,a)=0$, where*

$$\gamma(p)=\begin{cases}\alpha+2, & \text{if } p\neq 2 \text{ or } p=2,\ \alpha=0;\\ \alpha+3, & \text{if } p=2,\ \alpha>0.\end{cases}$$

**Proof.** See Lemma 8.3 of Hua [7]. □

For $k\geqslant 1$, we define

$$S_k(q,a)=\sum_{m=1}^{q}e\left(\frac{am^k}{q}\right).$$

**Lemma 8.** *Suppose that $(p, a) = 1$. Then*

$$S_k(p, a) = \sum_{\chi \in \mathscr{A}_k} \overline{\chi(a)} \tau(\chi),$$

*where $\mathscr{A}_k$ denotes the set of non–principal characters $\chi$ modulo $p$ for which $\chi^k$ is principal, and $\tau(\chi)$ denotes the Gauss sum*

$$\sum_{m=1}^{p} \chi(m) e\Big(\frac{m}{p}\Big).$$

*Also, there hold $|\tau(\chi)| = p^{1/2}$ and $|\mathscr{A}_k| = (k, p-1) - 1$.*

**Proof.** See Lemma 4.3 of Vaughan [1]. □

**Lemma 9.** *For $(p, n) = 1$, we have*

$$\left| \sum_{a=1}^{p-1} \left( \prod_{k=1}^{4} \frac{S_k(p, a)}{p} \right) e\Big(-\frac{an}{p}\Big) \right| \leqslant 24 p^{-\frac{3}{2}}. \tag{11}$$

**Proof.** We denote by $\mathcal{S}$ the left-hand side of (11). It follows from Lemma 8 that

$$\mathcal{S} = \frac{1}{p^4} \sum_{a=1}^{p-1} \left( \prod_{k=1}^{4} \left( \sum_{\chi_k \in \mathscr{A}_k} \overline{\chi_k(a)} \tau(\chi_k) \right) \right) e\Big(-\frac{an}{p}\Big).$$

If $|\mathscr{A}_k| = 0$ for some $k \in \{1, 2, 3, 4\}$, then $\mathcal{S} = 0$. If this is not the case, then

$$\begin{aligned} \mathcal{S} = \frac{1}{p^4} & \sum_{\chi_1 \in \mathscr{A}_1} \sum_{\chi_2 \in \mathscr{A}_2} \sum_{\chi_3 \in \mathscr{A}_3} \sum_{\chi_4 \in \mathscr{A}_4} \tau(\chi_1)\tau(\chi_2)\tau(\chi_3)\tau(\chi_4) \\ & \times \sum_{a=1}^{p-1} \overline{\chi_1(a)\chi_2(a)\chi_3(a)\chi_4(a)} e\Big(-\frac{an}{p}\Big). \end{aligned}$$

From Lemma 8, the quadruple outer sums have no more than $4! = 24$ terms. For each of these terms, there holds

$$|\tau(\chi_1)\tau(\chi_2)\tau(\chi_3)\tau(\chi_4)| = p^2.$$

Since in any one of these terms $\overline{\chi_1(a)\chi_2(a)\chi_3(a)\chi_4(a)}$ is a Dirichlet character $\chi \pmod p$, the inner sum is

$$\sum_{a=1}^{p-1} \chi(a) e\Big(-\frac{an}{p}\Big) = \overline{\chi(-n)} \sum_{a=1}^{p-1} \chi(-an) e\Big(-\frac{an}{p}\Big) = \overline{\chi(-n)} \tau(\chi).$$

By noting the fact that $\tau(\chi^0) = -1$ for principal character $\chi^0 \bmod p$, we derive that

$$|\overline{\chi(-n)} \tau(\chi)| \leqslant p^{\frac{1}{2}}.$$

From the above arguments, we deduce that

$$|\mathcal{S}| \leqslant \frac{1}{p^4} \cdot 24 \cdot p^2 \cdot p^{\frac{1}{2}} = 24 p^{-\frac{3}{2}},$$

which completes the proof of Lemma 9. □

**Lemma 10.** *Let $\mathcal{L}(p, n)$ denote the number of solutions of the congruence*

$$x_1 + x_2^2 + x_3^3 + x_4^4 \equiv n \pmod p, \qquad 1 \leqslant x_1, x_2, x_3, x_4 \leqslant p-1.$$

*Then, for $n \equiv 0\,(\bmod 2)$, we have $\mathcal{L}(p,n) > 0$.*

**Proof.** We have

$$p \cdot \mathcal{L}(p,n) = \sum_{a=1}^{p} C_1(p,a)C_2(p,a)C_3(p,a)C_4(p,a)e\Big(-\frac{an}{p}\Big) = (p-1)^4 + E_p,$$

where

$$E_p = \sum_{a=1}^{p-1} C_1(p,a)C_2(p,a)C_3(p,a)C_4(p,a)e\Big(-\frac{an}{p}\Big).$$

By Lemma 8, we obtain

$$|E_p| \leqslant (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)(3\sqrt{p}+1).$$

It is easy to check that $|E_p| < (p-1)^4$ for $p \geqslant 7$. Therefore, we obtain $\mathcal{L}(p,n) > 0$ for $p \geqslant 7$. For $p = 2,3,5$, we can check $\mathcal{L}(p,n) > 0$ one by one. This completes the proof of Lemma 10. □

**Lemma 11.** *$A(n,q)$ is multiplicative in $q$.*

**Proof.** From the definition of $A(n,q)$ in (10), it is sufficient to show that $B(n,q)$ is multiplicative in $q$. Suppose $q = q_1q_2$ with $(q_1,q_2) = 1$. Then we obtain

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a=1 \\ (a,q_1q_2)=1}}^{q_1q_2} \Big(\prod_{k=1}^{4} C_k(q_1q_2,a)\Big)e\Big(-\frac{an}{q_1q_2}\Big) \\ &= \sum_{\substack{a_1=1 \\ (a_1,q_1)=1}}^{q_1} \sum_{\substack{a_2=1 \\ (a_2,q_2)=1}}^{q_2} \Big(\prod_{k=1}^{4} C_k(q_1q_2,a_1q_2+a_2q_1)\Big)e\Big(-\frac{a_1n}{q_1}\Big)e\Big(-\frac{a_2n}{q_2}\Big). \end{aligned} \tag{12}$$

For $(q_1,q_2) = 1$, there holds

$$\begin{aligned} C_k(q_1q_2,a_1q_2+a_2q_1) &= \sum_{\substack{m=1 \\ (m,q_1q_2)=1}}^{q_1q_2} e\Big(\frac{(a_1q_2+a_2q_1)m^k}{q_1q_2}\Big) \\ &= \sum_{\substack{m_1=1 \\ (m_1,q_1)=1}}^{q_1} \sum_{\substack{m_2=1 \\ (m_2,q_2)=1}}^{q_2} e\Big(\frac{(a_1q_2+a_2q_1)(m_1q_2+m_2q_1)^k}{q_1q_2}\Big) \\ &= \sum_{\substack{m_1=1 \\ (m_1,q_1)=1}}^{q_1} e\Big(\frac{a_1(m_1q_2)^k}{q_1}\Big) \sum_{\substack{m_2=1 \\ (m_2,q_2)=1}}^{q_2} e\Big(\frac{a_2(m_2q_1)^k}{q_2}\Big) \\ &= C_k(q_1,a_1)C_k(q_2,a_2). \end{aligned} \tag{13}$$

Putting (13) into (12), we deduce that

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a_1=1 \\ (a_1,q_1)=1}}^{q_1} \Big(\prod_{k=1}^{4} C_k(q_1,a_1)\Big)e\Big(-\frac{a_1n}{q_1}\Big) \sum_{\substack{a_2=1 \\ (a_2,q_2)=1}}^{q_2} \Big(\prod_{k=1}^{4} C_k(q_2,a_2)\Big)e\Big(-\frac{a_2n}{q_2}\Big) \\ &= B(n,q_1)B(n,q_2). \end{aligned}$$

This completes the proof of Lemma 11. □

**Lemma 12.** *Let $A(n,q)$ be as defined in (10). Then* *(i)* *we have*

$$\sum_{q>Z}|A(n,q)| \ll Z^{-\frac{1}{2}+\varepsilon}d(n),$$

*and thus the singular series $\mathfrak{S}(n)$ is absolutely convergent and satisfies $\mathfrak{S}(n) \ll d(n)$.*

*(ii)* *there exists an absolute positive constant $c^* > 0$, such that, for $n \equiv 0 \,(\bmod 2)$,*

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}.$$

**Proof.** From Lemma 11, we know that $B(n,q)$ is multiplicative in $q$. Therefore, there holds

$$B(n,q) = \prod_{p^t\|q} B(n,p^t) = \prod_{p^t\|q} \sum_{\substack{a=1\\(a,p)=1}}^{p^t} \left(\prod_{k=1}^{4} C_k(p^t,a)\right) e\left(-\frac{an}{p^t}\right). \tag{14}$$

From (14) and Lemma 7, we deduce that $B(n,q) = \prod_{p\|q} B(n,p)$ or 0 according to $q$ is square–free or not. Thus, one has

$$\sum_{q=1}^{\infty} A(n,q) = \sum_{\substack{q=1\\ q \text{ square–free}}}^{\infty} A(n,q). \tag{15}$$

Write

$$\mathcal{R}(p,a) := \prod_{k=1}^{4} C_k(p,a) - \prod_{k=1}^{4} S_k(p,a).$$

Then

$$A(n,p) = \frac{1}{(p-1)^4}\sum_{a=1}^{p-1}\left(\prod_{k=1}^{4} S_k(p,a)\right)e\left(-\frac{an}{p}\right) + \frac{1}{(p-1)^4}\sum_{a=1}^{p-1}\mathcal{R}(p,a)e\left(-\frac{an}{p}\right). \tag{16}$$

Applying Lemma 6 and noticing that $S_k(p,a) = C_k(p,a)+1$, we get $S_k(p,a) \ll p^{\frac{1}{2}}$, and thus $\mathcal{R}(p,a) \ll p^{\frac{3}{2}}$. Therefore, the second term in (16) is $\leqslant c_1 p^{-\frac{3}{2}}$. On the other hand, from Lemma 9, we can see that the first term in (16) is $\leqslant 2^4 \cdot 24 p^{-\frac{3}{2}} = 384 p^{-\frac{3}{2}}$. Let $c_2 = \max(c_1, 384)$. Then we have proved that, for $p \nmid n$, there holds

$$|A(n,p)| \leqslant c_2 p^{-\frac{3}{2}}. \tag{17}$$

Moreover, if we use Lemma 6 directly, it follows that

$$\begin{aligned}|B(n,p)| &= \left|\sum_{a=1}^{p-1}\left(\prod_{k=1}^{4} C_k(p,a)\right)e\left(-\frac{an}{p}\right)\right| \leqslant \sum_{a=1}^{p-1}\prod_{k=1}^{4}|C_k(p,a)| \\ &\leqslant (p-1)\cdot 2^4 \cdot p^2 \cdot 24 = 384p^2(p-1),\end{aligned}$$

and therefore

$$|A(n,p)| = \frac{|B(n,p)|}{\varphi^4(p)} \leqslant \frac{384p^2}{(p-1)^3} \leqslant \frac{2^3\cdot 384 p^2}{p^3} = \frac{3072}{p}. \tag{18}$$

Let $c_3 = \max(c_2, 3072)$. Then, for square–free $q$, we have

$$\begin{aligned}|A(n,q)| &= \left(\prod_{\substack{p|q\\ p\nmid n}}|A(n,p)|\right)\left(\prod_{\substack{p|q\\ p|n}}|A(n,p)|\right) \leqslant \left(\prod_{\substack{p|q\\ p\nmid n}}(c_3 p^{-\frac{3}{2}})\right)\left(\prod_{\substack{p|q\\ p|n}}(c_3 p^{-1})\right) \\ &= c_3^{\omega(q)}\left(\prod_{p|q} p^{-\frac{3}{2}}\right)\left(\prod_{p|(n,q)} p^{\frac{1}{2}}\right) \ll q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}}.\end{aligned}$$

Hence, by (15), we obtain

$$\sum_{q>Z}|A(n,q)| \ll \sum_{q>Z} q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}} = \sum_{d|n}\sum_{q>\frac{Z}{d}}(dq)^{-\frac{3}{2}+\varepsilon}d^{\frac{1}{2}} = \sum_{d|n} d^{-1+\varepsilon}\sum_{q>\frac{Z}{d}} q^{-\frac{3}{2}+\varepsilon}$$
$$\ll \sum_{d|n} d^{-1+\varepsilon}\Big(\frac{Z}{d}\Big)^{-\frac{1}{2}+\varepsilon} = Z^{-\frac{1}{2}+\varepsilon}\sum_{d|n} d^{-\frac{1}{2}+\varepsilon} \ll Z^{-\frac{1}{2}+\varepsilon}d(n).$$

This proves (i) of Lemma 12.
To prove (ii) of Lemma 12, by Lemma 11, we first note that

$$\begin{aligned}\mathfrak{S}(n) &= \prod_p\Big(1+\sum_{t=1}^{\infty}A(n,p^t)\Big) = \prod_p(1+A(n,p)) \\ &= \Bigg(\prod_{p\leqslant c_3}(1+A(n,p))\Bigg)\Bigg(\prod_{\substack{p>c_3\\ p\nmid n}}(1+A(n,p))\Bigg)\Bigg(\prod_{\substack{p>c_3\\ p|n}}(1+A(n,p))\Bigg).\end{aligned} \tag{19}$$

From (17), we have

$$\prod_{\substack{p>c_3\\ p\nmid n}}(1+A(n,p)) \geqslant \prod_{p>c_3}\Big(1-\frac{c_3}{p^{3/2}}\Big) \geqslant c_4 > 0. \tag{20}$$

By (18), we know that there are $c_5 > 0$ such that

$$\prod_{\substack{p>c_3\\ p|n}}(1+A(n,p)) \geqslant \prod_{\substack{p>c_3\\ p|n}}\Big(1-\frac{c_3}{p}\Big) \geqslant \prod_{p|n}\Big(1-\frac{c_3}{p}\Big) \gg (\log\log n)^{-c_5}. \tag{21}$$

On the other hand, it is easy to see that

$$1+A(n,p) = \frac{p\cdot\mathcal{L}(p,n)}{\varphi^4(p)}.$$

By Lemma 10, we know that $\mathcal{L}(p,n) > 0$ for all $p$ with $n \equiv 0 \pmod 2$, and thus $1+A(n,p) > 0$. Therefore, there holds

$$\prod_{p\leqslant c_3}(1+A(n,p)) \geqslant c_6 > 0. \tag{22}$$

Combining the estimates (19)–(22), and taking $c^* = c_5 > 0$, we derive that

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}.$$

This completes the proof Lemma 12. □

## 5. Proof of Proposition 2

In this section, we shall give the proof of Proposition 2. We denote by $\mathcal{Z}_j(N)$ the set of integers $n$ satisfying $n \in [N/2, N]$ and $n \equiv 0 \pmod 2$ for which the following estimate

$$\left|\int_{\mathfrak{m}_j}\Bigg(\prod_{k=1}^{4}f_k(\alpha)\Bigg)e(-n\alpha)\mathrm{d}\alpha\right| \gg \frac{n^{\frac{13}{12}}}{\log^5 n} \tag{23}$$

holds. For convenience, we use $\mathcal{Z}_j$ to denote the cardinality of $\mathcal{Z}_j(N)$ for abbreviation. Also, we define the complex number $\xi_j(n)$ by taking $\xi_j(n) = 0$ for $n \notin \mathcal{Z}_j(N)$, and when $n \in \mathcal{Z}_j(N)$ by means of the equation

$$\left| \int_{\mathfrak{m}_j} \left( \prod_{k=1}^{4} f_k(\alpha) \right) e(-n\alpha) \mathrm{d}\alpha \right| = \xi_j(n) \int_{\mathfrak{m}_j} \left( \prod_{k=1}^{4} f_k(\alpha) \right) e(-n\alpha) \mathrm{d}\alpha. \tag{24}$$

Plainly, one has $|\xi_j(n)| = 1$ whenever $\xi_j(n)$ is nonzero. Therefore, we obtain

$$\sum_{n \in \mathcal{Z}_j(N)} \xi_j(n) \int_{\mathfrak{m}_j} \left( \prod_{k=1}^{4} f_k(\alpha) \right) e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{m}_j} \left( \prod_{k=1}^{4} f_k(\alpha) \right) \mathcal{K}_j(\alpha) \mathrm{d}\alpha, \tag{25}$$

where the exponential sum $\mathcal{K}_j(\alpha)$ is defined by

$$\mathcal{K}_j(\alpha) = \sum_{n \in \mathcal{Z}_j(N)} \xi_j(n) e(-n\alpha).$$

For $j = 1, 2$, set

$$I_j = \int_{\mathfrak{m}_j} \left( \prod_{k=1}^{4} f_k(\alpha) \right) \mathcal{K}_j(\alpha) \mathrm{d}\alpha.$$

By (23)–(25), we derive that

$$I_j \gg \sum_{n \in \mathcal{Z}_j(N)} \frac{n^{\frac{13}{12}}}{\log^5 n} \gg \frac{\mathcal{Z}_j N^{\frac{13}{12}}}{\log^5 N}, \qquad j = 1, 2. \tag{26}$$

By Lemma 2.1 of Wooley [24] with $k = 2$, we know that, for $j = 1, 2$, there holds

$$\int_0^1 \left| f_2(\alpha) \mathcal{K}_j(\alpha) \right|^2 \mathrm{d}\alpha \ll N^{\varepsilon} \big( \mathcal{Z}_j N^{\frac{1}{2}} + \mathcal{Z}_j^2 \big). \tag{27}$$

It follows from Cauchy's inequality, Lemma 4 and (27) that

$$\begin{aligned} I_1 &\ll \left( \sup_{\alpha \in \mathfrak{m}_1} |f_3(\alpha)| \right) \left( \sup_{\alpha \in \mathfrak{m}_1} |f_4(\alpha)| \right) \left( \int_0^1 |f_2(\alpha) \mathcal{K}_1(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \left( \int_0^1 |f_1(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \\ &\ll N^{\frac{11}{36}+\varepsilon} \cdot N^{\frac{23}{96}+\varepsilon} \cdot \Big( N^{\varepsilon} \big( \mathcal{Z}_1 N^{\frac{1}{2}} + \mathcal{Z}_1^2 \big) \Big)^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \\ &\ll N^{\frac{301}{288}+\varepsilon} \Big( \mathcal{Z}_1^{\frac{1}{2}} N^{\frac{1}{4}} + \mathcal{Z}_1 \Big) \ll \mathcal{Z}_1^{\frac{1}{2}} N^{\frac{373}{288}+\varepsilon} + \mathcal{Z}_1 N^{\frac{301}{288}+\varepsilon}. \end{aligned} \tag{28}$$

Combining (26) and (28), we get

$$\mathcal{Z}_1 N^{\frac{13}{12}} \log^{-5} N \ll I_1 \ll \mathcal{Z}_1^{\frac{1}{2}} N^{\frac{373}{288}+\varepsilon} + \mathcal{Z}_1 N^{\frac{301}{288}+\varepsilon},$$

which implies

$$\mathcal{Z}_1 \ll N^{\frac{61}{144}+\varepsilon}. \tag{29}$$

Next, we give the upper bound for $\mathcal{Z}_2$. By (9), we obtain

$$\begin{aligned} I_2 \ll & \int_{\mathfrak{m}_2} \left| f_1(\alpha) f_2(\alpha) V_3(\alpha) f_4(\alpha) \mathcal{K}_2(\alpha) \right| \mathrm{d}\alpha \\ & + N^{\frac{4}{15}+\varepsilon} \cdot \int_{\mathfrak{m}_2} \left| f_1(\alpha) f_2(\alpha) f_4(\alpha) \mathcal{K}_2(\alpha) \right| \mathrm{d}\alpha \\ = & \, I_{21} + I_{22}, \end{aligned} \tag{30}$$

say. For $\alpha \in \mathfrak{m}_2$, we have either $Q_0^{100} < q < Q_1$ or $Q_0^{100} < N|q\alpha - a| < NQ_2^{-1} = Q_1$. Therefore, by Lemma 1, we get

$$\sup_{\alpha \in \mathfrak{m}_2} |f_4(\alpha)| \ll \frac{N^{\frac{1}{4}}}{\log^{40A} N}. \tag{31}$$

In view of the fact that $\mathfrak{m}_2 \subseteq \mathcal{I}$, where $\mathcal{I}$ is defined by (8), Hölder's inequality, the trivial estimate $\mathcal{K}_2(\alpha) \ll \mathcal{Z}_2$ and Theorem 4 of Hua (See [7], p. 19), we obtain

$$\begin{aligned} I_{21} &\ll \mathcal{Z}_2 \sup_{\alpha \in \mathfrak{m}_2} |f_4(\alpha)| \times \left( \int_0^1 |f_1(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \left( \int_0^1 |f_2(\alpha)|^4 \mathrm{d}\alpha \right)^{\frac{1}{4}} \left( \int_{\mathcal{I}} |V_3(\alpha)|^4 \mathrm{d}\alpha \right)^{\frac{1}{4}} \\ &\ll \mathcal{Z}_2 \cdot \frac{N^{\frac{1}{4}}}{\log^{40A} N} \cdot N^{\frac{1}{2}} \cdot (N \log^c N)^{\frac{1}{4}} \cdot (N^{\frac{1}{3}} \log^c N)^{\frac{1}{4}} \ll \frac{\mathcal{Z}_2 N^{\frac{13}{12}}}{\log^{30A} N}. \end{aligned} \tag{32}$$

Moreover, it follows from (27), (31) and Cauchy's inequality that

$$\begin{aligned} I_{22} &\ll N^{\frac{4}{15}+\varepsilon} \cdot \sup_{\alpha \in \mathfrak{m}_2} |f_4(\alpha)| \times \left( \int_0^1 |f_1(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \left( \int_0^1 |f_2(\alpha)\mathcal{K}_2(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \\ &\ll N^{\frac{4}{15}+\varepsilon} \cdot \frac{N^{\frac{1}{4}}}{\log^{40A} N} \cdot N^{\frac{1}{2}} \cdot \left( N^{\varepsilon} \big( \mathcal{Z}_2 N^{\frac{1}{2}} + \mathcal{Z}_2^2 \big) \right)^{\frac{1}{2}} \\ &\ll N^{\frac{61}{60}+\varepsilon} \big( \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{1}{4}} + \mathcal{Z}_2 \big) \ll \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{19}{15}+\varepsilon} + \mathcal{Z}_2 N^{\frac{61}{60}+\varepsilon}. \end{aligned} \tag{33}$$

Combining (26), (30), (32) and (33), we deduce that

$$\frac{\mathcal{Z}_2 N^{\frac{13}{12}}}{\log^5 N} \ll I_2 = I_{21} + I_{22} \ll \frac{\mathcal{Z}_2 N^{\frac{13}{12}}}{\log^{30A} N} + \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{19}{15}+\varepsilon} + \mathcal{Z}_2 N^{\frac{61}{60}+\varepsilon},$$

which implies

$$\mathcal{Z}_2 \ll N^{\frac{11}{30}+\varepsilon}. \tag{34}$$

From (29) and (34), we have

$$\mathcal{Z}(N) \ll \mathcal{Z}_1 + \mathcal{Z}_2 \ll N^{\frac{61}{144}+\varepsilon},$$

which completes the proof of Proposition 2.

**Author Contributions:** All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research work is supported by National Natural Science Foundation of China (Grant No. 11901566, 11971476), the Fundamental Research Funds for the Central Universities (Grant No. 2019QS02), and National Training Program of Innovation and Entrepreneurship for Undergraduates (Grant No. 201911413071, C201907622).

**Acknowledgments:** The authors would like to express the most sincere gratitude to the referee for his/her patience in refereeing this paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Vaughan, R.C. *The Hardy-Littlewood Method*; Cambridge University Press: Cambridge, UK, 1997.
2. Davenport, H. On Waring's problem for fourth powers. *Ann. Math.* **1939**, *40*, 731–747. [CrossRef]
3. Hardy, G.H.; Ramanujan, S. Asymptotic formulae in combinatory analysis. *Proc. Lond. Math. Soc.* **1918**, *17*, 75–115. [CrossRef]
4. Vinogradov, I.M. Representation of an odd number as a sum of three primes. *Dokl. Akad. Nauk SSSR* **1937**, *15*, 6–7.

5. Vinogradov, I.M. Some theorems concerning the theory of primes. *Mat. Sb.* **1937**, *44*, 179–195.
6. Hua, L.K. *Additive Primzahltheorie*; B. G. Teubner Verlagsgesellschaft: Leipzig, Germany, 1959.
7. Hua, L.K. *Additive Theory of Prime Numbers*; American Mathematical Society: Providence, RI, USA, 1965.
8. Jagy, W.C.; Kaplansky, I. Sums of squares, cubes, and higher powers. *Exp. Math.* **1995**, *4*, 169–173. [CrossRef]
9. Brüdern, J. On Waring's problem for cubes and biquadrates. *J. Lond. Math. Soc.* **1988**, *37*, 25–42. [CrossRef]
10. Brüdern, J.; Wooley, T.D. On Waring's problem: Three cubes and a sixth power. *Nagoya Math. J.* **2001**, *163*, 13–53. [CrossRef]
11. Davenport, H.; Heilbronn, H. Note on a result in the additive theory of numbers. *Proc. Lond. Math. Soc.* **1937**, *2*, 142–151. [CrossRef]
12. Davenport, H.; Heilbronn, H. On Waring's problem: Two cubes and one square. *Proc. Lond. Math. Soc.* **1937**, *2*, 73–104.
13. Roth, K.F. Proof that almost all positive integers are sums of a square, a positive cube and a fourth power. *J. Lond. Math. Soc.* **1949**, *24*, 4–13. [CrossRef]
14. Vaughan, R.C. A ternary additive problem. *Proc. Lond. Math. Soc.* **1980**, *41*, 516–532. [CrossRef]
15. Hooley, C. On a new approach to various problems of Waring's type. In *Recent Progress in Analytic Number Theory*; Academic Press: London, UK, 1981; Volume 1, pp. 127–191.
16. Davenport, H. On Waring's problem for cubes. *Acta Math.* **1939**, *71*, 123–143. [CrossRef]
17. Lu, M.G. On Waring's problem for cubes and fifth power. *Sci. China Ser. A* **1993**, *36*, 641–662.
18. Kawada, K.; Wooley, T.D. Sums of fourth powers and related topics. *J. Reine Angew. Math.* **1999**, *512*, 173–223. [CrossRef]
19. Vaughan, R.C. A new iterative method in Waring's problem. *Acta Math.* **1989**, *162*, 1–71. [CrossRef]
20. Brüdern, J.; Kawada, K. Ternary problems in additive prime number theory. In *Analytic Number Theory*; Jia, C., Matsumoto, K., Eds.; Kluwer: Dordrecht, The Netherlands, 2002; pp. 39–91.
21. Zhao, L. On unequal powers of primes and powers of 2. *Acta Math. Hungar.* **2015**, *146*, 405–420. [CrossRef]
22. Liu, Z.; Lü, G. On unlike powers of primes and powers of 2. *Acta Math. Hung.* **2011**, *132*, 125–139. [CrossRef]
23. Lü, X.D. On unequal powers of primes and powers of 2. *Ramanujan J.* **2019**, *50*, 111–121. [CrossRef]
24. Wooley, T.D. Slim exceptional sets and the asymptotic formula in Waring's problem. *Math. Proc. Camb. Philos. Soc.* **2003**, *134*, 193–206. [CrossRef]
25. Zhao, L. On the Waring-Goldbach problem for fourth and sixth powers. *Proc. Lond. Math. Soc.* **2014**, *108*, 1593–1622. [CrossRef]
26. Pan, C.D.; Pan, C.B. *Goldbach Conjecture*; Science Press: Beijing, China, 1981.
27. Ren, X.M. On exponential sums over primes and application in Waring–Goldbach problem. *Sci. China Ser. A* **2005**, *48*, 785–797. [CrossRef]
28. Vinogradov, I.M. *Elements of Number Theory*; Dover Publications: New York, NY, USA, 1954.