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Viscous Matter in FRW Cosmology

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Abstract: We investigate the dynamics of dust matter with bulk viscosity effects. We explored the analogy dynamical problem to Chaplygin gas. Due to this analogy we give exact solutions for the FRW cosmology with viscosity coefficient parameterized by the Belinskii–Khalatnikov power law dependence with respect to energy density. These exact solutions are given in the form of hypergeometrical functions. We proved simple theorem which illustrated as viscosity effects can solved the initial singularity problem present in standard cosmological model.

Keywords: cosmological model; bulk viscosity; Chaplygin gas; dynamical systems

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1. Introduction

The viscous fluid is an important subject of study in fluid mechanics [1], but starting from the 1960s this idea becomes attractive in cosmology, too. Misner was the first to study the role of viscosity during the cosmic evolution [2,3]. Effects of viscosity were also investigated by Weinberg [4] and Nightingale [5]. Murphy studied bulk viscosity effects in FRW cosmology and demonstrated that the constant bulk viscosity can solve the problem of an initial singularity in the FRW cosmology [6]. Effects of bulk viscosity can naturally remove the initial singularity. Different possibilities how the bulk viscosity effects give rise to avoiding the initial singularity were demonstrated by Heller et al. [7]. Belinskii et al. [8] studied dynamical effects of bulk viscosity, introduced a non-constant coefficient of bulk viscosity parameterized by energy density in a power-law relation, and developed dynamical system methods for study different evolutional paths in the phase space. This paper was very important from the theoretical point of view because effect of bulk viscosity was investigated beyond the constant coefficient of bulk viscosity and the language of dynamical systems opened the discussion how dynamical effects such as a singularity avoiding can depend on initial conditions. In cosmology we do not know the initial conditions for the universe, so we should study all evolutional paths admissible for all initial conditions.

In this paper, we return to these pioneering papers on bulk viscosity (see also [9]) in order to investigate the problem of bulk viscous cosmology in the modern context. In contemporary cosmology the effect of acceleration of the current Universe is investigated in terms of the standard cosmological model or ΛCDM model. In this model role of substantial dark energy assumes the cosmological constant parameter Λ. In the present epoch this parameter is negligible small as we know from astronomical data. Why is it not zero? it is the reflection of the cosmological constant parameter. We can describe exactly a current state of the Universe but we cannot explain why the value of the cosmological constant is so small. Another difficulty of the ΛCDM model is the explanation why density parameters for dark energy and matter are comparable? This problem is known as the coincidence problem. For the state-of-the-art review of investigation of viscous cosmology see comprehensive reviews by Bamba et al. [10] and Brevik et al. [11]. The contemporary studies of viscous cosmological models are related to investigation of inflation [12], dark matter and dark energy problem, singularities etc. it can be also mentioned that the cosmological models with running Lambda maybe interesting [13].

In the present paper, we study dynamics of FRW Universe filled by dark matter, which is viscous. The idea of Chaplygin gas is attractive [14] because in the cosmological context it unifies effectively both dark matter (pressure) and dark energy (the parameter Λ) [15,16]. We show that in the same way Chaplygin gas unifies dark matter with viscosity. Obtained in such a way fluid with negative pressure we called viscous dark mater. We assume that the viscosity coefficient is parameterized by the Belinskii–Khalatnikov power law form but in a realistic case the parameterization can depend on the cosmological epoch [17].

The dynamical effects of presence of dust matter in the FRW universe is a main subject of the paper. We discuss theoretical aspects of an extended equation of state (including bulk viscosity effects). There is a strict relation between the equation of state and symmetries of dynamical equation because symmetries of the FRW equation enforce the form of the equation of state [18]. For example, the FRW equation with viscosity parameterized by $\rho^{1/2}$ and the equation of state $p = w\rho$ admit scaling symmetries and vice versa scaling symmetries enforce the form of the equation of state as well as the parameterization of the coefficient of the equation of state.

2. Bulk Viscosity in FRW Cosmology

The energy-momentum tensor in the present of bulk viscosity was given by Landau and Lifshitz [1]

$$T_i^j = (\rho + p)v_i v^j + p g_i^j - \xi\theta(g_i^j + v_i v^j) \tag{1}$$

where ρ, p and ξ are the energy density, pressure and bulk viscous coefficient, respectively, and v_i is the flow vector vector satisfying the relation $g_{ij}v^i v^j = -1$.

The basic dynamical equations which govern the FRW dynamics are

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p_{\text{eff}}) \tag{2}$$

$$\dot{\rho} = -3H(\rho + p_{\text{eff}}). \tag{3}$$

In Equation (3) $H = d(\ln a)/dt$ is the Hubble parameter, ρ_{eff} and p_{eff} are effective energy density and pressure

$$\rho_{\text{eff}} = \rho \tag{4}$$

and

$$p_{\text{eff}} = p - 3\xi(\rho)H \tag{5}$$

where $\xi(\rho) \sim \rho^m$, $m = \text{const}$ is the bulk viscosity coefficient in the correction term $-3\xi(\rho)H$ to the effective pressure.

$$p = w\rho, \qquad w = \text{const} \tag{6}$$

is the form of the equation of state for dust matter.

Equation (3) is the conservation condition. Equations (2) and (3) admit the first integral in the form

$$\rho - 3H^2 = 3k/a^2 \tag{7}$$

where $k = 0, +1, -1$ is curvature constant, $a(t)$ is the scale factor from the Robertson-Walker metric space of constant curvature.

For simplicity let us consider flat model $k = 0$. Then

$$\rho = 3H^2 = 3(da/dt)^2/a^2. \tag{8}$$

Now the effective pressure can be rewritten to the form

$$p_{\text{eff}} = 0 - 3\alpha\rho^m H = -\frac{A}{\rho^{-m-\frac{1}{2}}} \tag{9}$$

where A is an arbitrary positive constant and we denote

$$\alpha = -\left(m + \frac{1}{2}\right) \quad \text{or} \quad 1 + \alpha = \frac{1}{2} - m. \tag{10}$$

This form equation of state is known as the generalized Chaplygin gas [19]. In the special case of $\alpha = 1$ we obtain the equation of state for Chaplygin gas. This form of matter is important in the context of explanation for the acceleration of the current Universe in terms of substantial matter which unifies dark matter with dark energy in the form of cosmological constant (the quartessence idea).

Therefore matter with viscosity satisfies an analogous equation to the generalized Chaplygin gas.

Using conservation condition (3) and assuming that $\rho(t) = \rho(a(t))$ we obtain

$$\rho(a) = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}. \tag{11}$$

where A, B are arbitrary constant which are useful to parameterize through parameter $A = A_s$, $B = 1 - A_s$ and parameter ρ_0. Energy density $\rho(a_0 = 1) = \rho_0 = [A_s + (1 - A_s)]^{1/(1+\alpha)}$. We assume A_s is positive, $c^2_{s,0} = \alpha A_s$, $\alpha = -\frac{1}{2} - m$.

It is possible to find the exact solution for viscous fluid in the form of $t(a)$ dependence

$$a^{\frac{3}{2}}{}_2F_1\left(\frac{1}{1-2m}, \frac{1}{1-2m}, \frac{2-2m}{1-2m}, -\frac{A_S}{1-A_S}a^{3\left(\frac{1}{2}-m\right)}\right) = \frac{\sqrt{3}}{2}(1 - A_s)^{\frac{1}{1-2m}}t. \tag{12}$$

We considered $w = 0$ dust viscous matter and vanishing cosmological parameter Λ.

The exact solution for the flat model filled by Chaplygin gas was found by Gorini et al. [20]. This case corresponds to $m = -3/2$.

In the same model, Chaplygin gas is a candidate for dark energy [15,16,21]. One can consider viscous fluid as natural explanation for acceleration of the current Universe.

Finally, the dimensionless density parameter for viscous fluid assumes the following form

$$\Omega_{\text{visc}} = \Omega_{\text{visc},0}\left(A_s + \frac{1 - A_s}{a^{3(1+w)\left(\frac{1}{2}-m\right)}}\right)^{\frac{1}{\frac{1}{2}-m}} \tag{13}$$

where $\Omega_{\text{visc},0}$ is the density parameter defined in the present epoch ($a = a_0 = 1$). Please note that because $c^2_s =, 1 + \alpha = -m$ (if $\alpha < 0$ velocity of sound $c^2_s = dp/d\rho$ becomes a complex number).

In a similar way, Chaplygin gas has an interpretation as a scalar field with the potential with energy density $\rho_\phi = (d\phi/dt)^2 + V(\phi)$ and pressure $p_\phi = d\phi/dt)^2 - V(\phi)$ the viscous dust fluid can possesses interpretation as some a self-interacting scalar field with the potential $V(\phi)$ [20]

$$\phi(a) = \frac{2}{\sqrt{3}\left(\frac{1}{2} - m\right)}\sinh^{-1}\left(\sqrt{\frac{1-A_s}{A_s}}\frac{1}{a^{\frac{3}{2}\left(\frac{1}{2}-m\right)}}\right) \tag{14}$$

$$V(\phi) = \frac{1}{2}(1 - A_s)^{\frac{2}{1-2m}}\cosh^{\frac{4}{1-2m}}\left(\frac{\sqrt{3(1-2m)}}{4}\phi\right) + \frac{1}{2}A_s^{\frac{2}{1-2m}}\cosh^{\frac{4}{1-2m}}\left(\frac{\sqrt{3(1-2m)}}{4}\phi\right). \tag{15}$$

Models with Chaplygin gas was tested by astronomical data (for example see [22]) and this study favors the values of the parameter m for which velocity of sound at present epoch does not exceeds velocity of light c. Note in the viscous dust fluid case $c^2_s = -(1/2 + m)A_s$. This condition is satisfied if $0 < A_s < 1$. In addition, such an interval for the parameter $0 < \alpha < 1$ (or $m \in (-3/2, -1/2)$) is favored by astronomical data. In this interval effects of viscosity are negligible in an early universe and are important for late stages of the evolution of the universe. The latest attempt to put constraints on cosmological models with the viscosity effects was made by Odintsov et al. [23].

3. FRW Dynamics as Motion of a Particle in the Potential Well

The Friedmann first integral can be rewritten to the form analogous to the motion of a particle of unit mass in the potential [24], namely

$$\left(\frac{da}{dt}\right)^2 + V(a) = E = -\frac{k}{2} \tag{16}$$

where

$$V(a) = -\frac{1}{6}\rho_{\text{eff}}a^2 \tag{17}$$

and E is the total energy preserved during the motion. Therefore, in a configuration space the motion of the system takes place over levels of constant energy depending on the curvature constant (or density parameter for curvature).

Due to this analogy, one can think about the evolution of the Universe as the motion of particles which mimics the evolution of the universe because the state variable is the scale factor. Therefore, the model can be identified by a potential form as a function of the scale factor.

The corresponding dynamical system assumes the form of a dynamical system of the Newtonian type

$$\frac{d^2a}{dt^2} = -\text{grad}V(a). \tag{18}$$

There is a large class of models of modern cosmology formulated in terms of the particle-like description (see Table 1).

The universe accelerates for some intervals of the scale factor (or redshift) if $V(a)$ is $V(a)$ is a decreasing function of the scale factor.

In Table 1 we complete potential forms of the theoretical models which offer the possibility of the explanation of acceleration of the Universe in the substantial dark energy or modified gravity.

Table 1. Cosmological models with a substantial form of dark energy and with modified gravity. The potential $V(a)$ is obtained from $V(a) = -H^2a^2/2$; $1 + z = a^{-1}$.

Type	Cosmological Models
models with substantial dark energy	**ΛCDM** $w_{\mathbf{X}} = -1$ $\frac{H^2(z)}{H_0^2} = \Omega_{\text{m},0}(1+z)^3 + (1-\Omega_{\text{m},0})$
	constant E.Q.S. $w_{\mathbf{X}} = w_0 < -1$ $\frac{H^2(z)}{H_0^2} = \Omega_{\text{m},0}(1+z)^3 + (1-\Omega_{\text{m},0})(1+z)^{3(1+w_0)}$
	dynamic E.Q.S. $w_{\mathbf{X}} = w_0 + w_1(1-a)$ $\frac{H^2(z)}{H_0^2} = \Omega_{\text{m},0}(1+z)^3 + (1-\Omega_{\text{m},0})(1+z)^{3(w_0+w_1+1)}\exp[-\frac{3w_1 z}{1+z}]$
	quintessence $\bar{w}_{\mathbf{X}}(a) = \int w_{\mathbf{X}}(a)\mathbf{d}(\ln a)/\int \mathbf{d}(\ln a) \equiv w_0 a^{\alpha}$ $\frac{H^2(z)}{H_0^2} = \Omega_{\text{m},0}(1+z)^3 + (1-\Omega_{\text{m},0})(1+z)^{3(1+w_0(1+z)^{-\alpha})}$
	oscillating E.Q.S. $w_{\mathbf{X}}(z) = -1 + (1+z)^3[C\cos(\ln(1+z))]$ $\frac{H^2(z)}{H_0^2} = \Omega_{\Lambda,0}\exp\left((1+z)^3 D_2\cos(\ln(1+z))\right) + \Omega_{\text{m},0}(1+z)^3$

Table 1. *Cont.*

Type	Cosmological Models
models with modified gravity	**interacting DE & DM** $\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{int},0}(1+z)^n + 1 - \Omega_{\mathrm{m},0} - \Omega_{\mathrm{int},0}$
	bounce ΛCDM $\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{m},0}(1+z)^3 - \Omega_{n,0}(1+z)^n + 1 - \Omega_{\mathrm{m},0} + \Omega_{n,0}$
	Cardassian $\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{r},0}(1+z)^4 + \Omega_{\mathrm{m},0}(1+z)^4 \left[\frac{1}{1+z} + (1+z)^k \left(\frac{\Omega_{\mathrm{C},0}}{\Omega_{\mathrm{m},0}}\right) E(z)\right]$
	DGP $\frac{H^2(z)}{H_0^2} = \left[\sqrt{\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{rc},0}} + \sqrt{\Omega_{\mathrm{rc},0}}\right]^2, \quad \Omega_{\mathrm{rc},0} = \frac{(1-\Omega_{\mathrm{m},0})^2}{4}$
	Sahni-Shtanov brane I $\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\sigma,0} + 2\Omega_{l,0} - 2\sqrt{\Omega_{l,0}}P(z)$

If we try to generalize (18) by including dissipation effects then we should add an additional term to the right hand sides in the simplest case proportional to velocity (da/dt for the model under consideration). The corresponding models with the bulk viscosity coefficient in the Belinskii–Khalalatnikov parameterization assumes the form of an autonomous two-dimensional dynamical system, which possesses different attractors.

The phase portrait the dynamical system representing the cosmological model with Chaplygin gas is shown in Figure 1 (drawn in XPPAUT [25]). The structure of the phase space of this model is equivalent to the structure of the phase space of the ΛCDM model. Any cosmological model which competes with the standard cosmological model, i.e., the ΛCDM model should have the same structure of the phase space. Moreover, this kind of phase space structure guarantees the structural stability of models. The notion of structural stability seems to be important in the context of a realistic description of the universe [26].

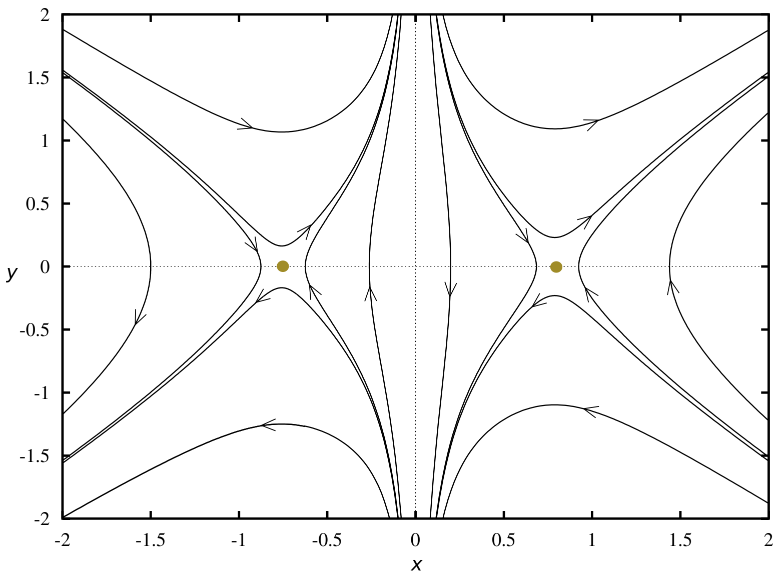

Figure 1. The phase portrait of the cosmological model with Chaplygin gas ($m = -3/2$) where $x = a$, $y = da/dt$. We postulate additionally the presence of non-interacting pressureless baryonic matter. Please note that the phase portrait is structurally stable and is topologically equivalent to the ΛCDM one.

4. Viscous Dark Energy Models—Dark Energy Models with Dissipation

In this section, we reached the class of cosmological models whose dynamics is represented by system (18) by adding bulk viscosity effect. For simplicity we assume that the model is flat and the viscosity parameter is constant.

Then motion of the system is over zero energy level and the Friedmann equation can be integrated by quadratures

$$t = \phi(a) = \pm \int_0^a \frac{da'}{\sqrt{-2V(a')}}. \tag{19}$$

One can prove the following theorem for simplest generalization of these systems in the case with bulk viscosity dissipation. Please note that now we admit the coefficient equation of state dependent on the scale factor $\rho = \rho(a)$ and $p_{\text{eff}} = p(a) - \alpha H$.

Theorem 1. *If $t = \phi(a)$ is solution of (19) for the scale factor dependence in conservative cosmology, then*

$$\exp(\alpha t(a)) = \phi(a) \tag{20}$$

is a solution dynamical problem with constant viscosity coefficient.
The scale factor for the model under consideration satisfies the generalized equation of Newtonian type with friction which describes bulk viscosity effect

$$\frac{d^2 a}{dt^2} = -\frac{\partial V}{\partial a} + \alpha \frac{da}{dt}. \tag{21}$$

Please note that in general the bulk viscosity parameter can be parameterized by the scale factor in consequence of dependence of energy density on the scale factor.

Proof. Let us differentiate both sides of (20) remember that $t = t(a)$. Then we obtain

$$\alpha \exp(\alpha t(a)) \frac{dt}{da} = (-2V(a))^{-1/2}.$$

Thus

$$\frac{1}{\alpha^2} \left(\frac{da}{dt} \right)^2 = \pm(-2V(a)) \exp(\alpha t(a)).$$

Now after differentiating both sides of above formula over time we obtain the equation for a second derivative of the scale factor in the form

$$\frac{d^2 a}{dt^2} = -\alpha^2 \exp(2\alpha t) \text{grad} V(a) + \alpha \frac{da}{dt}.$$

After introducing new time variable τ: $dt/\psi(t) = d\tau$ we can find such a function ψ which form

$$\psi = \pm \alpha^{-1} \exp(-\alpha t)$$

guarantees dropping the dissipative term with da/dt and reducing the equation to the corresponding form in the conservative cosmology

$$\frac{d^2 a}{dt^2} = -\text{grad} V(a).$$

□

This theorem has a simple application if we have the exact solution for case case of conservative cosmology then the solution with bulk viscosity can be obtained by taking $\ln \phi(a)$. Equivalently, if we replace the cosmic time t by the new time parameter $\tau = 1/\alpha \ln t$, $\tau = \phi(a(\tau))$ will be a solution with bulk viscosity. The parameter τ can be interpreted as the Milne atomic time [27]. Finally, if we replace the cosmological time in solutions of conservative cosmology by the Milne atomic time $\tau = 1/\alpha \ln t$, then we obtain exact solution with bulk viscosity.

Let us consider an anatomy of the mechanism for avoiding the initial singularity. If we start from some solution without bulk viscosity (conservative cosmology), then a generic property of its solutions is the presence of an initial singularity and generic asymptotic near this state. If the initial singularity is matter dominated, then the scale factor behave near the initial singularity following the Einstein-de Sitter solution $a(t) = (t - t_0)^{2/3}$, where t_0 is the moment when the metric is singular. Therefore, inverting this solution we have

$$t - t_0 = a(t)^{3/2}$$

and corresponding solution for viscous matter is

$$t - t_0 = \frac{1}{\alpha}\frac{3}{2}\ln(a(t)).$$

Therefore, as the scale factor goes to zero, time t is shifted to minus infinity. The scale factor behaves quasi-exponential and we obtain inflation phase for any constant coefficient of viscosity. This situation is illustrated in Figure 2 where we compare the Einstein-de Sitter model with dust matter ($a(t) = t^{2/3}$) and the corresponding solution for viscous dust matter $a(t) = \exp(3/2\alpha t)$.

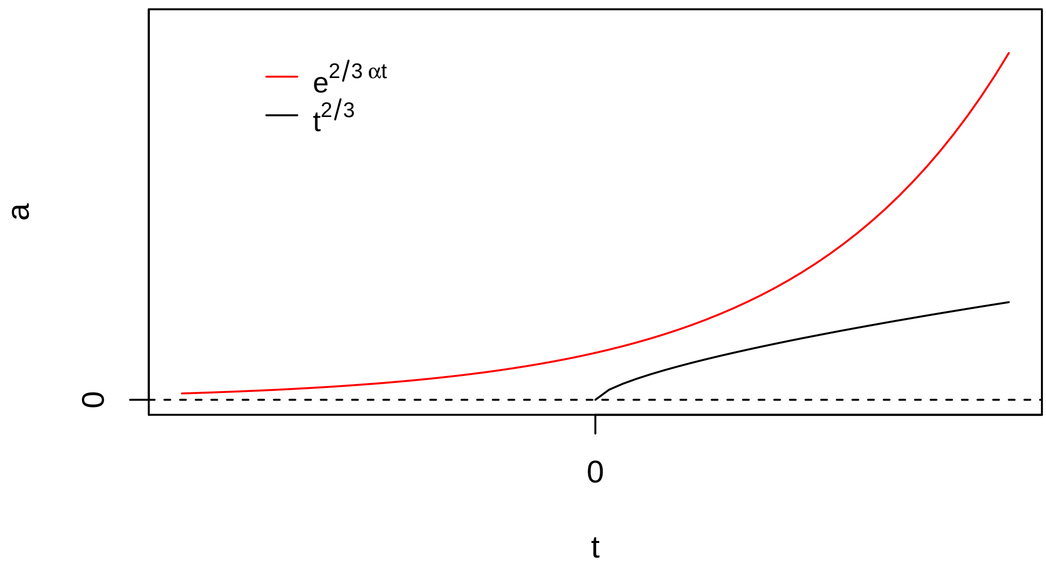

Figure 2. The illustration of a generic mechanism of avoiding the initial singularity in the model with viscous matter and without the cosmological parameter Λ. This mechanism can be interpreted in terms of the Milne atomic time.

5. Conclusions

In this theoretical paper, we explored an analogy of viscous FRW models with the FRW equation with Chaplygin gas. We have founded that both models are dual in this sense that there exist a 1-1 correspondence of both models. This opens the possibility to apply different analogies between them.

We introduced bulk viscosity to the dark energy models and then bulk viscosity effects are described by dynamical system of Newtonian type with friction (in principle anti-friction because energy is created in the comoving volume).

We proved the simple theorem which gives us possibility to obtain new solutions with constant bulk viscosity if we know solutions without bulk viscosity. This theorem shows explicitly how the initial singularity problem is solved by bulk viscosity effects, namely the initial singularity which is a generic property of conservative cosmology is removed to minus infinity. it is a generic effect of generation inflation phase in dark energy models. Therefore due to viscosity we realize the quintessence inflation idea.

In the paper, we showed how the initial singularity can be removed after including effect of viscous matter. It is shifted to minus infinity and the matter dominated singularity is replaced by a quasi de-Sitter solution analogous to the case found in the famous Starobinski's paper [28]. This solution plays a very important role in the context of inflation mechanism. The same character has the exact solution obtained in our paper.

We demonstrate that for $0 < A_s < 1$ and $m < 1/2$ the cosmological model with bulk viscosity is structurally stable and in the finite domain of the phase space its phase space structure is equivalent to the phase space structure of the ΛCDM model.

The results presented in the paper have an application to the broad class of FRW models with dark energy.

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