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A Note on the Degenerate Poly-Cauchy Polynomials and Numbers of the Second Kind

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Abstract: In this paper, we consider the degenerate Cauchy numbers of the second kind were defined by Kim (2015). By using modified polyexponential functions, first introduced by Kim-Kim (2019), we define the degenerate poly-Cauchy polynomials and numbers of the second kind and investigate some identities and relationship between various polynomials and the degenerate poly-Cauchy polynomials of the second kind. Using this as a basis of further research, we define the degenerate unipoly-Cauchy polynomials of the second kind and illustrate their important identities.

Keywords: polylogarithm functions; unipoly functions; Cauchy polynomials; poly-Cauchy polynomials; unipoly-Cauchy polynomials

1. Introduction

We first introduce the Cauchy polynomials $C_n(x)$ (or the Bernoulli polynomials of the second kind) derived from the integral as follows (see References [1–4]):

$$\int_0^1 (1+t)^{x+y} dy = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!}, \tag{1}$$

When $x = 0$, $C_n = C_n(0)$ are called the Cauchy numbers. The Daehee polynomial $D_n(x)$ was defined by Kim as the following generating function (see References [5–10]):

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \tag{2}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers. Kim [2] defined the degenerate Cauchy polynomials $C_{n,\lambda}(x)$ as follows:

$$\begin{aligned}\int_0^1 \left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x+y} dy &= \frac{\frac{1}{\lambda}\log(1+\lambda t)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x} \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{3}$$

When $x = 0$, $C_{n,\lambda} = C_{n,\lambda}(0)$ are the degenerate Cauchy numbers. The degenerate Cauchy polynomials $C_{n,\lambda,2}(x)$ of the second kind are introduced by Kim to be (see References [11–13]):

$$\frac{t}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x} = \sum_{n=0}^{\infty} C_{n,\lambda,2}(x)\frac{t^n}{n!}, \tag{4}$$

When $x = 0$, $C_{n,\lambda,2} = C_{n,\lambda,2}(0)$ are called the degenerate Cauchy polynomials of the second kind. Here, we note that a major study is to define these polynomials formal with more interesting conditions. From now on, we introduce the following polynomials studied by many researchers to find the identity and relationship between the various polynomials and new defined polynomials. Pyo-Kim-Kim defined the degenerate Cauchy polynomials $C_{n,\lambda,3}(x)$ of the third kind as follows (see References [11–13]):

$$\frac{\lambda\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1\right)}{\log(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} C_{n,\lambda,3}(x)\frac{t^n}{n!}, \tag{5}$$

and also defined the degenerate Cauchy polynomials $C_{n,\lambda,4}(x)$ of the fourth kind as follows (see References [14–16]):

$$\frac{\lambda t}{\log(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} C_{n,\lambda,4}(x)\frac{t^n}{n!}, \ . \tag{6}$$

Let us introduce the following numbers to find the identity and relationship between various polynomials and new defined polynomials. It is well known that the Stirling numbers of the first kind are defined by (see References [3,11,17,18]):

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \ , \tag{7}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\ldots(x-n+1)$, $(n \geq 1)$. From (7), it is easy to see that (see References [3,17]):

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \tag{8}$$

In the inverse expression to (7), for $n \geq 0$, the Stirling numbers of the second kind are defined by (see References [4,6,19–24]):

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \tag{9}$$

From (10), it is easy to see that (see References [17,25]):

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{10}$$

In this paper, we consider the degenerate Cauchy numbers of the second kind were defined by Kim (2015). By using modified polyexponential functions, first introduced by Kim-Kim (2019), we define the degenerate poly-Cauchy polynomials and numbers of the second kind and investigate some identities and relationship between various polynomials and the degenerate poly-Cauchy polynomials of the second kind. Using this as a basis of further research, we define the degenerate unipoly-Cauchy polynomials of the second kind and illustrate their important identities.

2. The Degenerate Poly-Cauchy Polynomials of the Second Kind

For $k \in \mathbb{Z}$, it is well known that the polylogarithm function $Li_k(x)$ is defined by a power series in x to be (see References [8,18,19,26]):

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = x + \frac{x^2}{2^k} + \frac{x^3}{3^k} + \cdots, \tag{11}$$

The polyexponential function was first studied by Hardy and then the polyexponential function modified by Kim-Kim was studied (see Reference [15]). Recently, Kim-Kim [19] considered the polyexponential function in the view of an inverse type of the polylogarithm function to be

$$Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}. \tag{12}$$

In (12), when $k = 1$, we get $Ei_1(x) = e^x - 1$. By using the modified polyexponential function, they also introduced type 2 poly-Bernoulli polynomials and unipoly-Bernoulli polynomials [19] .

In the same motivation of type 2 poly-Bernoulli polynomials arising from modified polyexponential function, we define the degenerate poly-Cauchy polynomials of the second kind as follows:

$$\frac{Ei_k(\log(1+t))}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x = \sum_{n=0}^{\infty} C_{n,\lambda,2}^{(k)}(x)\frac{t^n}{n!}. \tag{13}$$

When $x = 0$, $C_{n,\lambda,2}^{(k)} = C_{n,\lambda,2}^{(k)}(0)$ are called the degenerate poly-Cauchy numbers of the second kind. For $k = 1$, by (12), we note that

$$\begin{aligned} Ei_1(\log(1+t)) &= \sum_{n=1}^{\infty} \frac{(\log(1+t))^n}{(n-1)!n} \\ &= e^{\log(1+t)} - 1 = t. \end{aligned} \tag{14}$$

By (13) and (14), we see that $C^{(1)}_{n,\lambda,2} = C_{n,\lambda,2}$. From (13) with $x = 0$, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} C^{(k)}_{n,\lambda,2}\frac{t^n}{n!} &= \frac{Ei_k(\log(1+t))}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \\
&= \frac{t}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\frac{1}{t}\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{(m-1)!m^k} \\
&= \frac{t}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\frac{1}{t}\sum_{m=0}^{\infty}\frac{(\log(1+t))^{m+1}}{(m+1)!(m+1)^{k-1}} \\
&= \frac{t}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\frac{1}{t}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\sum_{l=m+1}^{\infty}S_1(l,m+1)\frac{t^l}{l!} \\
&= \frac{t}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\sum_{l=m}^{\infty}S_1(l+1,m+1)\frac{t^l}{(l+1)!} \\
&= \left(\sum_{s=0}^{\infty}C_{s,\lambda,2}\frac{t^s}{s!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}\frac{S_1(l+1,m+1)}{l+1}\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}C_{n-l,\lambda,2}\frac{S_1(l+1,m+1)}{(l+1)(m+1)^{k-1}}\right)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem which is an identity between the Stirling numbers of the first kind and the degenerate poly-Cauchy numbers of the second kind.

Theorem 1. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$C^{(k)}_{n,\lambda,2} = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}C_{n-l,\lambda,2}\frac{S_1(l+1,m+1)}{(l+1)((m+1)^{k-1}}. \tag{16}$$

Let us take $k = 1$. From (16), we get a very interesting recurrence relation as follows.

$$\sum_{l=1}^{n}\sum_{m=0}^{l}\binom{n}{l}C_{n-l,\lambda,2}\frac{S_1(l+1,m+1)}{(l+1)} = 0. \tag{17}$$

In Reference [19], it is well known that

$$\frac{d}{dx}Ei_k(\log(1+x)) = \frac{1}{(1+x)\log(1+x)}Ei_{k-1}(\log(1+x)). \tag{18}$$

From (18), we note that

$$\begin{aligned}
Ei_k(\log(1+x)) &= \int_0^x \frac{1}{(1+t)\log(1+t)}Ei_{k-1}(\log(1+t))dt \\
&= \int_0^x \frac{1}{(1+t)\log(1+t)}\underbrace{\int_0^t \frac{1}{(1+t)\log(1+t)}\int_0^t \cdots \int_0^t}_{(k-2)times}\frac{t}{(1+t)\log(1+t)}dtdt\cdots dt.
\end{aligned} \tag{19}$$

It is well known that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{l=0}^{\infty} B_l^{(l)} \frac{t^l}{l!}. \tag{20}$$

The above formula is very important and is used at the core of finding the relational formula. From (14), (19), and (20), we can find the following relational formula.

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,\lambda,2}^{(k)} \frac{x^n}{n!} &= \frac{Ei_k(\log(1+x))}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda x)\right)} \\
&= \frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda x)\right)} \int_0^x \frac{1}{(1+t)\log(1+t)} Ei_{k-1}(\log(1+t))dt \\
&= \frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda x)\right)} \int_0^x \frac{1}{(1+t)\log(1+t)} \\
&\quad \times \underbrace{\int_0^t \frac{1}{(1+t)\log(1+t)} \int_0^t \cdots \int_0^t}_{(k-2) times} \frac{t}{(1+t)\log(1+t)} dt \cdots dt dt.
\end{aligned} \tag{21}$$

Let us take $k = 2$. Then we can find a clearer case relationship in Equation (21)

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n,\lambda,2}^{(2)} \frac{t^n}{n!} \\
&= \frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda x)\right)} \int_0^x \frac{t}{(1+t)\log(1+t)} dt \\
&= \frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda x)\right)} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l!} \int_0^x t^l dt \\
&= \frac{x}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda x)\right)} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!} \\
&= \left(\sum_{m=0}^{\infty} C_{m,\lambda,2} \frac{x^m}{m!}\right) \left(\sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{C_{n-l,\lambda,2} B_l^{(l)}}{l+1}\right) \frac{x^n}{n!}.
\end{aligned} \tag{22}$$

Therefore, by (22), we found an identity equation that could calculate the degenerate poly-Cauchy numbers of the second kind from the degenerate Cauchy numbers of the second kind when $k = 2$ as the following theorem.

Theorem 2. *Let $n \geq 0$. Then we have*

$$C_{n,\lambda,2}^{(2)} = \sum_{l=0}^{n} \binom{n}{l} \frac{C_{n-l,\lambda,2} B_l^{(l)}}{l+1}. \tag{23}$$

From (13), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n,\lambda,2}^{(k)}(x)\frac{t^n}{n!} \\
&= \frac{Ei_k(\log(1+t))}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x \\
&= \left(\sum_{l=0}^{\infty} C_{l,\lambda,2}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{x}{m}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m\right) \\
&= \left(\sum_{l=0}^{\infty} C_{l,\lambda,2}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_m\lambda^{-m}\sum_{s=m}^{\infty}S_1(s,m)\lambda^s\frac{t^s}{s!}\right) \\
&= \left(\sum_{l=0}^{\infty} C_{l,\lambda,2}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\sum_{m=0}^{s}(x)_m\lambda^{-m}S_1(s,m)\lambda^s\frac{t^s}{s!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}C_{l,\lambda,2}^{(k)}(x)_m\lambda^{n-l-m}S_1(n-l,m)\right)\frac{t^n}{n!}. \qquad (24)
\end{aligned}$$

By comparing the coefficients on both sides of (24), we found an recurrence relation between the degenerate poly-Cauchy polynomials of the second kind as the following theorem.

Theorem 3. *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have*

$$C_{n,\lambda,2}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}C_{l,\lambda,2}^{(k)}(x)_m\lambda^{n-l-m}S_1(n-l,m). \qquad (25)$$

3. The Degenerate Unipoly-Cauchy Polynomials of the Second Kind

Let $p(n)$ be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Then Kim-Kim [19] defined the unipoly function attached to p by

$$u_k(x|p) = \sum_{n=1}^{\infty}\frac{p(n)x^n}{n^k}, \quad (k \in \mathbb{Z}). \qquad (26)$$

It is well known that

$$u_k(x|1) = \sum_{n=1}^{\infty}\frac{x^n}{n^k} = Li_k(x) \qquad (27)$$

is ordinary polylogarithm function, and for $k \geq 2$,

$$\frac{d}{dx}u_k(x|p) = \frac{1}{x}u_{k-1}(x|p), \qquad (28)$$

and

$$u_k(x|p) = \int_0^x \frac{1}{t}\underbrace{\int_0^t\cdots\int_0^t}_{(k-2)times}\frac{1}{t}u_1(t|p)dt\cdots dtdt, \quad \text{(see Reference [19])}. \qquad (29)$$

By using (26), we define the degenerate unipoly-Cauchy polynomials of the second kind as follows:

$$\frac{u_k(\log(1+t)|p)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}C_{n,\lambda,2}^{(k,p)}(x)\frac{t^n}{n!}. \tag{30}$$

When $x=0$, $C_{n,\lambda,2}^{(k,p)}=C_{n,\lambda,2}^{(k,p)}(0)$ are called the degenerate unipoly-Cauchy numbers of the second kind. Let us take $p(n)=\frac{1}{\Gamma(n)}$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}C_{n,\lambda,2}^{(k,p)}(x)\frac{t^n}{n!}&=\frac{u_k\left(\log(1+t)|\frac{1}{\Gamma}\right)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}}\\&=\frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{m^k(m-1)!}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}}\\&=\frac{Ei_k(\log(1+t))}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}C_{n,\lambda,2}^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Thus, by (31), we have the following theorem.

Theorem 4. *Let $n\geq 0$ and $k\in\mathbb{Z}$, and $\Gamma(n)$ be the Gamma function. Then, we have*

$$C_{n,\lambda,2}^{(k,\frac{1}{\Gamma})}(x)=C_{n,\lambda,2}^{(k)}(x). \tag{32}$$

From (30), we get

$$\begin{aligned}\sum_{n=0}^{\infty}C_{n,\lambda,2}^{(k,p)}\frac{t^n}{n!}&=\frac{u_k(\log(1+t)|p)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\\&=\frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+t))^m\\&=\frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m+1}^{\infty}S_1(m+1,l)\frac{t^l}{l!}\\&=\left(\sum_{j=0}^{\infty}C_{j,\lambda,2}\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{p(m+1)(m+1)!}{(m+1)^k}S_1(m+1,l)\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(m+1,l)C_{n-l,\lambda,2}}{l+1}\right)\frac{t^n}{n!}.\end{aligned} \tag{33}$$

Therefore, by comparing the coefficients on both sides of (33), we found an recurrence relation between the degenerate unipoly-Cauchy numbers of the second kind as the following theorem.

Theorem 5. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then we have*

$$C_{n,\lambda,2}^{(k,p)} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(m+1,l)C_{n-l,\lambda,2}}{l+1}. \tag{34}$$

In particular,

$$C_{n,\lambda,2}^{(k,\frac{1}{\Gamma})} = C_{n,\lambda,2}^{(k)} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{S_1(m+1,l)C_{n-l,\lambda,2}}{(m+1)^{k-1}(l+1)}. \tag{35}$$

From (30), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,\lambda,2}^{(k,p)}(x)\frac{t^n}{n!} &= \frac{u_k(\log(1+t)|p)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}} \\
&= \frac{u_k(\log(1+t)|p)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \\
&= \left(\sum_{l=0}^{\infty} C_{l,\lambda,2}^{(k,p)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\lambda^{-2m}\sum_{s=m}^{\infty}S_1(s,m)\frac{t^s}{s!}\right) \\
&= \left(\sum_{l=0}^{\infty} C_{l,\lambda,2}^{(k,p)}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\sum_{m=0}^{s}(x)_{m,\lambda}\lambda^{-2m}S_1(s,m)\frac{t^s}{s!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{n-l} C_{l,\lambda,2}^{(k,p)}(x)_{m,\lambda}\lambda^{-2m}S_1(n-l,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{36}$$

From (36) , we found an recurrence relation between the degenerate unipoly-Cauchy polynomials of the second kind as the following theorem.

Theorem 6. *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have*

$$C_{n,\lambda,2}^{(k,p)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{n-l} C_{l,\lambda,2}^{(k,p)}(x)_{m,\lambda}\lambda^{-2m}S_1(n-l,m). \tag{37}$$

From (30), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n,\lambda,2}^{(k,p)} \frac{t^n}{n!} \\
&= \frac{u_k(\log(1+t)|p)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \\
&= \frac{1}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} \frac{m!}{m!} (\log(1+t))^{m+1} \\
&= \frac{\log(1+t)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} \frac{m!}{m!} (\log(1+t))^{m} \\
&= \frac{\log(1+t)}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} \frac{m!}{m!} (\log(1+t))^{m} \\
&= \frac{\log(1+t)}{t} \frac{t}{\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \sum_{m=0}^{\infty} \frac{p(m+1)m!}{(m+1)^k} \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!} \\
&= \left(\sum_{s=0}^{\infty} D_s \frac{t^s}{s!}\right) \left(\sum_{a=0}^{\infty} C_{a,\lambda,2} \frac{t^a}{a!}\right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}\right) \\
&= \left(\sum_{b=0}^{\infty} \sum_{a=0}^{b} \binom{b}{a} D_{b-a} C_{a,\lambda,2} \frac{t^b}{b!}\right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{a=0}^{n-l} \sum_{m=0}^{l} \binom{n}{l} D_{n-l-a} C_{a,\lambda,2} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m)\right) \frac{t^n}{n!}.
\end{aligned} \tag{38}$$

By comparing coefficients on both sides of (38), we obtain the following theorem which is a relationship between Daehee numbers and the degenerate unipoly-Cauchy numbers of the second kind.

Theorem 7. *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have*

$$C_{n,\lambda,2}^{(k,p)} = \sum_{l=0}^{n} \sum_{a=0}^{n-l} \sum_{m=0}^{l} \binom{n}{l} D_{n-l-a} C_{a,\lambda,2} \frac{p(m+1)m!}{(m+1)^k} S_1(l.m). \tag{39}$$

4. Conclusions

In 2019 Kim-Kim considered the polyexponential functions and poly-Bernoulli polynomials and Kim [2] introduced the degenerate Cauchy numbers of the second kind. In the same view as these functions and polynomials, we defined the degenerate poly-Cauchy polynomials of the second kind Equation (13) and obtained some identities of the degenerate poly-Cauchy numbers of the second kind (Theorems 1 and 2). In particular, we obtained an identity of the degenerate poly-Cauchy polynomials of the second kind in Theorem 3. Furthermore, by using the unipoly functions, we defined the degenerate unipoly-Cauchy polynomials of the second kind Equation (30) and obtained some properties of the degenerate unipoly-Cauchy numbers of the second kind (Theorems 4 and 5). Finally, we obtained an identity of the degenerate unipoly-Cauchy polynomials of the second kind in Theorem 6 and gave the identity indicating the relationship of the degenerate unipoly-Cauchy numbers of the second kind and the Daehee numbers and degenerate Cauchy numbers of the second kind in Theorem 7. In Bayad-Hamahata studied the multi-poly-Bernoulli polynomials [27] , defined in analogy with the poly-case, using instead the multiple polylogarithm functions. In addition, as a good application of the results of this paper, we would recommend readers to see to references [28–30].

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