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Fixed Point Problems on Generalized Metric Spaces in Perov's Sense

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Received: 9 March 2020; Accepted: 6 May 2020; Published: 22 May 2020

Abstract: The aim of this paper is to give some fixed point results in generalized metric spaces in Perov's sense. The generalized metric considered here is the w-distance with a symmetry condition. The operators satisfy a contractive weakly condition of Hardy–Rogers type. The second part of the paper is devoted to the study of the data dependence, the well-posedness, and the Ulam–Hyers stability of the fixed point problem. An example is also given to sustain the presented results.

Keywords: fixed point; coupled fixed points; Perov space; generalized w-distance; Ulam–Hyers stability; well-posedness; data dependence

1. Introduction and Preliminaries

The well-known Banach contraction principle was extended by Perov in 1964 to the case of spaces endowed with vector-valued metrics. In [1], Perov introduced the concept of vector-valued metric as follows.

Let X be a nonempty set. A mapping $\widetilde{d} : X \times X \to \mathbb{R}^m$ where $\widetilde{d} = \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_m(x,y) \end{pmatrix}$ for every $m \in \mathbb{N}$ is called vector-valued metric on X if the following properties are satisfied.

(1) $\widetilde{d}(x,y) \geq 0$ for all $x, y \in X$, and $\widetilde{d}(x,y) = 0$ implies $x = y$;
(2) $\widetilde{d}(x,y) = \widetilde{d}(y,x)$;
(3) $\widetilde{d}(x,y) \leq \widetilde{d}(x,z) + \widetilde{d}(z,y)$ for all $x, y, z \in X$.

In this case, the pair $(X, \widetilde{d})$ is called a generalized metric space in Perov's sense. Some examples of fixed points on the sense of vector-valued metric are given in [2–6]. Throughout this paper $\mathcal{M}_{m,m}(\mathbb{R}_+)$ will denote the set of all $m \times m$ matrices with positive elements. We also denote by Θ the zero $m \times m$ matrix and $0_{1\times m} = \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}$, by I the identity $m \times m$ matrix and $I_{1\times m} = \begin{pmatrix} 1 \\ \cdots \\ 0 \end{pmatrix}$ and by U the unity $m \times m$ matrix and $U_{1\times m} = \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix}$. If $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, then the symbol A^{τ} stands for the transpose matrix of A.

Recall that a matrix A is said to be convergent to zero if and only if $A^n \to \Theta$ as $n \to \infty$.

Let us recall the following theorem, which is useful for the proof of the main result, see [7].

Theorem 1. *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. The following assertions are equivalent.*

(i) *A is a matrix convergent to zero;*
(ii) $A^n \to \Theta$ *as* $n \to \infty$;
(iii) *The eigenvalues of A are in the open unit disc, i.e.,* $|\lambda| < 1$, *for each* $\lambda \in \mathbb{C}$ *with* $det(A - \lambda I) = 0$;
(iv) *The matrix* $I - A$ *is non-singular and*

$$(I - A)^{-1} = I + A + ... + A^n + ...;$$

(v) *The matrix* $I - A$ *is non-singular and the matrix* $(I - A)^{-1}$ *has nonnenegative elements.*

In [8], one can find that the notion of K-metric, which is an extension of the Perov's metric. Huang and Zhang reconsidered in [9] the notion of K-metric under the name *cone metric*.

Hardy and Rogers [10] proved in 1973 a generalization of Reich fixed point theorem. Having this as a starting point, many authors obtained fixed point results for Hardy–Rogers type operators.

Let (X, d) be a metric space. Throughout this paper we use the following notations.

$P(X)$: *the set of all nonempty subsets of* X;
$P_{cl}(X)$: *the set of all nonempty closed subsets of* X;
$P_{cp}(X)$: *the set of all nonempty compact subsets of* X;
$Fix(F) := \{x \in X \mid x \in F(x)\}$: *the set of the fixed points of* F;
$SFix(F) := \{x \in X \mid \{x\} = F(x)\}$: *the set of the strict fixed points of* F.

We denote by $\mathbb{N}$ the set of all natural numbers. We also denote by $\mathbb{N}^* := \mathbb{N} - \{0\}$ the set of all natural numbers without 0.

Let $(X, \widetilde{d})$ be a generalized metric space in the sense of Perov. Here, if $v, r \in \mathbb{R}^m$ have the form $v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then by the inequality $v \leq r$ we mean $v_i \leq r_i$, for each $i \in \{1, 2, \cdots, m\}$, whereas by the inequality $v < r$, we mean $v_i < r_i$, for each $i \in \{1, 2, \cdots, m\}$. Moreover, $|v| := (|v_1|, |v_2|, \cdots, |v_m|)$ and, if $c \in \mathbb{R}$ then $v \leq c$ means $v_i \leq c$, for each $i \in \{1, 2, \cdots, m\}$.

We can notice that, in a generalized metric space, some concepts are similar to those given for metric space. Some of these concepts are Cauchy sequence, convergent sequence, completeness, and open and closed subsets.

In [11], Kada et al. introduced the concept of w-distance and improved several results replacing the involved metric by a generalized distance. On the other hand, the notions of single-valued and multivalued weakly contractive maps with respect to w-distance was introduced by Suzuki and Takahashi in [12]. Some recent fixed point results involving the w-distance can be found in [12–19].

Definition 1. *A mapping* $w : X \times X \to [0, \infty)$ *is a w-distance on X if it satisfies the following conditions for any* $x, y, z \in X$.

(1) $w(x, z) \leq w(x, y) + w(y, z)$;
(2) *the function* $w(x, .) : X \to [0, \infty)$ *is lower semicontinuous;*
(3) *for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $w(z, x) \leq \delta$ *and* $w(z, y) \leq \delta$ *imply* $d(x, y) \leq \varepsilon$.

In [20], we find the definition of w_0-distance as follows.

Definition 2. *Let* (X, d) *be a metric space. A mapping* $w : X \times X \to [0, \infty)$ *is called* w_0*-distance if it is w-distance on X with* $w(x, x) = 0$ *for every* $x \in \mathbb{R}$.

Remark 1. *Each metric is a* $\widetilde{w_0}$*-distance, but the reverse is not true.*

For the following notations see I.A. Rus [21,22], I.A. Rus, A. Petruşel, A. Sîntămărian [23], and A. Petruşel [24].

Definition 3. *Let (X,d) be a metric space and $f : X \to X$ be a single-valued operator. f is a weakly Picard operator (briefly WPO) if the sequence of successive approximations for f starting from $x \in X$, $(f^n(x))_{n\in\mathbb{N}}$, converges, for all $x \in X$ and its limit is a fixed point for f.*

If f is a WPO, then we consider the operator

$$f^{\infty} : X \to X \text{ defined by } f^{\infty}(x) := \lim_{n\to\infty} f^n(x).$$

Notice that $f^{\infty}(X) = Fix(f)$.

Definition 4. *Let (X,d) be a metric space, $f : X \to X$ be a WPO and $c > 0$ be a real number. By definition, the single-valued operator f is c-weakly Picard operator (briefly c-WPO) if and only if the following inequality holds,*

$$d(x, f^{\infty}(x)) \leq cd(x, f(x)), \text{ for all } x \in X.$$

For the theory of weakly Picard operators, for single-valued operators, see [21].

I.A. Rus gave in [22] the definition of Ulam–Hyers stability as follows.

Definition 5. *Let (X,d) be a metric space and $f : X \to X$ be a single-valued operator. By definition, the fixed point equation*

$$x = f(x) \tag{1}$$

is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{2}$$

there exists a solution x^* of Equation (1) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

Remark 2. *If f is a c-weakly Picard operator, then the fixed point Equation (1) is Ulam–Hyers stable.*

The Ulam stability of different functional type equations have been investigated by many authors (see [25–35]).

We present in the first part of this paper some fixed point results in generalized metric spaces in Perov's sense. The operator satisfies a contractive condition of Hardy–Rogers type. In the second part of the paper, we study the data dependence of the fixed point set. The well-posedness of the fixed point problem and the Ulam–Hyers stability are also studied.

2. Fixed Point Results

First, let us we recall the notion of generalized w-distance defined in [36] by L. Guran.

Definition 6. *Let $(X, \widetilde{d})$ be a generalized metric space. The mapping $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ is called generalized w-distance on X if it satisfies the following conditions.*

(1) $\widetilde{w}(x, y) \leq \widetilde{w}(x, z) + \widetilde{w}(z, y)$*, for every* $x, y, z \in X$*;*
(2) $\widetilde{w}$ *is lower semicontinuous with respect to the second variable.;*
(3) *For any* $\varepsilon := \begin{pmatrix} \varepsilon_1 \\ \cdots \\ \varepsilon_m \end{pmatrix} > 0$*, there exists* $\delta := \begin{pmatrix} \delta_1 \\ \cdots \\ \delta_m \end{pmatrix} > 0$*, such that* $\widetilde{w}(z, x) \leq \delta$ *and* $\widetilde{w}(z, y) \leq \delta$ *implies* $\widetilde{d}(x, y) \leq \varepsilon$*.*

Examples of generalized w-distance and some of its useful properties are also given in [36] and [37]. In the same framework, let us give the definition of generalized w_0-distance.

Definition 7. *Let $(X, \widetilde{d})$ be a generalized metric space. A mapping $\widetilde{w} : X \times X \to [0, \infty)$ is called generalized $\widetilde{w_0}$-distance if it is generalized w-distance on X with $\widetilde{w}(x, x) = 0_{1\times m}$ for every $x \in \mathbb{R}$.*

Let us recall the following useful result.

Lemma 1. *Let $(X, \widetilde{d})$ be a generalized metric space, and let $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let (x_n) and (y_n) be two sequences in X, let $\alpha_n := \begin{pmatrix} \alpha_{n1} \\ \cdots \\ \alpha_{nm} \end{pmatrix} \in \mathbb{R}^m_+$ and $\beta_n = \begin{pmatrix} \beta_{n1} \\ \cdots \\ \beta_{nm} \end{pmatrix} \in \mathbb{R}^m_+$ be two sequences such that $\alpha_{n(i)}$ and $\beta_{n(i)}$ converge to zero for each $i \in \{1, 2, \ldots, m\}$. Let $x, y, z \in X$. Then, the following assertions hold, for every $x, y, z \in X$.*

(1) If $\widetilde{w}(x_n, y) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.
(2) If $\widetilde{w}(x_n, y_n) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.
(3) If $\widetilde{w}(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.
(4) If $\widetilde{w}(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

Next, let us give the definition of single-valued weakly Hardy–Rogers type operator on generalized metric space in Perov's sense.

Definition 8. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance, and $f : X \to X$ be a single-valued operator. We say that f is a weakly Hardy–Rogers type operator if the following inequality is satisfied,*

$$\widetilde{w}(f(x), f(y)) \leq A\widetilde{w}(x, y) + B[\widetilde{w}(x, f(x)) + \widetilde{w}(y, f(y))] + C[\widetilde{w}(x, f(y)) + \widetilde{w}(y, f(x))],$$

for all $x, y \in \mathbb{R}$ and $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$.

The first fixed point result of this paper is the following.

Theorem 2. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance. Let $f : X \to X$ be a single-valued weakly Hardy–Rogers type operator such that*

(a) f is continuous;
(b) there exist matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that

(i) $M = (I - (B + C))^{-1}(A + B + C)$ converges to Θ;
(ii) $I - (B + C)$ is nonsingular and $(I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;
(iii) $I - (A + 2B + 2C)$ is nonsingular and $[I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$.

Then, $Fix(f) \neq \emptyset$. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.

Proof. Fix $x_0 \in X$. Let $x_1 = f(x_0)$ and $x_2 = f(x_1)$. Then, we have

$$\begin{aligned} \widetilde{w}(x_1, x_2) = \widetilde{w}(f(x_0), f(x_1)) A\widetilde{w}(x_0, x_1) + B[\widetilde{w}(x_0, f(x_0)) + \widetilde{w}(x_1, f(x_1))] + C[\widetilde{w}(x_0, f(x_1)) \\ + \widetilde{w}(x_1, f(x_0))] = A\widetilde{w}(x_0, x_1) + B[\widetilde{w}(x_0, x_1) + \widetilde{w}(x_1, x_2)] + C[\widetilde{w}(x_0, x_2) + \widetilde{w}(x_1, x_1)] \\ = (A + B)\widetilde{w}(x_0, x_1) + B(\widetilde{w}(x_1, x_2)) + C[\widetilde{w}(x_0, x_1) + \widetilde{w}(x_1, x_2)] \\ = (A + B + C)\widetilde{w}(x_0, x_1) + (B + C)\widetilde{w}(x_1, x_2). \end{aligned}$$

Then, we have $[I - (B + C)]\widetilde{w}(x_1, x_2) \leq (A + B + C)\widetilde{w}(x_0, x_1)$.

We get the inequality

$$\widetilde{w}(x_1, x_2) \leq [I - (B + C)]^{-1}(A + B + C)\widetilde{w}(x_0, x_1) = M\widetilde{w}(x_0, x_1). \tag{3}$$

For the next step, we have

$$\widetilde{w}(x_2, x_3) = \widetilde{w}(f(x_1), f(x_2))A\widetilde{w}(x_1, x_2) + B[\widetilde{w}(x_1, f(x_1)) + \widetilde{w}(x_2, f(x_2))] + C[\widetilde{w}(x_1, f(x_2))$$
$$+\widetilde{w}(x_2, f(x_1))] = A\widetilde{w}(x_1, x_2) + B[\widetilde{w}(x_1, x_2) + \widetilde{w}(x_2, x_3)] + C[\widetilde{w}(x_1, x_3) + \widetilde{w}(x_2, x_2)]$$
$$= (A + B)\widetilde{w}(x_1, x_2) + B(\widetilde{w}(x_2, x_3)) + C[\widetilde{w}(x_1, x_2) + \widetilde{w}(x_2, x_3)]$$
$$= (A + B + C)\widetilde{w}(x_1, x_2) + (B + C)\widetilde{w}(x_2, x_3).$$

Then, we have $[I - (B + C)]\widetilde{w}(x_2, x_3) \leq (A + B + C)\widetilde{w}(x_1, x_2)$.
Using (3) we obtain the inequality

$$\widetilde{w}(x_2, x_3) \leq [I - (B + C)]^{-1}(A + B + C)\widetilde{w}(x_1, x_2) = M\widetilde{w}(x_1, x_2) \leq M^2\widetilde{w}(x_0, x_1). \tag{4}$$

By induction we obtain a sequence $(x)_{n\in\mathbb{N}} \in X$, with $x_n = f(x_{n-1})$ such that

$$\widetilde{w}(x_n, x_{n+1}) \leq M^n\widetilde{w}(x_0, x_1), \tag{5}$$

with $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $n \in \mathbb{N}$.

We will prove next that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, by estimating $\widetilde{w}(x_n, x_m)$, for every $m, n \in \mathbb{N}$ with $m > n$.

$$\widetilde{w}(x_n, x_m) \leq \widetilde{w}(x_n, x_{n+1}) + \widetilde{w}(x_{n+1}, x_{n+2}) + ... + \widetilde{w}(x_{m-1}, x_m)$$
$$\leq M^n(\widetilde{w}(x_0, x_1)) + M^{n+1}(\widetilde{w}(x_0, x_1)) + ... + M^{m-1}(\widetilde{w}(x_0, x_1))$$
$$\leq M^n(I + M + M^2 + ... + M^{m-n-1})(\widetilde{w}(x_0, x_1)) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1)).$$

Note that $(I - M)$ is nonsingular since M is convergent to zero. This implies

$$\lim_{n\to\infty} w(x_n, x_m) \leq \lim_{n\to\infty} M^n(I - M)^{-1}\widetilde{w}(x_0, x_1)) \xrightarrow{d} 0_{1\times m}.$$

By Lemma 1 (3) the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

By (a) we have $\widetilde{w}(f(x_{n-1}), f(x^*)) \xrightarrow{d} 0_{1\times m}$, as $n \to \infty$. As (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n \xrightarrow{d} x^*$ as $n \to \infty$. From the continuity of f, it follows that $x_{n+1} = f(x_n) \xrightarrow{d} f(x^*)$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = f(x^*)$, that is, x^* is a fixed point of f. Then $Fix(f) \neq \emptyset$.

Let $x^* \in X$ such that $x^* = f(x^*)$. Then, we have

$$\widetilde{w}(x^*, x^*) = \widetilde{w}(f(x^*), f(x^*)) \leq A\widetilde{w}(x^*, x^*)$$
$$+B[\widetilde{w}(x^*, f(x^*)) + \widetilde{w}(x^*, f(x^*))] + C[\widetilde{d}(x^*, f(x^*)) + \widetilde{d}(x^*, f(x^*))]$$
$$= A\widetilde{w}(x^*, x^*) + 2B\widetilde{w}(x^*, x^*) + 2C\widetilde{w}(x^*, x^*). \tag{6}$$

This implies $[I - (A + 2B + 2C)]\widetilde{w}(x^*, x^*) \leq 0_{1\times m}$. By hypothesis (iii) we get $\widetilde{w}(x^*, x^*) = 0_{1\times m}$. □

We can replace the continuity condition on the operator f and we obtain the following fixed point theorem.

Theorem 3. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov's sense and $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance. Let $f : X \to X$ be a single-valued weakly Hardy–Rogers type operator such that the following conditions are satisfied,*

(a) $inf\{\widetilde{w}(x, y) + \widetilde{w}(x, f(x)) : x \in X\} > 0$;
(b) *there exist matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that:*

- *(i)* $M = (I - (B + C))^{-1}(A + B + C)$ *converges to* Θ;
- *(ii)* $I - (B + C)$ *is nonsingular and* $(I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;
- *(iii)* $I - (A + 2B + 2C)$ *is nonsingular and* $[I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$.

Then $Fix(f) \neq \emptyset$. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.

Proof. Following the same steps as in the previous theorem, Theorem 2, we have the estimation

$$\widetilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1) \tag{7}$$

with $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $n \in \mathbb{N}$.

By Lemma 1 (3), the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. As $(X, \widetilde{d})$ is complete, there exists $x^* \in X$ such that $x_n \xrightarrow{d} x^*$. Let $n \in \mathbb{N}$ be fixed. Then, as $(x_m)_{m\in\mathbb{N}} \xrightarrow{d} x^*$ and $\widetilde{w}(x_n, \cdot)$ is lower semicontinuous, we have

$$\widetilde{w}(x_n, x^*) \leq \liminf_{m\to\infty} \widetilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1). \tag{8}$$

Assume that $x^* \neq f(x^*)$. Then, for every $x \in X$, by hypothesis (a) we have

$$0 < \inf\{\widetilde{w}(x, x^*) + \widetilde{w}(x, f(x)) : x \in X\} \leq \inf\{\widetilde{w}(x_n, x^*) + \widetilde{w}(x_n, x_{n+1}) : n \in \mathbb{N}\}$$

$$\leq \inf\{M^n(I - M)^{-1}\widetilde{w}(x_0, x_1) + M^n\widetilde{w}(x_0, x_1)\} = 0.$$

This is a contradiction. Therefore $x^* = f(x^*)$, so $Fix(f) \neq \emptyset$. For the proof of the last part of this theorem we use the same steps as is the previous theorem, Theorem 2. □

Further we give a more general fixed point result concerning this new type of operators.

Theorem 4. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance, and $f : X \to X$ be a single-valued weakly Hardy–Rogers type operator. There exist matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that*

(i) $M = (I - (B + C))^{-1}(A + B + C)$ *converges to* Θ;
(ii) $I - (B + C)$ *is nonsingular and* $(I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;
(iii) $I - (A + 2B + 2C)$ *is nonsingular and* $[I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$.

Then $Fix(f) \neq \emptyset$. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.

Proof. Following the same steps as in Theorem 2, we get the estimation

$$\widetilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1) \tag{9}$$

with $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $n \in \mathbb{N}$.

By Lemma 1 (3) the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence; since $(X, \widetilde{d})$ is complete there exists $x^* \in X$ such that $x_n \xrightarrow{d} x^*$.

Let $n \in \mathbb{N}$ be fixed. Then, as $(x_m)_{m\in\mathbb{N}} \xrightarrow{d} x^*$, $\widetilde{w}(x_n, \cdot)$ is lower semicontinuous and letting $n \to \infty$ we have

$$\widetilde{w}(x_n, x^*) \leq \liminf_{m\to\infty} \widetilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1) \xrightarrow{d} 0_{1\times m}. \tag{10}$$

Let $f(x^*) \in X$. By triangle inequality and using (6) we obtain

$$\widetilde{w}(x_n, f(x^*)) = \widetilde{w}(x_n, x^*) + \widetilde{w}(x^*, f(x^*)) \leq \widetilde{w}(x_n, x^*) + \widetilde{w}(f(x^*), f(x^*))$$

$$\leq M^n(I-M)^{-1}\widetilde{w}(x_0, x_1) + [I - (A + 2B + 2C)]\widetilde{w}(x^*, x^*) \xrightarrow{d} 0_{1\times m}. \tag{11}$$

Using Lemma 1(1), by Equations (10) and (11), we get $x^* = f(x^*)$. Then, $Fix(f) \neq \emptyset$.
For the last part of the proof we use the same steps as in Theorem 2. □

Another fixed point result concerning the single-valued weakly Hardy–Rogers operators in generalized metric space is the following.

Theorem 5. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov' sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance and $f : X \to X$ be a single-valued Hardy–Rogers type operator. Suppose that all the hypothesis of Theorem 2 hold. Then, we have*

(1) $Fix(f) \neq \emptyset$.
(2) *There exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$ and converge to a fixed point of f.*
(3) $\widetilde{d}(x_n, x^*) \leq M^n\widetilde{d}(x_0, x_1)$, *where* $x^* \in Fix(f)$.

Example 1. *Let $X = \mathbb{R}^2$ be a normed linear space endowed with the generalized norm $\widetilde{d}$ defined by $\widetilde{d}(x,y)(= \begin{pmatrix} ||x_1 - y_1|| \\ ||x_2 - y_2|| \end{pmatrix}$ and $\widetilde{w}$ a generalized w_0-distance defined by $\widetilde{w}(x,y)(= \begin{pmatrix} ||y_1|| \\ ||y_2|| \end{pmatrix}$, for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an operator given by*

$$f(x,y) = \begin{cases} \frac{4x}{5} + \frac{6y}{5} - 1, \frac{6y}{5} - 1, & \text{for } (x,y) \in \mathbb{R}^2, \text{ with } x \leq 5; \\ \frac{x}{5} + \frac{y}{3} - 1, \frac{y}{5}, & \text{for } (x,y) \in \mathbb{R}^2, \text{ with } x > 5. \end{cases}$$

We take $f(x,y) = (f_1(x,y), f_2(x,y))$ where $f_1(x,y) = \begin{cases} \frac{4x}{5} + \frac{6y}{5} - 1, & \text{for } (x,y) \in \mathbb{R}^2, \text{ with } x \leq 5; \\ \frac{x}{5} + \frac{y}{3} - 1, & \text{for } (x,y) \in \mathbb{R}^2, \text{ with } x > 5. \end{cases}$ and $f_2(x,y) = \begin{cases} \frac{6y}{5} - 1, & \text{for } (x,y) \in \mathbb{R}^2, \text{ with } x \leq 5; \\ \frac{y}{5}, & \text{for } (x,y) \in \mathbb{R}^2, \text{ with } x > 5. \end{cases}$

Next, we show that weakly Hardy–Rogers type condition takes place.

Let $A = \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix}$.

Case 1. If $1 \leq x_1, x_2, y_1, y_2 \leq 5$ we have

$$\widetilde{w}(f(x), f(y)) = \begin{pmatrix} ||f_1(y_1, y_2)|| \\ ||f_2(y_1, y_2)|| \end{pmatrix} = \begin{pmatrix} ||\frac{4}{5}y_1 + \frac{6}{5}y_2 - 1|| \\ ||0 \cdot y_1 + \frac{6}{5}y_2 - 1|| \end{pmatrix} \leq \begin{pmatrix} \frac{4}{5}||y_1|| + \frac{6}{5}||y_2|| - 1 \\ 0 \cdot ||y_1|| + \frac{6}{5}||y_2|| - 1 \end{pmatrix}$$

$$\leq \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix} \begin{pmatrix} ||y_1|| \\ ||y_2|| \end{pmatrix} = A\widetilde{w}(x,y).$$

Case 2. If $x_1, x_2, y_1, y_2 > 5$ we have

$$\widetilde{w}(f(x), f(y)) = \begin{pmatrix} ||f_1(y_1, y_2)|| \\ ||f_2(y_1, y_2)|| \end{pmatrix} = \begin{pmatrix} ||\frac{1}{5}y_1 + \frac{1}{3}y_2 - 1|| \\ ||0 \cdot y_1 + \frac{1}{5}y_2|| \end{pmatrix} \leq \begin{pmatrix} \frac{1}{5}||y_1|| + \frac{1}{3}||y_2|| - 1 \\ 0 \cdot ||y_1|| + \frac{1}{5}||y_2|| \end{pmatrix}$$

$$\leq \begin{pmatrix} \frac{1}{5} & \frac{1}{3} \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} ||y_1|| \\ ||y_2|| \end{pmatrix} < \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix} \begin{pmatrix} ||y_1|| \\ ||y_2|| \end{pmatrix} = A\widetilde{w}(x,y).$$

Case 3. For other choices of x_1, x_2, y_1, y_2 *we have*

$$\widetilde{w}(f(x), f(y)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix} \begin{pmatrix} ||y_1|| \\ ||y_2|| \end{pmatrix} = A\widetilde{w}(x, y).$$

Thus, the weakly Hardy–Rogers type condition is satisfied for $A = \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix}$ *and* $B = C = \Theta$ *or* $B + C = \Theta$.

As all the hypothesis of Theorem 3 hold, f *has a fixed point and it is easy to check that* $x = f(x) = (f_1(x), f_2(x))$, *where* $x = (1, 1)$.

Next, let us give some common fixed point results.

Theorem 6. *Let* $(X, \widetilde{d})$ *be a complete generalized metric space in Perov's sense,* $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ *be a generalized w-distance, and* $f, g : X \to X$ *be two continuous single-valued weakly Hardy–Rogers type operators. There exist matrices* $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that*

(i) $I - (B + C)$ *is nonsingular and* $(I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;
(ii) $M = (I - (B + C))^{-1}(A + B + C)$ *converges to* Θ.

Then, f *and* g *have a common fixed point* $x^* \in X$.

Proof. (1) Let $x_0 \in X$. We consider $(x_n)_{n \in \mathbb{N}}$ the sequence of successive approximations for f and g, defined by

$$x_{2n+1} = f(x_{2n}), n = 0, 1, ...$$

$$x_{2n+2} = g(x_{2n+1}), n = 0, 1, ...$$

Then, we have

$$\widetilde{w}(x_{2n}, x_{2n+1}) = \widetilde{w}(g(x_{2n-1}), f(x_{2n})) \leq A\widetilde{w}(x_{2n-1}, f(x_{2n})$$

$$+B[\widetilde{w}(x_{2n}, f(x_{2n})) + \widetilde{w}(x_{2n-1}, g(x_{2n-1}))] + C[\widetilde{w}(x_{2n}, g(x_{2n-1})) + \widetilde{w}(x_{2n-1}, f(x_{2n}))]$$

$$= A\widetilde{w}(x_{2n-1}, x_{2n}) + B[\widetilde{w}(x_{2n}, x_{2n+1}) + \widetilde{w}(x_{2n-1}, x_{2n})] + C\widetilde{w}(x_{2n-1}, x_{2n+1})$$

$$\leq A\widetilde{w}(x_{2n-1}, x_{2n}) + B[\widetilde{w}(x_{2n}, x_{2n+1}) + \widetilde{w}(x_{2n-1}, x_{2n})] + C[\widetilde{w}(x_{2n-1}, x_{2n}) + \widetilde{w}(x_{2n}, x_{2n+1})].$$

Then, we have $\widetilde{w}(x_{2n}, x_{2n+1}) \leq (I - (B + C))^{-1}(A + B + C)\widetilde{w}(x_{2n-1}, x_{2n}) = M\widetilde{w}(x_{2n-1}, x_{2n})$.
By the same argument as above, we get

$$\widetilde{w}(x_{2n+1}, x_{2n+2}) = \widetilde{w}(f(x_{2n}), g(x_{2n+1})) \leq A\widetilde{d}(x_{2n}, f(x_{2n+1})$$

$$+B[\widetilde{w}(x_{2n}, f(x_{2n})) + \widetilde{w}(x_{2n+1}, g(x_{2n+1}))] + C[\widetilde{w}(x_{2n}, g(x_{2n+1})) + \widetilde{w}(x_{2n+1}, f(x_{2n}))]$$

$$= A\widetilde{w}(x_{2n}, x_{2n+1}) + B[\widetilde{w}(x_{2n}, x_{2n+1}) + \widetilde{w}(x_{2n+1}, x_{2n+2})] + C\widetilde{w}(x_{2n}, x_{2n+2})$$

$$\leq A\widetilde{w}(x_{2n}, x_{2n+1}) + B[\widetilde{w}(x_{2n}, x_{2n+1}) + \widetilde{w}(x_{2n+1}, x_{2n+2})] + C[\widetilde{w}(x_{2n}, x_{2n+1}) + \widetilde{w}(x_{2n+1}, x_{2n+2})].$$

Then, we have $\widetilde{w}(x_{2n+1}, x_{2n+2}) \leq (I - (B + C))^{-1}(A + B + C)\widetilde{w}(x_{2n}, x_{2n+1}) = M\widetilde{w}(x_{2n}, x_{2n+1})$.
Further, we obtain $\widetilde{w}(x_n, x_{n+1}) \leq M^n\widetilde{w}(x_0, x_1)$ for each $n \in \mathbb{N}$.

Following the same steps as in the proof of Theorem 2 we estimate $\widetilde{w}(x_n, x_m)$, for every $m, n \in \mathbb{N}$ with $m > n$.

$$\widetilde{w}(x_n, x_m) \leq \widetilde{w}(x_n, x_{n+1}) + \widetilde{w}(x_{n+1}, x_{n+2}) + ... + \widetilde{w}(x_{m-1}, x_m)$$

$$\leq M^n(\widetilde{w}(x_0, x_1)) + M^{n+1}(\widetilde{w}(x_0, x_1)) + ... + M^{m-1}(\widetilde{w}(x_0, x_1))$$

$$\leq M^n(I + M + M^2 + ... + M^{m-n-1})(\widetilde{w}(x_0, x_1)) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1)).$$

Note that $(I - M)$ is nonsingular since M is convergent to Θ. Using Lemma 1 (3) the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Using the lower semicontinuity of the generalized w-distance, by relation (8) we have $\widetilde{w}(x_n, x^*) \xrightarrow{d} 0_{1\times m}$ as $n \to \infty$. Then, we have $\widetilde{w}(x_{2n}, x^*) \xrightarrow{d} 0_{1\times m}$ as $n \to \infty$. By the continuity of f it follows $x_{2n+1} = f(x_{2n}) \xrightarrow{d} f(x^*)$ as $n \to \infty$. By the uniqueness of the limit we get $x^* = f(x^*)$.

By $\widetilde{w}(x_n, x^*) \xrightarrow{d} 0_{1\times m}$ as $n \to \infty$ we have that $\widetilde{w}(x_{2n+1}, x^*) \xrightarrow{d} 0_{1\times m}$ as $n \to \infty$. By the continuity of g it follows $x_{2n+2} = g(x_{2n+1}) \xrightarrow{d} g(x^*)$ as $n \to \infty$. By the uniqueness of the limit we get $x^* = g(x^*)$.

Then, x^* is a common fixed point for f and g. □

By replacing the continuity condition for the mappings f and g, we can state the following result.

Theorem 7. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance, and $f, g : X \to X$ be two single-valued Hardy–Rogers type operators. There exist matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that*

(i) *$I - (B + C)$ is nonsingular and $(I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;*
(ii) *$I - (A + 2B + 2C)$ is nonsingular and $[I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;*
(iii) *$M = (I - (B + C))^{-1}(A + B + C)$ converges to Θ.*

Then f and g have a common fixed point $x^* \in X$.

Proof. (1) As in the proof of the previous theorem, Theorem 6, for $x_0 \in X$ we consider $(x_n)_{n\in\mathbb{N}}$ the sequence of successive approximations for f and g, defined by

$$x_{2n+1} = f(x_{2n}), n = 0, 1, ...$$

$$x_{2n+2} = g(x_{2n+1}), n = 0, 1, ...$$

We define the sequence $(x_n)_{n\mathbb{N}} \in X$ such that

$$\widetilde{w}(x_{2n+1}, x_{2n+2}) \leq (I - (B + C))^{-1}(A + B + C)\widetilde{w}(x_{2n}, x_{2n+1}) = M\widetilde{w}(x_{2n}, x_{2n+1}).$$

Further, we obtain $\widetilde{w}(x_n, x_{n+1}) \leq M^n \widetilde{d}(x_0, x_1)$ for each $n \in \mathbb{N}$.

Following the same steps as in the proof of Theorem 6 we estimate $\widetilde{w}(x_n, x_m)$, for every $m, n \in \mathbb{N}$ with $m > n$ and we get $\widetilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1))$.

Note that $(I - M)$ is nonsingular since M is convergent to Θ. By Lemma 1 (3), the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Using the lower semicontinuity of the generalized w-distance, by relation (8), we have $\widetilde{w}(x_n, x^*) \xrightarrow{d} 0_{1\times m}$, as $n \to \infty$. By (11) we have $\widetilde{w}(x_n, f(x^*)) \xrightarrow{d} 0_{1\times m}$, as $n \to \infty$. Then, using Lemma 1 (2), we get $x^* = f(x^*)$.

Let us show that $g(x^*) = x^*$. Then, by the definition of Hardy–Rogers type operators we have

$$\widetilde{w}(x^*, g(x^*)) = \widetilde{d}(f(x^*), g(x^*))$$

$$\leq A\widetilde{w}(x^*, x^*) + B[\widetilde{w}(x^*, f(x^*)) + \widetilde{w}(x^*, g(x^*)] + C[\widetilde{w}(x^*, g(x^*)) + \widetilde{w}(x^*, f(x^*))].$$

Then, we get

$$\widetilde{w}(x^*, g(x^*)) \leq (I - (B + C))^{-1}(A + B + C)\widetilde{w}(x^*, x^*). \tag{12}$$

By (6) we get $\widetilde{w}(x^*, g(x^*)) = 0_{1\times m}$.

Let $g(x^*) \in X$. By triangle inequality and using (12) we obtain

$$\widetilde{w}(x_n, g(x^*)) = \widetilde{w}(x_n, x^*) + \widetilde{w}(x^*, g(x^*)) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1) + 0_{1\times m} \xrightarrow{d} 0_{1\times m}. \tag{13}$$

Using (8) and (13), by Lemma 1 (2), we obtain $x^* = g(x^*)$. Then x^* is a common fixed point for f and g. $\square$

Remark 3. *In the case of common fixed points, the generalized $\widetilde{w}$-distance must not necessarily be a generalized $\widetilde{w_0}$-distance.*

3. Ulam–Hyers Stability, Well-Posedness, and Data Dependence of Fixed Point Problem

We begin this section with the extension of Ulam–Hyers stability for fixed point equation for the case of single-valued operators on generalized metric space in Perov's sense. Then, let us recall the definition of weakly Ulam–Hyers stability.

Definition 9. *Let $(X, \widetilde{d})$ be a metric space, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance, and $f : X \to X$ be an operator. By definition, the fixed point equation*

$$x = f(x) \tag{14}$$

is weakly Ulam–Hyers stable if there exists a real positive matrix $N \in \mathcal{M}_{m,m}(\mathbb{R}+)$ such that, for each $\varepsilon > 0$ and each solution y^* of the inequation

$$\widetilde{w}(y, f(y)) \leq \varepsilon I_{1\times m} \tag{15}$$

there exists a solution x^* of the Equation (14) such that

$$\widetilde{d}(y^*, x^*) \leq N\varepsilon I_{1\times m}.$$

Theorem 8. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance and $f : X \to X$ be a single-valued Hardy–Rogers type operator defined in (8). There exist matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that*

(i) $N = M^n(I - M)^{-1}$ is nonsingular and $N = M^n(I - M)^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where $M = (I(B + C))^{-1}(A + B + C)$ converges to Θ;
(ii) $I - (A + 2B + 2C)$ is nonsingular and $[I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;
(iii) $I - P^2$ is nonsingular and $I - P^2 \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ where $P = [I - (A + C)]^{-1}C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$.

Then, the fixed point Equation (14) is weakly Ulam–Hyers stable.

Proof. Let $\delta I_{1\times m} > 0_{1\times m}$ such that $\widetilde{w}(x_0, x_1) \leq \delta I_{1\times m}$, for every $x_0, x_1 \in X$ with $x_1 = f(x_0)$. Let $Fix(f) = \{x^*\}$ and $u^* \in X$ be a solution of Equation (14). Then, $\widetilde{w}(u^*, f(u^*)) \leq \varepsilon I_{1\times m}$. By the definition of the weakly Hardy–Rogers type operator we obtain

$$\begin{aligned} \widetilde{w}(x^*, u^*) &\leq \widetilde{w}(f(x^*), f(u^*)) \leq A\widetilde{w}(x^*, u^*) + B[\widetilde{w}(x^*, f(x^*)) + \widetilde{w}(u^*, f(u^*))] + C[\widetilde{w}(x^*, f(u^*)) \\ &+ \widetilde{w}(u^*, f(x^*))] = A\widetilde{w}(x^*, u^*) + B[\widetilde{w}(x^*, x^*) + \widetilde{w}(u^*, u^*)] + C[\widetilde{w}(x^*, u^*) + \widetilde{w}(u^*, x^*)] \\ &= (A + C)\widetilde{w}(x^*, u^*) + B[\widetilde{w}(x^*, x^*) + \widetilde{w}(u^*, u^*)] + C\widetilde{w}(u^*, x^*). \end{aligned} \tag{16}$$

By (6) we get

$$\widetilde{w}(x^*, x^*) = \widetilde{w}(f(x^*), f(x^*)) \leq (A + 2B + 2C)\widetilde{w}(x^*, x^*) \text{ and} \tag{17}$$

$$\widetilde{w}(u^*, u^*) = \widetilde{w}(f(u^*), f(u^*)) \leq (A + 2B + 2C)\widetilde{w}(u^*, u^*).$$

Using hypothesis (ii) we get $\widetilde{w}(x^*, x^*) = \widetilde{w}(u^*, u^*) = 0_{1\times m}$.
By (16) we obtain

$$\widetilde{w}(x^*, u^*) \leq [I - (A + C)]^{-1}C\widetilde{w}(u^*, x^*). \tag{18}$$

By the definition of the weakly Hardy–Rogers type operator we get

$$\widetilde{w}(u^*, x^*) \leq [I - (A + C)]^{-1} C \widetilde{w}(x^*, u^*)$$

and using (18) we obtain

$$\widetilde{w}(x^*, u^*) \leq ([I - (A + C)]^{-1} C)^2 \widetilde{w}(x^*, u^*) = P^2 \widetilde{w}(x^*, u^*). \tag{19}$$

Then, $(I - P^2)\widetilde{w}(x^*, u^*) \leq 0_{1\times m}$. By hypothesis (iii) we get $\widetilde{w}(x^*, u^*) = 0_{1\times m}$.
Let $x_n \in X$ such that, by Equations (8) and (19) we have

$$\widetilde{w}(x_n, x^*) \leq M^n (I - M)^{-1} \widetilde{w}(x_0, x_1) \leq N\delta I_{1\times m} \text{ and} \tag{20}$$

$$\widetilde{w}(x_n, u^*) \leq \widetilde{w}(x_n, x^*) + \widetilde{w}(x^*, u^*) \leq M^n (I - M)^{-1} \widetilde{w}(x_0, x_1) + 0_{1\times m} \leq N\delta I_{1\times m}.$$

Then, using the definition of generalized w-distance, there exists $\varepsilon I_{1\times m} > 0_{1\times m}$ such that

$$\widetilde{d}(x^*, u^*) \leq \varepsilon I_{1\times m} \leq N\varepsilon I_{1\times m}.$$

Then, the fixed point Equation (14) is weakly Ulam–Hyers stable.
□

The following result assures the well-posedness of the fixed point problem with respect to the generalized w_0-distance $\widetilde{w}$.

Theorem 9. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance, and $f : X \to X$ be a single-valued Hardy–Rogers type operator defined in Equation (8). If all the hypothesis of Theorem 2 (respectively, 3 and 4) are satisfied, the fixed point Equation (14) is well-posed with respect to the generalized w_0-distance $\widetilde{w}$, i.e., if $Fix(f) = \{x^*\}$ and $x_n \in \mathbb{N}$, with $n \in \mathbb{N}$, such that $\widetilde{w}(x_n, f(x_n)) \to 0_{1\times m}$ as $n \to \infty$, then $x_n \to x^*$ as $n \to \infty$.*

Proof. Let $x^* \in Fix(f)$ and let $(x)_{n\in\mathbb{N}} \in X$ such that $\widetilde{w}(x_n, f(x_n)) \xrightarrow{d} 0_{1\times m}$ as $n \to \infty$. That means $\widetilde{w}(x_{n-1}, x_n) \xrightarrow{d} 0_{1\times m}$ as $n \to \infty$.

By the lower semicontinuity of the generalized w-distance, using (8) we have

$$\widetilde{w}(x_{n-1}, x^*) \leq \liminf_{m\to\infty} \widetilde{w}(x_n, x_m) \leq M^n (I - M)^{-1} \widetilde{w}(x_0, x_1) \xrightarrow{d} 0_{1\times m}.$$

Then, using Lemma 1 (3) we get $x_n \xrightarrow{d} x^*$ as $n \to \infty$. □

The next theorem presents a data dependence result.

Theorem 10. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense, $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w_0-distance, and $f_1, f_2 : X \to X$ be single-valued operators, which satisfy the following conditions,*

(i) for $A, B, C, M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ with $M = [I - (B + C)]^{-1}(A + B + C)$ a matrix convergent to Θ such that, for every $x, y \in X$ and $i \in \{1, 2\}$, we have:
$\widetilde{w}(f_i(x), f_i(y)) \leq A\widetilde{w}(x, y) + B[\widetilde{w}(x, f_i(x)) + \widetilde{w}(y, f_i(y))] + C[\widetilde{w}(x, f_i(y)) + \widetilde{w}(y, f_i(x))]$;
(ii) there exists $\eta > 0$ such that $\widetilde{w}(f_1(x), f_2(x)) \leq \eta I$, for all $x \in X$.

Then, for $x_1^* = f_1(x_1^*)$ there exists $x_2^* = f_2(x_2^*)$ such that $\widetilde{d}(x_1^*, x_2^*) \leq (I - M)^{-1}\eta I_{1\times m}$; (respectively, for $x_2^* = f_2(x_2^*)$ there exists $x_1^* = f_1(x_1^*)$ such that $\widetilde{w}(x_2^*, x_1^*) \leq (I - M)^{-1}\eta I_{1\times m}$).

Proof. As in the proof of Theorem 2 (respectively, Theorem 3) we construct the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \in X$ of f_2 with $x_0 := x_1^*$ and $x_1 = f_2(x_1^*)$ having the property $\widetilde{w}(x_n, x_{n+1}) \leq M^n\widetilde{w}(x_0, x_1)$, where $M = [I - (B + C)]^{-1}(A + B + C)$.

If we consider the sequence $(x_n)_{n\in\mathbb{N}} \in X$ converges to x_2^*, we have $x_2^* = f(x_2^*)$. Moreover, for each $n, p \in \mathbb{N}$ we have $\widetilde{w}(x_n, x_{n+p}) \leq M^n(I - M)^{-1}\widetilde{w}(x_0, x_1)$.

Letting $p \to 0$ we get $\widetilde{w}(x_n, x_2^*) \leq I(I - M)^{-1}\widetilde{w}(x_0, x_1)$.

Choosing $n = 0$ we get $\widetilde{w}(x_0, x_2^*) \leq I(I - M)^{-1}\widetilde{w}(x_0, x_1)$ and using above the notations we get our conclusion $\widetilde{w}(x_1^*, x_2^*) \leq (I - M)^{-1}\eta I_{1\times m}$. □

4. Conclusions

The purpose of this paper is to establish some fixed point results in generalized metric spaces in Perov's sense. The generalized metric considered here is the w-distance, for which the symmetry condition is not satisfied. The operators satisfy a contractive weakly condition of Hardy–Rogers type. The second part of the paper is devoted to the study of the data dependence, as well as the well-posedness and the Ulam–Hyers stability of the fixed point problem. In order to prove our main results we had to impose a symmetry condition for the w-distance. The results presented in this paper generalize some recent ones.

Author Contributions: Conceptualization, L.G. and M.-F.B.; methodology, A.N.; software, L.G.; validation, L.G., M.-F.B. and A.N.; formal analysis, M.-F.B.; investigation, L.G.; resources, A.N.; data curation, A.N.; writing—original draft preparation, L.G.; writing—review and editing, M.-F.B., L.G.; visualization, A.N.; supervision, L.G.; project administration, L.G., M.-F.B.; funding acquisition, M.-F.B. All authors have read and agreed to the published version of the manuscript.

Funding: The second author wish to thanks Babeș-Bolyai University, Cluj-Napoca, Romania, for the financial support.

Conflicts of Interest: The authors declare no conflicts of interest.

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