Some New $q$–Integral Inequalities Using Generalized Quantum Montgomery Identity via Preinvex Functions

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Abstract: In this work the authors establish a new generalized version of Montgomery’s identity in the setting of quantum calculus. From this result, some new estimates of Ostrowski type inequalities are given using preinvex functions. Given the generality of preinvex functions, particular $q$–integral inequalities are established with appropriate choice of the parametric bifunction. Some new special cases from the main results are obtained and some known results are recaptured as well. At the end, a briefly conclusion is given.

Keywords: Quantum Montgomery identity; $ϕ$-convex functions; integral inequalities

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1. Introduction

Quantum calculus, or $q$–calculus, has had an important development in recent decades, both in pure mathematics and its applicability, for example in Physics [1]. The convexity of a function has played an important role as a tool in the development of inequalities. Some fields of Mathematics have used this property: harmonic analysis, interpolation theory, and control theory, as can be seen in the works of C.P. Niculescu [2], C. Bennett and R. Sharpley [3], S. Mititelu and S. Trenţă [4], S. Trenţă [5,6].

Furthermore, it is important to note that in recent decades, the evolution of the concept of convexity has been extended and its evolution has been subject of many studies as is shown in the works of Ben-Israel A. and Mond B. [7], Hernández Hernández, J. E. [8,9], Niculescu C.P. [2], Mitrović D.S. et al. [10], Noor M. et.al. [11,12], Sarikaya M.Z. et. al. [13], Vivas-Cortez M.J. et al. [14], Weir, T.; Mond, B. [15] and others.

Recently, Tariboon et al. in [16], defined $q$-derivative and $q$-integral as follows:

**Definition 1.** Let $Y : [a_1, a_2] \to \mathbb{R}$ be a continuous function and let $x \in [a_1, a_2]$ and $0 < q < 1$ be a constant. Then the $q$-derivative on $[a_1, a_2]$ of function $Y$ at $x$ is defined as

$$a_1 D_q Y(x) = \frac{Y(x) - Y(qx + (1-q)a_1)}{(1-q)(x-a_1)}, \quad x \neq a_1,$$  \hspace{1cm} (1)
We say that $Y$ is q-differentiable on $[a_1, a_2]$ provided $a_1 D_q Y(x)$ exists for all $x \in [a_1, a_2]$.

**Definition 2.** Let $Y : [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on $[a_1, a_2]$ is defined as

$$\int_{a_1}^{x} Y(v) a_1 d_q v = (1 - q)(x - a_1) \sum_{n=0}^{\infty} q^n Y(q^n x + (1 - q^n)a_1),$$

(2)

for $x \in [a_1, a_2]$.

Some properties of interest regarding these definitions are the following.

**Theorem 1.** ([17]) Let $f : I \to \mathbb{R}$ be a q-differentiable functions. Then we have

1. The sum $(f + g)$ is q-differentiable on $I$ with

$$a D_q (f(t) + g(t)) = a D_q f(t) + a D_q g(t)$$

2. For any constant $\alpha \in \mathbb{R}$, the function $(\alpha f)$ is q-differentiable and

$$a D_q (\alpha f(t)) = \alpha_a D_q f(t)$$

3. The function $(fg)$ is q-differentiable with

$$a D_q (fg)(t) = f(t) a D_q g(t) + (qt + (1 - q)a) D_q f(t)$$

$$= g(t) a D_q f(t) + f(qt + (1 - q,a) D_q g(t)$$

**Lemma 1.** ([17]). Let $\alpha \in \mathbb{R}$, then we have

$$a D_q (x-a)^\alpha = \left(\frac{1 - q^\alpha}{1 - q}\right) (x-a)^{\alpha - 1}$$

**Theorem 2.** ([17]) Let $f : I \to \mathbb{R}$ be a continuous function. Then we have

1. $a D_q \int_{c}^{x} f(t) d_q t = f(x)$
2. $\int_{c}^{x} a D_q f(t) d_q t = f(x) - f(c)$ for $c \in (a, x)$

**Theorem 3.** ([17]) Let $f, g : I \to \mathbb{R}$ be a continuous functions and $\alpha \in \mathbb{R}$. Then, for $x \in I$ we have

1. $\int_{c}^{x} (f(t) + g(t)) d_q t = \int_{c}^{x} f(t) d_q t + \int_{c}^{x} g(t) d_q t$
2. $\int_{c}^{x} (\alpha f(t)) d_q t = \alpha \int_{c}^{x} f(t) d_q t$
3. $\int_{c}^{x} f(t) a D_q g(t) d_q t = f(t) g(t) \bigg|_{c}^{x} - \int_{c}^{x} g(qt + (1 - q)a) D_q f(t) d_q t$

For more details on q-calculus and certain q-analogues of classical inequalities, see [16, 18–29].

The following famous identity in [10], is called Montgomery identity:

$$f(x) = \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(v) dv + \frac{1}{a_2 - a_1} \int_{a_1}^{x} f'(v) (v - a_1) dv + \frac{1}{a_2 - a_1} \int_{x}^{a_2} f'(v) (v - a_2) dv,$$

(3)

where $f(x)$ is continuous function on $[a_1, a_2]$ with a continuous first derivative in $(a_1, a_2)$. By changing the variable, the Montgomery identity (3) could be expressed as follows:

$$f(x) = \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(v) dv = (a_2 - a_1) \int_{0}^{1} K(v) f'((1 - v)a_1 + va_2) dv,$$

(4)
where

\[ K(\nu) = \begin{cases} \nu, & \nu \in \left[0, \frac{x-a_1}{a_2-a_1}\right], \\ \nu - 1, & \nu \in \left(\frac{x-a_1}{a_2-a_1}, 1\right]. \end{cases} \]

This identity has been used in various works to establish bounds for quadrature rules via specialized algorithms [30].

We recall now some basic definitions for our study as follows:

Let \( E \subset \mathbb{R}^n \) be a non-empty set, \( \Upsilon : E \rightarrow \mathbb{R} \) be a continuous function and \( \theta : E \times E \rightarrow \mathbb{R}^n \) be a continuous bifunction.

**Definition 3.** [7] A set \( E \subset \mathbb{R}^n \) is said to be invex with respect to bifunction \( \theta \), if

\[ a_1 + \nu \theta(a_2, a_1) \in E, \quad \forall a_1, a_2 \in E, \nu \in [0, 1]. \]

**Definition 4.** [15] A function \( \Upsilon : E \rightarrow \mathbb{R} \) is said to be preinvex with respect to bifunction \( \theta \), if

\[ \Upsilon(a_1 + \nu \theta(a_2, a_1)) \leq (1 - \nu) \Upsilon(a_1) + \nu \Upsilon(a_2), \quad \forall a_1, a_2 \in E, \nu \in [0, 1]. \]

Some properties of this class of functions can be found in [31].

Motivated by the above literatures, the main objective of this article is to obtain a generalization of the Montgomery identity given in (4) using the concepts of q-calculus. From this identity, several new and known q-analogues of integral inequalities involving preinvex functions will be obtained. We also discuss some new special cases of the main results. At the end, a briefly conclusion is provided as well.

2. Main Results

In this section, before we derive our main results, for brevity we define the following notations:

\[ P = [a_1, a_1 + \theta(a_2, a_1)], \quad \wp_\theta(x) = \frac{x - a_1}{\theta(a_2, a_1)}, \quad \text{where } \theta(a_2, a_1) > 0. \]

**Lemma 2.** (Generalized quantum Montgomery identity) If \( \Upsilon : P \rightarrow \mathbb{R} \) is a q-differentiable function such that \( \mathcal{e}D_q \Upsilon \) is quantum integrable on \( P^o \) (the interior of \( P \)), then the following identity holds:

\[ \Upsilon(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} \Upsilon(v) a_1 d_q v = \theta(a_2, a_1) \int_0^1 T_q(v) a_1 D_q \Upsilon(a_1 + v \theta(a_2, a_1)) d_q v, \]

where

\[ T_q(\nu) = \begin{cases} \nu, & \nu \in [0, \wp_\theta(x)]; \\ \nu - 1, & \nu \in (\wp_\theta(x), 1]. \end{cases} \]
Proof. By using Definitions 1 and 2, we have

$$\theta(a_2, a_1) \int_0^1 T_q(\nu) a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q$$

$$= \theta(a_2, a_1) \left[ \int_0^{\varphi(x)} q \nu a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q + \int_0^{\varphi(x)} (q \nu - 1) a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q \right]$$

$$= \theta(a_2, a_1) \left[ \int_0^{\varphi(x)} q \nu a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q + \int_0^{\varphi(x)} (q \nu - 1) a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q \right]$$

$$- \int_0^{\varphi(x)} (q \nu - 1) a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q$$

$$= \theta(a_2, a_1) \left[ \int_0^{\varphi(x)} (q \nu - 1) a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q + \int_0^{\varphi(x)} a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q \right]$$

$$= \theta(a_2, a_1) \left[ \int_0^{\varphi(x)} q \nu a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q + \int_0^{\varphi(x)} a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q \right]$$

$$= \theta(a_2, a_1) \left[ \int_0^{\varphi(x)} a_1 D_q Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q \right]$$

$$= \frac{1}{1 - q} \left[ q \int_0^{\varphi(x)} Y(a_1 + \nu \theta(a_2, a_1)) \, d_\nu q - \int_0^{\varphi(x)} Y(a_1 + q \nu \theta(a_2, a_1)) \, d_\nu q \right]$$

$$- \left[ \int_0^{\varphi(x)} \frac{Y(a_1 + \nu \theta(a_2, a_1))}{\nu} \, d_\nu q - \int_0^{\varphi(x)} \frac{Y(a_1 + q \nu \theta(a_2, a_1))}{\nu} \, d_\nu q \right]$$

$$+ \left[ \int_0^{\varphi(x)} \frac{Y(a_1 + \nu \theta(a_2, a_1))}{\nu} \, d_\nu q - \int_0^{\varphi(x)} \frac{Y(a_1 + q \nu \theta(a_2, a_1))}{\nu} \, d_\nu q \right]$$
\[= \frac{1}{1-q} \left[ q(1-q) \left( \sum_{n=0}^{\infty} q^n Y(a_1 + q^n \theta(a_2, a_1)) - \sum_{n=0}^{\infty} q^n Y(a_1 + q^{n+1} \theta(a_2, a_1)) \right) \right. \\
- (1-q) \left( \sum_{n=0}^{\infty} q^n \frac{Y(a_1 + q^n \theta(a_2, a_1))}{q^n} - \sum_{n=0}^{\infty} q^n \frac{Y(a_1 + q^{n+1} \theta(a_2, a_1))}{q^n} \right) \\
\left. + (1-q) \psi_\theta(x) \left( \sum_{n=0}^{\infty} q^n \frac{Y(a_1 + q^n \psi_\theta(x) \theta(a_2, a_1))}{q^n \psi_\theta(x)} - \sum_{n=0}^{\infty} q^n \frac{Y(a_1 + q^{n+1} \psi_\theta(x) \theta(a_2, a_1))}{q^n \psi_\theta(x)} \right) \right] \\
= q \left( \sum_{n=0}^{\infty} q^n Y(a_1 + q^n \theta(a_2, a_1)) - \sum_{n=0}^{\infty} q^n Y(a_1 + q^{n+1} \theta(a_2, a_1)) \right) \\
- \left( \sum_{n=0}^{\infty} Y(a_1 + q^n \theta(a_2, a_1)) - \sum_{n=0}^{\infty} Y(a_1 + q^{n+1} \theta(a_2, a_1)) \right) \\
+ \left( \sum_{n=0}^{\infty} Y(a_1 + q^n \psi_\theta(x) \theta(a_2, a_1)) - \sum_{n=0}^{\infty} Y(a_1 + q^{n+1} \psi_\theta(x) \theta(a_2, a_1)) \right) \\
= q \left[ (1-q) \sum_{n=0}^{\infty} q^n Y(a_1 + q^n \theta(a_2, a_1)) + \frac{Y(a_1 + \theta(a_2, a_1))}{q} \right] \\
- Y(a_1 + \theta(a_2, a_1)) + Y(a_1 + \psi_\theta(x) \theta(a_2, a_1)) \\
= Y(x) - (1-q) \sum_{n=0}^{\infty} q^n Y(a_1 + q^n \theta(a_2, a_1)) \\
= Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1+\theta(a_2, a_1)} Y(v) \, dv \, d\theta_{a_2, a_1} \, v \, d\psi_{a_2, a_1}.
\]

The proof is complete. \(\square\)

**Remark 1.** Taking \(q \to 1^-\) in Lemma 2, we have

\[Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1+\theta(a_2, a_1)} Y(v) \, dv \]

\[= \theta(a_2, a_1) \left[ \int_{0}^{\psi_\theta(x)} v Y'(a_1 + v \theta(a_2, a_1)) \, dv + \int_{\psi_\theta(x)}^{1} (v - 1) Y'(a_1 + v \theta(a_2, a_1)) \, dv \right].\]

**Remark 2.** Taking \(\theta(a_2, a_1) = a_2 - a_1\) in Lemma 2, we get ([20], Lemma 3).

**Remark 3.** Taking \(q \to 1^-\) and \(\theta(a_2, a_1) = a_2 - a_1\) in Lemma 2, we obtain Montgomery identity given in (4).
Remark 4. Taking \( x = \frac{3a_1 + \theta(a_2, a_1)}{2} \) in Remark 1, we get ([11], Lemma 3.10).

\[
Y \left( \frac{2a_1 + \theta(a_2, a_1)}{2} \right) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v) \, dv \\
= \theta(a_2, a_1) \left[ \int_{0}^{1} v Y'(a_1 + v \theta(a_2, a_1)) \, dv + \int_{1}^{2} (v - 1) Y'(a_1 + v \theta(a_2, a_1)) \, dv \right].
\]

Remark 5. Taking \( x = \frac{a_1 + a_2}{1 + q} \) and \( \theta(a_2, a_1) = a_2 - a_1 \) in Lemma 2, we obtain equality (4.1) of [18].

\[
Y \left( \frac{a_1 + a_2}{1 + q} \right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} Y(v) \, dv \\
= (a_2 - a_1) \left[ \int_{0}^{\frac{1}{1 + q}} qv \, D_q Y((1 - v)a_1 + v a_2) \, dq \\
+ \int_{\frac{1}{1 + q}}^{q} (qv - 1) a_1 D_q Y((1 - v)a_1 + v a_2) \, dq \right].
\]

Remark 6. Taking \( x = \frac{a_1 + q(a_1 + \theta(a_2, a_1))}{1 + q} \) in Lemma 2, we have

\[
Y \left( \frac{a_1 + q(a_1 + \theta(a_2, a_1))}{1 + q} \right) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v) \, dv \\
= \theta(a_2, a_1) \left[ \int_{0}^{q} qv \, a_1 D_q Y(a_1 + v \theta(a_2, a_1)) \, dq \\
+ \int_{q}^{1} (qv - 1) a_1 D_q Y(a_1 + v \theta(a_2, a_1)) \, dq \right].
\]

Now using Lemma 2, we are in position to derive our main results for the class of preinvex functions.

Theorem 4. Let \( Y : P \to \mathbb{R} \) be a function such that \( a_1 D_q Y \) is q-integrable on \( P^o \) (the interior of \( P \)). If \( |a_1 D_q Y|' \) is preinvex function on \( P \), then for \( r > 1 \) and \( p^{-1} + r^{-1} = 1 \), the following inequality holds:

\[
\left| Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v) \, dv \right| \\
\leq q^{\theta(a_2, a_1)} \left[ \left( L_1(q, a_1, a_2, x) \right)^{\frac{1}{p}} \left( |a_1 D_q Y(a_1)|' L_2(q, a_1, a_2, x) + |a_1 D_q Y(a_2)|' L_3(q, a_1, a_2, x) \right)^{\frac{1}{2}} \\
+ \left( L_4(q, a_1, a_2, x) \right)^{\frac{1}{2}} \left( |a_1 D_q Y(a_1)|' L_5(q, a_1, a_2, x) + |a_1 D_q Y(a_2)|' L_6(q, a_1, a_2, x) \right)^{\frac{1}{2}} \right],
\]

Remark 1, we get ([11], Lemma 3.10).
where

\[ L_1(q, a_1, a_2, x) = |\varphi(x)|^{p+1} \frac{1-q}{1-q^{p+1}}, \]
\[ L_2(q, a_1, a_2, x) = \varphi(x) - \frac{1}{1+q} [\varphi(x)]^2, \]
\[ L_3(q, a_1, a_2, x) = \frac{1}{1+q} [\varphi(x)]^2, \]
\[ L_4(q, a_1, a_2, x) = (1-q) \left[ \sum_{n=0}^{\infty} q^n \left( q^n - \frac{1}{q} \right)^p \right] - \varphi(x) \sum_{n=0}^{\infty} q^n \left( q^n \varphi(x) - \frac{1}{q} \right)^p, \]
\[ L_5(q, a_1, a_2, x) = \frac{q}{1+q} - \psi \vartheta(x) + \frac{1}{1+q} [\varphi(x)]^2, \]
\[ L_6(q, a_1, a_2, x) = \frac{1}{1+q} \left( 1 - [\varphi(x)]^2 \right). \]

**Proof.** Using Lemma 2, preinvexity of \(|a_1, D_q Y|^{\nu}\) and Hölder’s inequality, we get

\[
\begin{aligned}
&\left| Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{x} Y(\nu) a_1 d_\nu \right| \\
\leq &\theta(a_2, a_1) \left[ \int_0^{\varphi(x)} (q \nu - 1)|a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))| \theta d_\nu \right] \\
&+ \left( \int_0^{\varphi(x)} (q \nu - 1)|a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))| \theta d_\nu \right)^{\frac{1}{p}} \left( \int_0^{\varphi(x)} \int_0^{\varphi(x)} [a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))]^{\prime} \theta d_\nu \right)^{\frac{1}{q}} \\
\leq &\theta(a_2, a_1) \left[ \int_0^{\varphi(x)} (q \nu - 1)|a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))| \theta d_\nu \right] \\
&+ \left( \int_0^{\varphi(x)} (q \nu - 1)|a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))| \theta d_\nu \right)^{\frac{1}{p}} \left( \int_0^{\varphi(x)} \int_0^{\varphi(x)} [a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))]^{\prime} \theta d_\nu \right)^{\frac{1}{q}} \left( \int_0^{\varphi(x)} \int_0^{\varphi(x)} [a_1, D_q Y(a_1 + \nu \theta(a_2, a_1))]^{\prime} \theta d_\nu \right)^{\frac{1}{q}} \\
\leq &\theta(a_2, a_1) \left[ \int_0^{\varphi(x)} (q \nu - 1)|a_1, D_q Y(a_1)| \theta d_\nu \right] \\
&+ \left( \int_0^{\varphi(x)} (q \nu - 1)|a_1, D_q Y(a_1)| \theta d_\nu \right)^{\frac{1}{p}} \left( \int_0^{\varphi(x)} \int_0^{\varphi(x)} [a_1, D_q Y(a_1)]^{\prime} \theta d_\nu \right)^{\frac{1}{q}} \left( \int_0^{\varphi(x)} \int_0^{\varphi(x)} [a_1, D_q Y(a_1)]^{\prime} \theta d_\nu \right)^{\frac{1}{q}} \left( \int_0^{\varphi(x)} \int_0^{\varphi(x)} [a_1, D_q Y(a_1)]^{\prime} \theta d_\nu \right)^{\frac{1}{q}}. \\
\end{aligned}
\]
Letting

\[ L_1(q, a_1, a_2, x) = \int_0^{\varphi_0(x)} \nu d_q = |\varphi_0(x)|^{p+1} \frac{1-a}{1-q^{p+1}}, \]

\[ L_2(q, a_1, a_2, x) = \int_0^{\varphi_0(x)} (1-\nu) d_q = \varphi_0(x) - \frac{1}{1+q} |\varphi_0(x)|^2, \]

\[ L_3(q, a_1, a_2, x) = \int_0^{\varphi_0(x)} \nu d_q = \frac{1}{1+q} |\varphi_0(x)|^2, \]

\[ L_4(q, a_1, a_2, x) = \int_0^{\varphi_0(x)} (1-q) \nu d_q = (1-q) \left[ \sum_{n=0}^{\infty} q^n \left( q^n - \frac{1}{q} \right) - \varphi_0(x) \sum_{n=0}^{\infty} q^n \left( q^n \varphi_0(x) - \frac{1}{q} \right) \right], \]

\[ L_5(q, a_1, a_2, x) = \int_0^{\varphi_0(x)} (1-q) \nu d_q = \frac{q}{1+q} - \varphi_0(x) + \frac{1}{1+q} |\varphi_0(x)|^2, \]

\[ L_6(q, a_1, a_2, x) = \int_0^{\varphi_0(x)} \nu d_q = \frac{1}{1+q} \left( 1 - |\varphi_0(x)|^2 \right), \]

we have the desired result. The proof is complete. \( \square \)

We point out some special cases of Theorem 4.

**Corollary 1.** Taking \( q \to 1^- \) in Theorem 4, we have

\[
|Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1+\theta(a_2, a_1)} Y(v) dv| \\
\leq \theta(a_2, a_1) \left[ |L_7(a_1, a_2, x)|^{\frac{1}{2}} \left[ |Y'(a_1)|^{\frac{1}{2}} L_8(a_1, a_2, x) + |Y'(a_2)|^{\frac{1}{2}} L_9(a_1, a_2, x) \right] \right]^{\frac{1}{2}} \\
+ \left[ L_{10}(a_1, a_2, x) \right]^{\frac{1}{2}} \left[ |Y'(a_1)|^{\frac{1}{2}} L_{11}(a_1, a_2, x) + |Y'(a_2)|^{\frac{1}{2}} L_{12}(a_1, a_2, x) \right]^{\frac{1}{2}},
\]
where

\[ L_7(a_1, a_2, x) = \int_0^{\psi(x)} \frac{1}{p+1} (|\psi(x)|)^{p+1}, \]

\[ L_8(a_1, a_2, x) = \int_0^{\psi(x)} (1 - \nu)^d \nu = \psi(x) - \frac{1}{2} (\psi(x))^2, \]

\[ L_9(a_1, a_2, x) = \int_0^{\psi(x)} \nu d\nu = \frac{1}{2} (\psi(x))^2, \]

\[ L_{10}(a_1, a_2, x) = \int_0^{\psi(x)} (1 - \nu)^d \nu = \frac{1}{p+1} \left( \frac{a_1 + \theta(a_2, a_1) - x}{\theta(a_2, a_1)} \right)^{p+1}, \]

\[ L_{11}(a_1, a_2, x) = \int_0^{\psi(x)} (1 - \nu)^d \nu = \frac{a_1 + \theta(a_2, a_1) - x}{\theta(a_2, a_1)} - \frac{1}{2} (1 - (\psi(x))^2), \]

\[ L_{12}(a_1, a_2, x) = \int_0^{\psi(x)} \nu d\nu = \frac{1}{2} (1 - (\psi(x))^2). \]

II. Taking \( q \to 1^- \) and \( x = \frac{2a_1 + \theta(a_2, a_1)}{2} \) in Theorem 4, we get ([13], Theorem 6).

\[ \left| Y \left( \frac{2a_1 + \theta(a_2, a_1)}{2} \right) - \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(\nu) d\nu \right| \leq \frac{\theta(a_2, a_1)}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{2}} \left[ (3|Y'(a_1)|^r + |Y'(a_2)|^r)^{\frac{1}{r}} + (|Y'(a_1)|^r + 3|Y'(a_2)|^r)^{\frac{1}{r}} \right]. \]

III. Taking \( x = \frac{qa_1 + a_2}{1+q} \) and \( \theta(a_2, a_1) = a_2 - a_1 \) in Theorem 4, we obtain ([18], Theorem 18).

\[ \left| Y \left( \frac{qa_1 + a_2}{1+q} \right) - \int_{a_1}^{a_2} Y(\nu) d\nu \right| \leq q(a_2 - a_1) \left[ \frac{1}{(1+q)^{p+1}} \left( \frac{1-q}{1-q} \right) \left[ \frac{(q^2 + 2q)|a_1 D_q Y(a_1)|^r + q^2 |a_1 D_q Y(a_2)|^r}{(1+q)^3} \right]^{\frac{1}{r}} \right] + \left( \int_{1+q}^{1} \left( 1 - \frac{1}{q} \right)^p a_1 d\nu \right)^{\frac{1}{r}} \left[ \frac{(q^3 + q^2 - q)|a_1 D_q Y(a_1)|^r + (q^2 + 2q)|a_1 D_q Y(a_2)|^r}{(1+q)^3} \right]^{\frac{1}{r}}, \]

where

\[ \int_{1+q}^{1} \left( 1 - \frac{1}{q} \right)^p a_1 d\nu = (1-q) \left[ \sum_{n=0}^{\infty} q^n \left( q^n - \frac{1}{q} \right)^p - \frac{1}{1+q} \sum_{n=0}^{\infty} q^n \left( q^n \left( \frac{1}{1+q} \right) - \frac{1}{q} \right)^p \right]. \]
IV. Taking \( x = \frac{a_1 + q(a_1 + \theta(a_2, a_1))}{1 + q^2} \) in Theorem 4, we get

\[
Y \left( \frac{a_1 + q(a_1 + \theta(a_2, a_1))}{1 + q} \right) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v) a_1 d_q v
\]

\[
\leq q\theta(a_2, a_1) \left[ \left( \frac{q^p}{(1 + q)^{p+1}} \right) \left( \frac{q^3 + q^2 + q}{(1 + q)^3} \right) + \frac{q^2 a_1 D_q Y(a_1)}{(1 + q)^3} \right]^{1/2}
+ \left( \int_{\frac{1}{q+1}}^{1} (\nu - \frac{1}{q}) \frac{p}{a_1} d_q v \right) \left[ \frac{q^2 a_1 D_q Y(a_1)}{(1 + q)^3} + \frac{q^2 a_1 D_q Y(a_2)}{(1 + q)^3} \right]^{1/2}
\]

where

\[
\int_{\frac{1}{q+1}}^{1} (\nu - \frac{1}{q}) \frac{p}{a_1} d_q v = (1 - q) \left[ \sum_{n=0}^{\infty} q^n \left( \frac{q^n}{1 + q^2} \right) - \frac{q}{1 + q} \sum_{n=0}^{\infty} q^n \left( \frac{q^n}{1 + q^2} - \frac{1}{q} \right) \right].
\]

**Theorem 5.** Let \( Y : P \rightarrow \mathbb{R} \) be a function such that \( a_1 D_q Y \) is \( q \)-integrable on \( P^0 \) (the interior of \( P \)). If \( |a_1 D_q Y|^r \) is preinvert function on \( P \), then for \( r \geq 1 \), the following inequality holds:

\[
\left| Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v) a_1 d_q v \right| \leq \theta(a_2, a_1) \times
\]

\[
\left[ |f_1(q, a_1, a_2, x)|^{1/2} \right. + \left. \left[ |a_1 D_q Y(a_1)|^r f_2(q, a_1, a_2, x) + |a_1 D_q Y(a_2)|^r f_3(q, a_1, a_2, x) \right]^{1/2}
+ \left[ f_4(q, a_1, a_2, x) \right]^{1/2} + \left[ f_5(q, a_1, a_2, x) \right]^{1/2} + \left[ a_1 D_q Y(a_2) \right]^{1/2} f_6(q, a_1, a_2, x) \right]^{1/2},
\]

where

\[
f_1(q, a_1, a_2, x) = \int_{\frac{\psi_0(x)}{q+1}}^{\psi_0(x)} q^2 \nu d_q v = \frac{q}{1 + q} [\psi_0(x)]^2,
\]

\[
f_2(q, a_1, a_2, x) = \int_{\frac{\psi_0(x)}{q+1}}^{\psi_0(x)} (q^2 - q^2) d_q v = f_1(q, a_1, a_2, x) - f_3(q, a_1, a_2, x),
\]

\[
f_3(q, a_1, a_2, x) = \int_{\frac{\psi_0(x)}{q+1}}^{\psi_0(x)} q^2 \nu d_q v = \frac{q}{1 + q + q^2} [\psi_0(x)]^3,
\]

\[
f_4(q, a_1, a_2, x) = \int_{\frac{\psi_0(x)}{q+1}}^{\psi_0(x)} (q^2 - 1) d_q v = \frac{q}{1 + q + q^2} \left( a_1 + \theta(a_2, a_1) - x \right)^2,
\]

\[
f_5(q, a_1, a_2, x) = \int_{\frac{\psi_0(x)}{q+1}}^{\psi_0(x)} (1 - q^2 - v + q^2) d_q v = f_4(q, a_1, a_2, x) - f_6(q, a_1, a_2, x),
\]

\[
f_6(q, a_1, a_2, x) = \int_{\frac{\psi_0(x)}{q+1}}^{\psi_0(x)} (q^2 - v) d_q v = \frac{1}{(1 + q)(1 + q + q^2)} - \frac{1}{1 + q} [\psi_0(x)]^2 \frac{1}{1 + q + q^2} [\psi_0(x)]^3.
\]
We point out some special cases of Theorem 5.

**Proof.** Using Lemma 2, preinvexity of $|a_1D_qY'|$ and the well-known power mean inequality, we have

$$
\left| Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1} a_1 + \theta(a_2, a_1) Y(v) a_1 d_qv \right|
\leq \theta(a_2, a_1) \left[ \int_{\phi(x)} qv |a_1D_qY(a_1 + v\theta(a_2, a_1))| a_1 d_qv + \int_{\phi(x)} (qv - 1) |a_1D_qY(a_1 + v\theta(a_2, a_1))| a_1 d_qv \right]
\leq \theta(a_2, a_1) \left[ \left( \int_{\phi(x)} qv a_1 d_qv \right)^{1-\alpha} \left( \int_{\phi(x)} qv |a_1D_qY(a_1 + v\theta(a_2, a_1))| a_1 d_qv \right)^\alpha \right]
\leq \theta(a_2, a_1) \left[ \left( \int_{\phi(x)} (qv - 1) a_1 d_qv \right)^{1-\alpha} \left( \int_{\phi(x)} (1 - qv + qv^2) a_1 d_qv + \int_{\phi(x)} qv |a_1D_qY(a_1 + v\theta(a_2, a_1))| a_1 d_qv \right)^\alpha \right].

The proof of Theorem 5 is completed. □

We point out some special cases of Theorem 5.

**Corollary 2. I.** Taking $r = 1$ in Theorem 5, we have

$$
\left| Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1} Y(v) a_1 d_qv \right| \leq \theta(a_2, a_1) \left[ |a_1D_qY(a_1)| [J_2(q, a_1, a_2, x) + J_5(q, a_1, a_2, x)] + |a_1D_qY(a_2)| [J_3(q, a_1, a_2, x) + J_6(q, a_1, a_2, x)] \right].
$$

**II.** Taking $r = 1$ and $q \to 1^-$ in Theorem 5, we get

$$
\left| Y(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1} Y(v) dv \right| \leq \theta(a_2, a_1) \left[ |Y'(a_1)| [J_7(a_1, a_2, x) + J_9(a_1, a_2, x)] + |Y'(a_2)| [J_8(a_1, a_2, x) + J_{10}(a_1, a_2, x)] \right],
$$
where

\[ J_{a_1}(a_2, x) = \int_0^{\phi(x)} v(1 - v)dv = \frac{1}{2}[\phi(x)]^2 - \frac{1}{3}[\phi(x)]^3, \]

\[ J_{b}(a_1, a_2, x) = \int_0^{\phi(x)} v^2dv = \frac{1}{3}[\phi(x)]^3, \]

\[ J_{a}(a_1, a_2, x) = \int_{\phi(x)} (1 - 2v + v^2)dv = \frac{1}{3} - \phi(x) + [\phi(x)]^2 - \frac{1}{3}[\phi(x)]^3, \]

\[ J_{10}(a_1, a_2, x) = \int_0^{\phi(x)} (v - v^2)dv = \frac{1}{6} - \frac{1}{2}[\phi(x)]^2 + \frac{1}{3}[\phi(x)]^3. \]

**III.** Taking \( q \to 1^- \) and \( x = \frac{2a_1 + \theta(a_2, a_1)}{2} \) in Theorem 5, we obtain ([13], Theorem 8).

\[
\left| Y \left( \frac{2a_1 + \theta(a_2, a_1)}{2} \right) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v)dv \right| \leq \frac{\theta(a_2, a_1)}{8} \left[ \left( \frac{2|Y'(a_1)| + |Y'(a_2)|}{3} \right)^\frac{1}{2} + \left( \frac{|Y'(a_1)|^2 + 2|Y'(a_2)|^2}{3} \right)^\frac{1}{2} \right].
\]

**IV.** Taking \( r = 1, q \to 1^- \) and \( x = \frac{2a_1 + \theta(a_2, a_1)}{2} \) in Theorem 5, we get ([13], Theorem 5).

\[
\left| Y \left( \frac{2a_1 + \theta(a_2, a_1)}{2} \right) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(v)dv \right| \leq \frac{\theta(a_2, a_1)}{8} \left[ |Y'(a_1)| + |Y'(a_2)| \right].
\]

**IV.** Taking \( x = \frac{a_0 + a_2}{1 + q} \) and \( \theta(a_2, a_1) = a_2 - a_1 \) in Theorem 5, we have the following inequalities, for more details, see [20].

\[
\left| Y \left( \frac{a_0 + a_2}{1 + q} \right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} Y(v)dv \right| \leq (a_2 - a_1) \left[ \frac{1}{(1 + q)^{\frac{3}{2}}} \left[ |a_1 D_q Y(a_1)|^r \frac{q^2(1 + q)}{(1 + q + q^2)(1 + q)^3} + |a_1 D_q Y(a_2)|^r \frac{q}{(1 + q + q^2)(1 + q)^3} \right]^\frac{1}{2} \right.
\]

\[
+ \left( \frac{q}{1 + q} \right)^{\frac{3}{2}} \left[ |a_1 D_q Y(a_1)|^r \frac{q(1 + q^4 + q^3 - 2q)}{(1 + q + q^2)(1 + q)^3} + |a_1 D_q Y(a_2)|^r \frac{2q}{(1 + q + q^2)(1 + q)^3} \right]^\frac{1}{2} \right].
\]

**V.** Taking \( r = 1, x = \frac{a_0 + a_2}{1 + q} \) and \( \theta(a_2, a_1) = a_2 - a_1 \) in Theorem 5, we obtain ([18], Theorem 13).

\[
\left| Y \left( \frac{a_0 + a_2}{1 + q} \right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} Y(v)dv \right| \leq (a_2 - a_1) \left[ |a_1 D_q Y(a_1)|^r \frac{q^3 + q^4 + q^3 - 2q}{(1 + q + q^2)(1 + q)^3} + |a_1 D_q Y(a_2)|^r \frac{3q}{(1 + q + q^2)(1 + q)^3} \right].
\]
VI. Taking $x = \frac{a_1 + q(a_1 + \theta(a_2, a_1))}{1 + q}$ in Theorem 5, we get

$$
\left| \frac{a_1 + q(a_1 + \theta(a_2, a_1))}{1 + q} - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1 + \theta(a_2, a_1)} Y(\nu) \, d\nu \right| \leq \theta(a_2, a_1) \times \\
\left[ \frac{q}{(1 + q)^3} \right]^{3 - \frac{1}{p}} \left| a_1 D_q Y(a_1) \right|^{r} \frac{q^3(1 + q^2)}{(1 + q + q^2)(1 + q)^3} + \left| a_1 D_q Y(a_2) \right|^{r} \frac{q^4}{(1 + q + q^2)(1 + q)^3} \right]^\frac{1}{r}.
$$

3. Applications to Special Means

Recalling the following means for arbitrary real numbers $a$ and $b$ with $a \neq b$:

$$
A(a_1, a_2) = \frac{a_1 + a_2}{2} \quad \text{Arithmetic mean}
$$

$$
L_p(a_1, a_2) = \left( \frac{a_2^{p+1} - a_1^{p+1}}{(p + 1)(a_2 - a_1)} \right)^{1/p}, \quad p \in \mathbb{R} \setminus \{ -1, 0 \} \quad \text{p- Logarithmic mean}
$$

it is possible to relate them through the previous results.

**Proposition 1.** Let $p \in \mathbb{R} \setminus \{ -1, 0 \}$ and $a_1, a_2$ real numbers such that $a_2 \geq a_2$, then

$$
\left| A^p(a_1, a_2) - \frac{1}{a_2 - a_1} L_p(a_1, a_2) \right| \leq \frac{|p|}{8} (a_2 - a_1) \left( |a_1^{p-1}| + |a_2^{p-1}| \right)
$$

**Proof.** Let $f(x) = x^p$ for some arbitrary $p \in \mathbb{R} \setminus \{ -1, 0 \}$, $x = (qa_1 + a_2)/(1 + q)$ and $r = 1$. Then

$$
f \left( \frac{qa_1 + a_2}{1 + q} \right) = \left( \frac{qa_1 + a_2}{1 + q} \right)^p,
$$

$$
\int_{a_1}^{a_2} t^p \, a_1 d_q t = \left[ \frac{1 - q}{1 - q^{p+1}} \right] \left( \frac{a_2^{p+1} - a_1^{p+1}}{a_2 - a_1} \right)
$$

and

$$
a_1 D_q Y(a_1) = \left[ \frac{1 - q^p}{1 - q} \right] a_1^{p-1} \quad \text{and} \quad a_1 D_q Y(a_2) = \left[ \frac{1 - q^p}{1 - q} \right] a_2^{p-1}
$$

So, using the Corollary 2 part IV, we have

$$
\left| \frac{qa_1 + a_2}{1 + q} \right|^{p} - \frac{1}{a_2 - a_1} \left[ \frac{1 - q}{1 - q^{p+1}} \right] \left( \frac{a_2^{p+1} - a_1^{p+1}}{a_2 - a_1} \right)
\leq (a_2 - a_1) \left[ \frac{1}{(1 + q)^{p+1}} \right] \left( \frac{1 - q^p}{1 - q} \right) \left| a_1^{p-1} \right| \frac{q^2(1 + q)}{(1 + q + q^2)(1 + q)^3} + \left[ \frac{1 - q^p}{1 - q} \right] a_1^{p-1} \left| \frac{q}{1 + q + q^2}(1 + q)^3 \right] \right]^\frac{1}{r}.
$$

$$
+ \left( \frac{q}{1 + q} \right)^{3 - \frac{1}{p}} \left[ \left| \frac{1 - q^p}{1 - q} \right| a_1^{p-1} \left| q^2 + q^4 + q^3 - 2q \right| (1 + q + q^2)(1 + q)^3 \right] + \left[ \frac{1 - q^p}{1 - q} \right] a_2^{p-1} \left| \frac{2q}{1 + q + q^2}(1 + q)^3 \right] \right]^\frac{1}{r}.
$$
Taking limit when \( q \to 1 \)

\[
\left| \left( \frac{a_1 + a_2}{2} \right)^p - \frac{1}{a_2 - a_1} \left( \frac{a_{p+1}^{p+1} - a_1^{p+1}}{(p+1)(a_2 - a_1)} \right) \right| \\
\leq (a_2 - a_1) \left| \frac{pa_1^{p-1}}{12} + \frac{pa_2^{p-1}}{12} + \frac{pa_1^{p-1}}{24} + \frac{pa_2^{p-1}}{24} + \frac{pa_1^{p-1}}{12} \right| \\
\leq \frac{|p|}{8} (a_2 - a_1) \left( |a_1^{p-1}| + |a_2^{p-1}| \right).
\]

The proof is complete. \( \square \)

4. Conclusions

It is expected that from the results obtained, and following the methodology applied, additional special functions may also be evaluated. Future works can be developed in the area of numerical analysis and even contributions using quantum algorithms, using the theorems and corollaries presented. Finally, our results can be applied to derive some inequalities using special means. The authors hope that the ideas and techniques of this paper will inspire interested readers working in this fascinating field.

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References


