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Exact Solutions and Conservation Laws of Time-Fractional Levi Equation

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Abstract: Exact solutions were derived for a time-fractional Levi equation with Riemann–Liouville fractional derivative. The methods involve, first, the reduction of the time-fractional Levi equation to fractional ordinary differential equations with Erdélyi-Kober fractional differential operator with respect to point symmetry groups, and second, use of the invariant subspace to reduce the time-fractional Levi equation into a system of fractional ordinary differential equations, which were solved by the symmetry group method. The obtained explicit solutions have interesting analytic behaviors connected with blow-up and dispersion. The conservation laws generated by the point symmetries of the time-fractional Levi equation are shown via nonlinear self-adjointness method.

Keywords: time-fractional Levi equation; conservation laws; invariant subspace; exact solutions

1. Introductions

Fractional differential equations (FDEs) [1–4] are of considerable importance due to their connection with real-world problems that depend not only on the instant time but also on the previous time, in particular, modeling the phenomena by means of fractals, random walk processes, control theory, signal processing, acoustics, etc. It has been shown that the fractional order models are much more adequate than the integer order models. A number of methods used to solve nonlinear partial differential equations (PDEs) have been successfully generalized to FDEs, such as the Adomain decomposition method, homotopy analysis method, variational iteration method, transform method, symmetry group method, and invariant subspace method.

The symmetry group method was first introduced by Lie [5] and later popularized by Ovsiannikov [6]. Since then, the symmetry properties of PDEs have been extensively studied (see, for example, [7–9]). The effectiveness of this method motivates the study of the symmetry group theory for FDEs. The first extension of the notion of invariance to FDEs, provided by Buckwar and Luchko [10], was under scaling transformation group. In [11,12], Gazizov et al. proposed the prolongation formulas for both the Riemann–Liouville fractional derivative operator and the Caputo fractional derivative. The theoretical framework of Lie theory for FDEs was set up in [13], where the authors proved the Lie theorem for FDEs and deduced that the proper space for the analysis of these symmetries was the infinite dimensional jet space. In [14], the authors presented the algorithm for the systematic calculation of Lie point symmetries for FDEs, which was implemented in the MAPLE package FracSym. Recently, Singla and Gupta investigated the Lie point symmetries for systems of FDEs in a series of work [15].

Conservation laws are important tools in the study of PDEs: for instance, the conserved quantities of energy are used to estimate classical solutions and define norms of weak solutions. Noether's method [7,9] is the principal systematic procedure for constructing conservation laws of differential equations with integer order. The lack of variational structures for the PDEs not having Lagrangians cuts the applications of Noether's method significantly. Taking into account this fact, several generalizations to the Noether's method were introduced, which include the direct construction method [16,17] and nonlinear self-adjointness method [18]. To study the conservation laws for FDEs, researchers generalized Lagrangians [19] and Euler–Lagrange equations [20] to fractional derivatives, which were used to prove several fractional generalizations of Noether's theorem [21]. Consequently, some number of fractional conservation laws were constructed for equations and systems with fractional Lagrangians. However, conservation laws could not yet get widely used because of the lack of Lagrangian for some type of FDEs. In [22], Lukashchuk generalized the nonlinear self-adjointness method to FDEs in such a way that the conservation laws of FDEs in the non-Lagrangian forms can be determined. Inspired by the preliminary results, several works [23,24] were devoted to investigating the conserved vectors to FDEs.

The method of invariant subspace initially presented by Galaktionov and his collaborators [25] is an effective way to perform reductions of nonlinear PDEs to finite-dimensional dynamical systems. The exact solutions obtained through invariant subspace method reveal an optimal description their behavior, including local and global existence, uniqueness and asymptotic, etc. One of the crucial points of this approach is the estimation of maximal dimension for the subspaces. It has been proved that the maximal dimension of invariant subspaces for a k-th order nonlinear ordinary differential operator is $2k+1$ [25]. In [26,27], Qu et al. generalized the result to systems with two-component nonlinear diffusion equations and a k-th order m component nonlinear vector operator, respectively. There it is concluded that the dimension of invariant subspaces admitted by such kind of operator is bounded by $2mk+1$. Such an estimate was further extended to a two-component nonlinear vector differential operators with two different orders [28]. The extension of the invariant subspace method to FDEs was carried out by Gazizov and Kasatkin in [29].

Let us consider time-fractional Levi system

$$\partial_t^\alpha u = -u_{xx} + 2uu_x + 2v_x, \tag{1a}$$

$$\partial_t^\alpha v = v_{xx} + 2uv_x + 2vu_x, \tag{1b}$$

for $u(t,x)$ and $v(t,x)$, where ∂_t^α is Riemann–Liouville fractional differential operator of order α with respect to variable t. When $\alpha = 1$, Equation (1) is one member of the Levi hierarchy. Symmetry-related methods have been applied to Equation (1) with $\alpha = 1$ in [30], where some interesting results were obtained. The purposes of the present paper are, firstly, to study the reductions of time-fractional Levi Equation (1) and, then to detect the admitted conservation laws, and secondly, to derive explicit solutions of Equation (1) by invariant subspace method.

The rest of this paper is organized as follows. Section 2 provides a short review of several definitions and results in fractional calculus. In Section 3, point symmetry groups admitted by Equation (1) are presented. We then write down the fractional ordinary differential equations (ODEs) given by reduction under each point symmetry group. In Section 4, we summarize the conservation laws that arise from the admitted point symmetries via nonlinear self-adjointness. Section 5 first applies the invariant subspace method to find explicit solutions of time-fractional Levi Equation (1). Then, we discuss some analytical behaviors of these solutions. Finally, Section 6 has some concluding remarks.

2. Preliminaries on Fractional Calculus

To proceed, we state several definitions of fractional integral operators and fractional derivative operators. Some useful properties of these operators are also listed in [2,3,31]. Assume $0 \le n-1 < \alpha \le n$ and $n \in \mathbb{Z}^+$.

The left-sided *fractional integral operator* of order $\mu > 0$ for a function $f(t) \in L^1([a,b],\mathbb{R}_+)$ is defined by

$${}_aI_t^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-\zeta)^{\mu-1} f(\zeta) d\zeta, \tag{2}$$

where and whereafter $\Gamma(z)$ is the Euler Gamma function, given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re}(z) > 0. \tag{3}$$

In particular, ${}_aI_t^0 f(t) = f(t)$. The left-sided *Riemann–Liouville fractional derivative* of order $\alpha > 0$ for a function $f(t) \in L^1([a,b], \mathbb{R}_+)$ is defined by

$${}_aD_t^\alpha f(t) = D_t^n({}_aI_t^{n-\alpha} f(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\zeta)^{n-\alpha-1} f(\zeta) d\zeta. \tag{4}$$

The left-sided *Caputo fractional derivative* of order $\alpha > 0$ for a function $f(t) \in L^1([a,b], \mathbb{R}_+)$ is defined by

$${}_a^C D_t^\alpha f(t) = {}_aI_t^{n-\alpha}(D_t^n f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\zeta)^{n-\alpha-1} f^{(n)}(\zeta) d\zeta. \tag{5}$$

The corresponding right-sided operators can be written out by considering integrals over interval (t, b). These fractional derivative operators possess the following properties

$${}_aD_t^\alpha k = \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}, \quad {}_a^C D_t^\alpha k = 0, \tag{6}$$

$${}_aD_t^\alpha t^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1-\alpha+\nu)} t^{\nu-\alpha}, \quad {}_a^C D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha}, \tag{7}$$

where k is any real constant including zero, $\nu > 0$, $\mathrm{Re}(\beta) > 0$. In what follows, we focus on the case when we assume $a = 0$ in (2), (4), and (5), in which case we have $I_t^\mu f(t)$, $D_t^\alpha f(t)$ and ${}^C D_t^\alpha f(t)$.

3. Symmetry Reduction

For the time-fractional Levi Equation (1), a *point symmetry* is a one-dimensional Lie group of transformations acting on (t, x, u, v)

$$t^* = t + \epsilon\tau(t,x,u,v) + O(\epsilon^2), \tag{8a}$$

$$x^* = x + \epsilon\xi(t,x,u,v) + O(\epsilon^2), \tag{8b}$$

$$u^* = u + \epsilon\eta(t,x,u,v) + O(\epsilon^2), \tag{8c}$$

$$v^* = v + \epsilon\zeta(t,x,u,v) + O(\epsilon^2), \tag{8d}$$

where ϵ is the group parameter and $(\tau(t,x,u,v), \xi(t,x,u,v), \eta(t,x,u,v), \zeta(t,x,u,v))$ is the infinitesimal, such that the $(\alpha,2)$-th order prolongation $\mathrm{pr}^{(\alpha,2)}\mathrm{X}$ of its infinitesimal generator

$$\mathrm{X} = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \zeta(t,x,u,v)\partial_v \tag{9}$$

leaves invariant the Equation (1). The operator $\mathrm{pr}^{(\alpha,2)}\mathrm{X}$ is defined by

$$\mathrm{pr}^{(\alpha,2)}\mathrm{X} = \mathrm{X} + \eta^{\alpha,t}\partial_{(u)_t^\alpha} + \zeta^{\alpha,t}\partial_{(v)_t^\alpha} + \eta^x\partial_{u_x} + \zeta^x\partial_{v_x} + \eta^{xx}\partial_{u_{xx}} + \zeta^{xx}\partial_{v_{xx}}, \tag{10}$$

where $(\cdot)_t^\alpha = \partial_t^\alpha \cdot$, η^x, ζ^x, η^{xx} and ζ^{xx} are the integer extended infinitesimals (see [7,8]), $\eta^{\alpha,t}$ and $\zeta^{\alpha,t}$ are the extended infinitesimals of order α, given by

$$\eta^{\alpha,t} = D_t^\alpha\eta + \xi D_t^\alpha u_x - D_t^\alpha(\xi u_x) + D_t^\alpha(uD_t\tau) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}u, \tag{11a}$$

$$\zeta^{\alpha,t} = D_t^\alpha\zeta + \xi D_t^\alpha v_x - D_t^\alpha(\xi v_x) + D_t^\alpha(vD_t\tau) - D_t^{\alpha+1}(\tau v) + \tau D_t^{\alpha+1}v. \tag{11b}$$

Here D_t^α is Riemann–Liouville fractional derivative operator. With the generalized Leibniz rule [1,2] and the generalized chain rule [3], $\eta^{\alpha,t}$ and $\zeta^{\alpha,t}$ in (11) are expressed in forms

$$\eta^{\alpha,t} =\partial_t^\alpha\eta + (\eta_u - \alpha D_t\tau)\partial_t^\alpha u - u\partial_t^\alpha\eta_u + \eta_v\partial_t^\alpha v - v\partial_t^\alpha\eta_v + \mu_1 + \mu_2 - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n\xi D_t^{\alpha-n}u_x + \sum_{n=1}^{\infty}\left(\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\tau\right)D_t^{\alpha-n}u + \sum_{n=1}^{\infty}\binom{\alpha}{n}\partial_t^n\eta_v D_t^{\alpha-n}v, \tag{12a}$$

$$\zeta^{\alpha,t} =\partial_t^\alpha\zeta + (\zeta_v - \alpha D_t\tau)\partial_t^\alpha v - v\partial_t^\alpha\zeta_v + \zeta_u\partial_t^\alpha u - u\partial_t^\alpha\zeta_u + \mu_3 + \mu_4 - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n\xi D_t^{\alpha-n}v_x + \sum_{n=1}^{\infty}\left(\binom{\alpha}{n}\partial_t^n\zeta_v - \binom{\alpha}{n+1}D_t^{n+1}\tau\right)D_t^{\alpha-n}v + \sum_{n=1}^{\infty}\binom{\alpha}{n}\partial_t^n\zeta_u D_t^{\alpha-n}u, \tag{12b}$$

where D_t is the total derivative operator with respect to t, μ_i $(i = 1,\ldots,4)$ are given by

$$\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-l}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r\partial_t^m u^{k-r}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}, \tag{13a}$$

$$\mu_2 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-l}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-v)^r\partial_t^m v^{k-r}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial v^k}, \tag{13b}$$

$$\mu_3 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-l}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r\partial_t^m u^{k-r}\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial u^k}, \tag{13c}$$

$$\mu_4 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-l}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-v)^r\partial_t^m v^{k-r}\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial v^k}, \tag{13d}$$

and

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1-n)}. \tag{14}$$

The invariance of the time-fractional Levi Equation (1) is then given by the conditions

$$\eta^{\alpha,t} + \eta^{x,x} - 2u\eta^x - 2\eta u_x - 2\zeta^x = 0, \tag{15a}$$

$$\zeta^{\alpha,t} - \zeta^{x,x} - 2u\zeta^x - 2\eta v_x - 2v\eta^x - 2\zeta u_x = 0 \tag{15b}$$

holding for all solutions of Equation (1). Each of these invariance conditions splits with respect to all derivatives of the dependent variables modulo Equation (1) and their differential consequences, yielding an overdetermined system in the functions ξ, τ, η and ζ of t, x, u, v

$$\xi_t = \xi_u = \xi_v = \tau_x = \tau_u = \tau_v = \eta_v = \zeta_u = \eta_{uu} = \zeta_{vv} = 0, \tag{16a}$$

$$\alpha D_t\tau - 2\xi_x = 0, \tag{16b}$$

$$\eta_u - \alpha D_t\tau - \zeta_v + \xi_x = 0, \tag{16c}$$

$$2v\zeta_v - 2\alpha v D_t\tau - 2v\eta_u + 2v\xi_x - 2\zeta = 0, \tag{16d}$$

$$2\eta_{xu} - 2\alpha u D_t\tau - \xi_{xx} + 2u\xi_x - 2\eta = 0, \tag{16e}$$

$$\zeta_{xv} + \eta_{xu} - \xi_{xx} = 0, \tag{16f}$$

$$\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u + \eta_{xx} - 2u\eta_x - 2\zeta_x = 0, \tag{16g}$$

$$\partial_t^\alpha\zeta - v\partial_t^\alpha\zeta_v - \zeta_{xx} - 2u\zeta_x - 2v\eta_x = 0, \tag{16h}$$

$$\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0, \quad n = 1,2,\ldots, \tag{16i}$$

$$\binom{\alpha}{n}\partial_t^n\zeta_v - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0, \quad n = 1,2,\ldots \tag{16j}$$

The infinitesimals must also satisfy an additional condition

$$\tau(t,x,u)|_{t=0} = 0 \tag{17}$$

to preserve the structure of Riemann–Liouville fractional derivative.

It is straightforward to solve system (16) combined with condition (17). The results are summarized as follows.

Theorem 1. *The point symmetries of the time-fractional Levi Equation (1) are generated by*

$$\text{space-translation} \quad \mathrm{X}_{\text{trans.}} = \partial_x, \tag{18a}$$
$$\text{scaling} \quad \mathrm{X}_{\text{scal.}} = 2t\partial_t + \alpha x\partial_x - \alpha u\partial_u - 2\alpha v\partial_v. \tag{18b}$$

The corresponding one-dimensional point transformation groups are given by

$$t \to t, x \to x + \epsilon, u \to u, v \to v, \tag{19a}$$
$$t \to \lambda^2 t, x \to \lambda^\alpha x, u \to \lambda^{-\alpha} u, v \to \lambda^{-2\alpha} v, \tag{19b}$$

with group parameters $-\infty < \epsilon < \infty$ *and* $0 < \lambda < \infty$.

Each one-dimensional point symmetry group admitted by the time-fractional Levi Equation (1) can be used to reduce the equation to obtain corresponding group-invariant solutions. The form of these solutions $u(t,x)$ and $v(t,x)$ for a given symmetry with generator X is determined by solving the invariance condition $\mathrm{X}(u - u(t,x))|_{u=u(t,x),v=v(t,x)} = \mathrm{X}(v - v(t,x))|_{u=u(t,x),v=v(t,x)} = 0$. When the given symmetry X does not leave both variables t and x invariant, the corresponding symmetry reduction leads to fractional ODEs formulated by invariants

$$z = \Theta(t,x,u,v), \quad U = \Upsilon(t,x,u,v), \quad V = \Omega(t,x,u,v) \tag{20}$$

determined by

$$\mathrm{X}z = \mathrm{X}U = \mathrm{X}V = 0. \tag{21}$$

Then each solution of the fractional ODEs will yield a group-invariant solution of Equation (1). It remains necessary to solve the fractional ODEs to obtain group-invariant solutions explicitly.

3.1. Reduction under Space-Translation

The symmetry group of space-translation transformation (19a) with the generator (18a) has invariants

$$z = t, \quad U = u, \quad V = v. \tag{22}$$

Therefore Equation (1) reduces to fractional ODEs

$$D_z^\alpha U = 0, \tag{23a}$$
$$D_z^\alpha V = 0, \tag{23b}$$

which have a trivial solution $U = V = 0$. Thus the time-fractional Levi Equation (1) has a trivial space-translation invariant solution $u = v = 0$.

3.2. Reduction under Scaling

The symmetry group of scaling transformation (19b) with the generator (18b) has invariants

$$z = xt^{-\alpha/2}, \quad U = t^{\alpha/2}u, \quad V = t^{\alpha}v. \tag{24}$$

Therefore Equation (1) is reduced to fractional ODEs

$$\left(\mathrm{P}_{2/\alpha}^{1-3\alpha/2,\alpha}U\right)(z) = \frac{1}{z}U'' - \frac{2}{z^2}V' - \frac{2}{z^2}(1+U)U' + \frac{2}{z^3}\left(U^2+U+2V\right), \tag{25a}$$

$$\left(\mathrm{P}_{2/\alpha}^{1-2\alpha,\alpha}V\right)(z) = -\frac{1}{z^2}V'' - \frac{2}{z^3}VU' - \frac{2}{z^3}(U-2)V' + \frac{6}{z^4}V(U-1), \tag{25b}$$

where $\left(\mathrm{P}_{\delta}^{\beta,\alpha}y\right)(z)$ is the Erdélyi-Kober fractional differential operator applied to the function $y = y(z)$, given by

$$\left(\mathrm{P}_{\delta}^{\beta,\alpha}y\right)(z) = \prod_{j=0}^{l-1}\left(\beta + j - \frac{1}{\delta}z\frac{d}{dz}\right)\left(\mathrm{K}_{\delta}^{\beta+\alpha,l-\alpha}y\right)(z), \quad \text{with } l = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N}; \\ \alpha, & \alpha \in \mathbb{N}. \end{cases} \tag{26}$$

Here

$$\left(\mathrm{K}_{\delta}^{\beta,\alpha}y\right)(z) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(s-1)^{\alpha-1}s^{-\beta-\alpha}y(zs^{1/\delta})ds, & \alpha > 0; \\ y(z), & \alpha = 0, \end{cases} \tag{27}$$

is the Erdélyi-Kober fractional integral operator applied to the function $y = y(z)$.

Equation (25) was obtained as follows. The definition of Riemann–Liouville fractional derivative (4) combined with the change of variables (24) yields

$$\partial_t^{\alpha}u = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\left(\int_0^t (t-s)^{n-\alpha-1}s^{-\alpha/2}U\left(xs^{-\alpha/2}\right)ds\right). \tag{28}$$

After the change of variable $s \to y = t/s$, Equation (28) takes the form of

$$\partial_t^{\alpha}u = \frac{\partial^n}{\partial t^n}\left(\frac{t^{n-3\alpha/2}}{\Gamma(n-\alpha)}\int_1^{\infty}(y-1)^{n-\alpha-1}y^{-n-1+3\alpha/2}U\left(zy^{\alpha/2}\right)dy\right) \tag{29}$$

$$= \frac{\partial^n}{\partial t^n}\left(t^{n-3\alpha/2}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha}U\right)(z)\right). \tag{30}$$

Applying the chain rule with (24) to Equation (30), we get

$$\partial_t^{\alpha}u = \frac{\partial^{n-1}}{\partial t^{n-1}}\left(t^{n-2\alpha-1}\left((n - \frac{3\alpha}{2} - \frac{\alpha}{2}z\frac{d}{dz}\right)\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha}U\right)(z)\right). \tag{31}$$

After differentiation $m-1$ times, Equation (31) becomes

$$\partial_t^{\alpha}u = t^{-3\alpha/2}\left(\mathrm{P}_{2/\alpha}^{1-3\alpha/2,\alpha}U\right)(z). \tag{32}$$

With a similar computation, we find

$$\partial_t^{\alpha}v = t^{-2\alpha}\left(\mathrm{P}_{2/\alpha}^{1-2\alpha,\alpha}V\right)(z). \tag{33}$$

Thus Equation (1) gets transformed into the ODEs (25) for the invariants (24).

4. Conservation Law

A *conservation law* of the time-fractional Levi Equation (1) is a continuity equation

$$D_t C^t + D_x C^x = 0 \tag{34}$$

holding for every solution, where the *conserved density* C^t and the *spacial flux* C^x are functions of t, x, u, v and all derivatives of the dependent variables u and v. Consider the so-called *formal Lagrangian* of Equation (1)

$$\mathcal{L} = f(t,x)F_1 + g(t,x)F_2, \tag{35}$$

where $F_1 = \partial_t^\alpha u + u_{xx} - 2uu_x - 2v_x$, $F_2 = \partial_t^\alpha v - v_{xx} - 2uv_x - 2vu_x$, $f(t,x)$ and $g(t,x)$ are new introduced dependent variables. The adjoint equations of Equation (1) are defined by

$$\frac{\delta \mathcal{L}}{\delta u} = (D_t^\alpha)^* f + 2f_x u + 2g_x v + f_{xx} = 0, \tag{36a}$$

$$\frac{\delta \mathcal{L}}{\delta v} = (D_t^\alpha)^* g + 2f_x + 2g_x u - g_{xx} = 0, \tag{36b}$$

where $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ are the Euler–Lagrange operators with respect to variables u and v, respectively, $(D_t^\alpha)^*$ is the adjoint operator to the Riemann–Liouville fractional differential operator D_t^α, defined by

$$(D_t^\alpha)^* = (-1)^n {}_tI_T^{n-\alpha}(D_t^n) = {}_t^C D_T^\alpha. \tag{37}$$

Here ${}_tI_T^{n-\alpha}$ and ${}_t^C D_T^\alpha$ are the right-sided fractional integral operator of order $n-\alpha$, and the right-sided Caputo fractional derivative operator of order α, respectively, see (2)–(5). Equation (1) is called *nonlinear self-adjoint* if the adjoint Equation (36) satisfy

$$\left.\frac{\delta \mathcal{L}}{\delta u}\right|_{\mathcal{E}} = \lambda_1 F_1 + \lambda_2 F_2, \tag{38a}$$

$$\left.\frac{\delta \mathcal{L}}{\delta v}\right|_{\mathcal{E}} = \lambda_3 F_1 + \lambda_4 F_2 \tag{38b}$$

for some

$$f = \phi(t,x,u,v), \quad g = \psi(t,x,u,v), \tag{39}$$

on the solution set $\mathcal{E}$ of Equation (1), where λ_i $(i = 1,2,3,4)$ are unknown coefficients to be determined. It is straightforward to set up and split the determining Equation (38) with f and g given by (39) into an overdetermined system for ϕ and ψ:

$$\phi_u - \lambda_1 = \phi_v + \lambda_2 = \psi_u + \lambda_3 = \psi_v - \lambda_4 = 0, \tag{40a}$$

$$v\psi_v + \lambda_1 = 0, \tag{40b}$$

$$\phi_u + \lambda_4 v = 0, \tag{40c}$$

$$v\psi_u + 2\lambda_1 u + \lambda_2 v = 0, \tag{40d}$$

$$\phi_v + 2\lambda_4 u + \lambda_3 = 0, \tag{40e}$$

$${}_t^C D_t^\alpha \phi + 2u\phi_x + 2v\psi_x + \phi_{xx} - \lambda_1 \partial_t^\alpha u + \lambda_1 \partial_t^\alpha v = 0, \tag{40f}$$

$${}_t^C D_t^\alpha \psi + 2\phi_x + 2u\psi_x - \phi_{xx} - \lambda_3 \partial_t^\alpha u - \lambda_4 \partial_t^\alpha v = 0. \tag{40g}$$

Solving system (40), we find

$$\phi = c_1, \quad \psi = c_2. \tag{41}$$

The nonlinear self-adjointness method proposed in [18] showed that the component of conserved vector could be determined by acting the so-called Noether operators to the formal Lagrangian

$\mathcal{L}$. Thus for the case when Riemann–Liouville fractional derivative is involved in Equation (1), the conserved density C^t and spacial flux C^x in (34) can be expressed explicitly as follows [22]

$$C^t = D_t^{\alpha-1}(W^1)\frac{\delta\mathcal{L}}{\delta(D_t^\alpha u)} + D_t^{\alpha-1}(W^2)\frac{\delta\mathcal{L}}{\delta(D_t^\alpha v)} + I\left(W^1, D_t\left(\frac{\delta\mathcal{L}}{\delta(D_t^\alpha u)}\right)\right) + I\left(W^2, D_t\left(\frac{\delta\mathcal{L}}{\delta(D_t^\alpha v)}\right)\right), \quad (42)$$

$$C^x = W^1\frac{\delta\mathcal{L}}{\delta u_x} + W^2\frac{\delta\mathcal{L}}{\delta v_x} + D_x(W^1)\frac{\delta\mathcal{L}}{\delta u_{xx}} + D_x(W^2)\frac{\delta\mathcal{L}}{\delta v_{xx}}, \quad (43)$$

where $W^1 = \eta - \xi u_x - \tau u_t$ and $W^2 = \zeta - \xi v_x - \tau v_t$ are the characteristic functions corresponding to the point symmetry generator $\mathrm{X} = \xi\partial_x + \tau\partial_t + \eta\partial_u + \zeta\partial_v$ and the integral $I(p,q)$ is defined by

$$I(p,q) = \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_t^T(\mu - s)^{-\alpha}p(s,x)q(\mu,x)d\mu ds. \quad (44)$$

The resulting conservation laws generated by all point symmetries admitted by Equation (1) in Theorem 1 are shown in Table 1. Observe that, in Table 1, $D_t^{\alpha-1} = I_t^{1-\alpha}$ holds for $0 < \alpha < 1$.

Table 1. Conservation laws for time-fractional Levi Equation (1).

Symm.	(W^1, W^2)	C^t	C^x
X_1	$(-u_x, -v_x)$	$-D_t^{\alpha-1}(u_x) - D_t^{\alpha-1}(v_x)$	$2(u+v)u_x + 2(u+1)v_x - u_{xx} + v_{xx}$
X_2	$\left(-tu_t - \frac{\alpha}{2}(xu_x + u), -tv_t - \alpha v - \frac{\alpha}{2}xu_x\right)$	$-\alpha x D_t^{\alpha-1}(u_x) - 2D_t^{\alpha-1}(tu_t) - \alpha D_t^{\alpha-1}v - \frac{\alpha}{2}D_t^{\alpha-1}u$	$(u+v)(\alpha u + \alpha x u_x + 2tu_t) + (1+u)(2\alpha v + \alpha x u_x + 2tu_t) - \alpha x u_{xx} - \alpha v_x - 2tu_{tx} - \frac{3\alpha}{2}u_x$

5. Invariant Subspace

Let us consider the time-fractional Levi Equation (1), and let us introduce the second order differential operators

$$F^1[u,v] = -u_{xx} + 2uu_x + 2v_x, \quad (45a)$$
$$F^2[u,v] = v_{xx} + 2uv_x + 2vu_x. \quad (45b)$$

Let

$$W_{n_q}^q = \mathfrak{L}\{f_1^q(x), \ldots, f_{n_q}^q(x)\} = \left\{\sum_{i=1}^{n_q} C_i^q f_i^q(x) \middle| C_i^q \in \mathbb{R}\right\} \quad (46)$$

be the space spanned by the linearly independent functions $f_1^q(x), \ldots, f_{n_q}^q(x)$, where $q = 1, 2$. The subspace $\mathcal{W} = W_{n_1}^1 \times W_{n_2}^2$ is said to be *invariant* under the vector differential operator (F^1, F^2), if $F^q[\mathcal{W}] \subseteq \mathcal{W}$, $(q = 1, 2)$, i.e.,

$$F^q\left[\sum_{i=1}^{n_1} C_j^1 f_i^1(x), \sum_{i=1}^{n_2} C_i^2 f_i^2(x)\right] = \sum_{i=1}^{n_q}\Psi_i^q(C_1^1, \ldots, C_{n_1}^1, \ldots, C_1^m, \ldots, C_{n_m}^m)f_i^q(x), \quad (47)$$

where Ψ_i^q, $(i = 1, \ldots, n_q)$ are functions to be determined. Thus the Equation (1) has a solution of the form

$$(u,v) = \left(\sum_{i=1}^{n_1} C_i^1(t)f_i^1(x), \sum_{i=1}^{n_2} C_i^2(t)f_i^2(x)\right), \quad (48)$$

if the coefficients $\{C_i^q(t)\}$ satisfy a system of fractional ODEs

$$D_t^\alpha C_i^q(t) = \Psi_i^q(C_1^1, \ldots, C_{n_1}^1, \ldots, C_1^m, \ldots, C_{n_m}^m), \quad i = 1, \ldots, n_q, \quad q = 1, 2. \quad (49)$$

Suppose that the space $W^q_{n_q} = \mathfrak{L}\{f^q_1(x), \ldots, f^q_{n_q}(x)\}$ is the solution space of a linear n_q-order ODE

$$L^q[y_q] \equiv y_q^{(n_q)} + a^q_{n_q-1}(x) y_q^{(n_q-1)} + \cdots + a^q_1(x) y'_q + a^q_0(x) y_q = 0, \quad q = 1, 2. \tag{50}$$

The invariance condition of the subspace $\mathcal{W}$ with respect to (F^1, F^2) is thereby given by

$$L^q[F^q]\big|_{[H^1] \cap [H^2]} \equiv 0, \quad q = 1, 2, \tag{51}$$

where $[H^1]$ denotes the solution set of equation $L^1[u] = 0$ and its differential consequences with respect to x, and $[H^2]$ denotes the solution set of equation $L^2[v] = 0$ and its differential consequences with respect to x.

To proceed, let us consider the maximal dimension of invariant subspace $\mathcal{W}$ for operator (F^1, F^2). Such a dimension can be straightforwardly determined by the results in [28]. Without loss of generality, we just consider the case of $n_1 \geq n_2 > 0$. The required conditions on n_1 and n_2 are thus given by

$$n_1 - n_2 \leq 2, \quad n_1 \leq 9. \tag{52}$$

Solving these conditions in (52), we find all possible cases for (n_1, n_2):

$$\begin{aligned}(n_1, n_2) =& (9,9),\ (9,8),\ (9,7),\ (8,8),\ (8,7),\ (8,6),\ (7,7),\ (7,6),\ (7,5),\ (6,6),\ (6,5),\ (6,4),\\ & (5,5),\ (5,4),\ (5,3),\ (4,4),\ (4,3),\ (4,2),\ (3,3),\ (3,2),\ (3,1),\ (2,2),\ (2,1),\ (1,1).\end{aligned} \tag{53}$$

For each case in (53), the invariance condition (51) with the operator L^q given by (50) yields an overdetermined system, which can be solved in a direct way. As a consequence, we have the following classification result.

Theorem 2.

(i) *The operators $F^1[u, v]$ and $F^2[u, v]$ in (45) admit no invariant subspace, if (n_1, n_2) is an element of the list (53) with $n_1 \geq 2$.*

(ii) *When $(n_1, n_2) = (1, 1)$, the operators $F^1[u, v]$ and $F^2[u, v]$ in (45) admit two invariant subspaces, given by*

$$W^1_1 \times W^2_1 = \mathfrak{L}\{1\} \times \mathfrak{L}\{1\}; \tag{54}$$

$$W^1_1 \times W^2_1 = \mathfrak{L}\{x\} \times \mathfrak{L}\{1\}. \tag{55}$$

5.1. Invariant Subspace $W^1_1 \times W^2_1 = \mathfrak{L}\{1\} \times \mathfrak{L}\{1\}$

In this case, Equation (1) has a solution

$$u = C_1(t), \quad v = C_2(t). \tag{56}$$

Therefore, in view of (49), Equation (1) reduces to the following system of fractional ODEs for coefficients $\{C_1(t), C_2(t)\}$:

$$D^\alpha_t C_1(t) = 0, \tag{57a}$$

$$D^\alpha_t C_2(t) = 0, \tag{57b}$$

which has only one trivial solution $(C_1(t), C_2(t)) = (0, 0)$. Thus we get a trivial solution of the time-fractional Levi Equation (1), that is, $u = v = 0$.

5.2. Invariant Subspace $W_1^1 \times W_1^2 = \mathfrak{L}\{x\} \times \mathfrak{L}\{1\}$

In this case, Equation (1) has a solution

$$u = C_1(t)x, \quad v = C_2(t). \tag{58}$$

Therefore, in view of (49), Equation (1) reduces to the following system of fractional ODEs for coefficients $\{C_1(t), C_2(t)\}$:

$$D_t^\alpha C_1(t) = 2C_1^2(t), \tag{59a}$$
$$D_t^\alpha C_2(t) = 2C_1(t)C_2(t). \tag{59b}$$

The symmetry group method will now be applied to obtain explicit solutions of system (59). In analogy with the analysis outlined in Section 3, the corresponding determining equations for point symmetries of system (59) can be set up and solved, which yield the following result.

Proposition 1. *System (59) admits four point symmetries*

$$Y_1 = t\partial_t - \alpha C_1 \partial_{C_1}, \tag{60a}$$
$$Y_2 = C_1 \partial_{C_2}, \tag{60b}$$
$$Y_3 = C_2 \partial_{C_2}, \tag{60c}$$
$$Y_4 = t^2\partial_t - (2/3)tC_1\partial_{C_1} - (2/3)tC_2\partial_{C_2}, \quad \textit{for } \alpha = 1/3. \tag{60d}$$

The corresponding transformation groups acting on a solution $(C_1(t), C_2(t))$ of system (59) are given by

$$(\hat{C}_1(t), \hat{C}_2(t)) = \left(\lambda^{-\alpha} C_1(\lambda^{-1}t), C_2(\lambda^{-1}t)\right), \tag{61a}$$
$$(\hat{C}_1(t), \hat{C}_2(t)) = \left(C_1(t), C_2(t) + \epsilon C_1(t)\right), \tag{61b}$$
$$(\hat{C}_1(t), \hat{C}_2(t)) = \left(C_1(t), \lambda C_2(t)\right), \tag{61c}$$
$$(\hat{C}_1(t), \hat{C}_2(t)) = \left((1+\epsilon t)^{-2/3} C_1(t/(1+\epsilon t)), (1+\epsilon t)^{-2/3} C_2(t/(1+\epsilon t))\right), \quad \textit{for } \alpha = 1/3, \tag{61d}$$

with group parameters $-\infty < \epsilon < +\infty$ and $0 < \lambda < +\infty$.

The scaling symmetry generator (60a) has invariants $C_1 = k_1 t^{-\alpha}$ and $C_2 = k_2$. By considering the system (59) and properties (7), we get

$$k_1 = \frac{\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)}, \quad k_2 = 0, \tag{62}$$

which gives us the corresponding group-invariant solution of system (59)

$$C_1(t) = \frac{\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)} t^{-\alpha}, \quad C_2(t) = 0. \tag{63}$$

Applying now transformation (61b) corresponding to symmetry Y_2 to the solution (63), we obtain an additional new solution for system (59)

$$C_1(t) = \frac{\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)} t^{-\alpha}, \quad C_2(t) = c_1 \frac{\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)} t^{-\alpha}, \tag{64}$$

where and whereafter c_i, $i = 1, 2$, are arbitrary constants. Hence, Equation (1) has the following solution

$$u(t,x) = \frac{\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)} t^{-\alpha} x, \tag{65a}$$

$$v(t,x) = \frac{c_1\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)} t^{-\alpha}. \tag{65b}$$

In the case of $\alpha = 1/3$, we apply the transformation (61d) to the solution (64), which yields one additional solution of system (59)

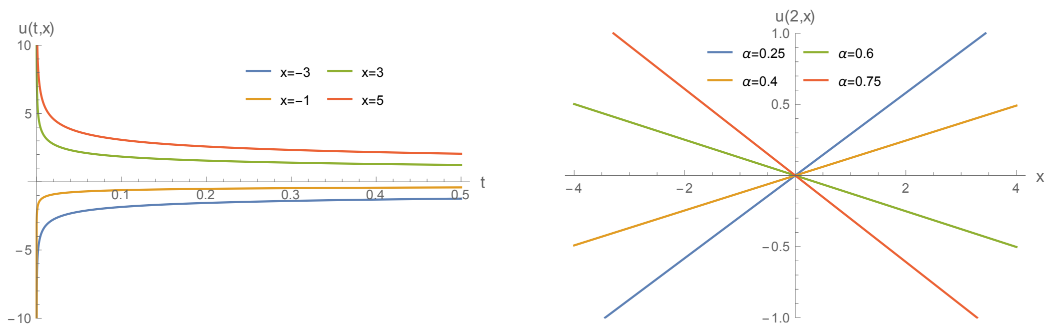
$$C_1(t) = \frac{\Gamma(2/3)}{2\Gamma(1/3)} t^{-1/3}(1+c_2 t)^{-1/3}, \quad C_2(t) = \frac{c_1\Gamma(2/3)}{2\Gamma(1/3)} t^{-1/3}(1+c_2 t)^{-1/3}. \tag{66}$$

Then we get a solution of Equation (1) with $\alpha = 1/3$:

$$u(t,x) = \frac{\Gamma(2/3)}{2\Gamma(1/3)} t^{-1/3}(1+c_2 t)^{-1/3} x, \tag{67a}$$

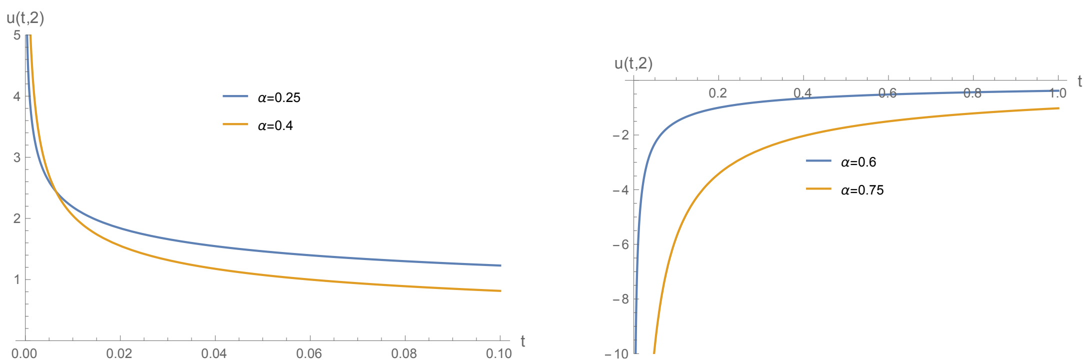
$$v(t,x) = \frac{c_1\Gamma(2/3)}{2\Gamma(1/3)} t^{-1/3}(1+c_2 t)^{-1/3}. \tag{67b}$$

Solutions (65) and (67) are separable regular solutions, which are dispersive as $t \to \infty$. In both of these two solutions, $u(t,x)$ is unbounded as $x \to \infty$. Interestingly, solution (67) has a blow-up for $t \to T$ with $0 < T = -c_2^{-1} < \infty$. In view of the relation between $u(t,x)$ and $v(t,x)$ in (65) and (67), we only sketch the behaviors of $u(t,x)$ in Figures 1–3.



(**a**) (**b**)

Figure 1. Behaviors of u in solution (65) with (**a**) $\alpha = 0.25$ and (**b**) $t = 2$.



(**a**) (**b**)

Figure 2. Behaviors of u in solution (65) with $x = 2$ and (**a**) $0 < \alpha < 0.5$, (**b**) $0.5 < \alpha < 1$.

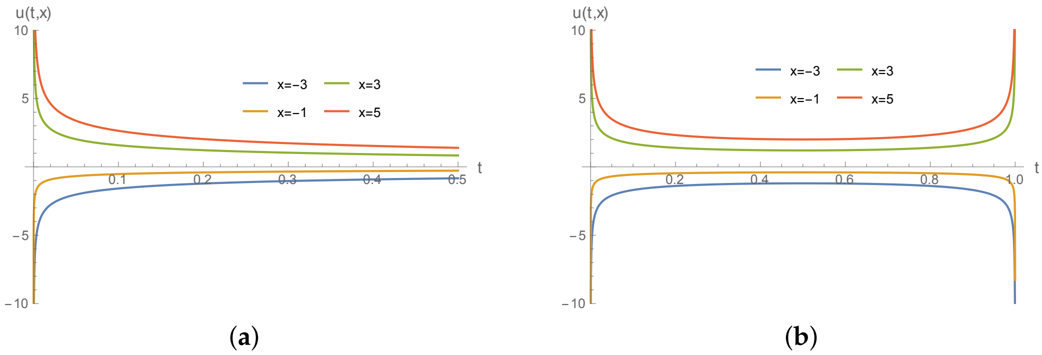


Figure 3. Behaviors of u in solution (67) with (**a**) $c_2 = 1$ and (**b**) $c_2 = -1$.

6. Conclusions

In this paper, we investigated the time-fractional Levi Equation (1) with Riemann–Liouville fractional derivative. Firstly, we found that Equation (1) admits two point symmetries, under which Equation (1) is reduced to a system of fractional ODEs. Secondly, we used the nonlinear adjointness method to list the conservation laws of Equation (1) in terms of the resulting point symmetries. Finally, we studied the invariant subspaces of Equation (1). Out of the 24 possible dimensions, Equation (1) admits two (1,1)-dimensional invariant subspaces. Then Equation (1) is reduced to a system of fractional ODEs, which are solved again by symmetry group method. As a result, we derived two explicit solutions of Equation (1). In the near future, we plan to generalize the group-foliation method [32] to FDEs, which will allow us to find more exact solutions to Equation (1) can be given out.

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