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Abstract: In this article, we propose a new modified extragradient-like method to solve pseudomonotone equilibrium problems in real Hilbert space with a Lipschitz-type condition on a bifunction. This method uses a variable stepsize formula that is updated at each iteration based on the previous iterations. The advantage of the method is that it operates without prior knowledge of Lipschitz-type constants and any line search method. The weak convergence of the method is established by taking mild conditions on a bifunction. In the context of an application, fixed-point theorems involving strict pseudo-contraction and results for pseudomonotone variational inequalities are considered. Many numerical results have been reported to explain the numerical behavior of the proposed method.

Keywords: pseudomonotone bifunction; convex optimization; equilibrium problems; variational inequality problems; weak convergence

1. Introduction

Let \( C \) be a nonempty, closed and convex subset of a real Hilbert space \( \mathcal{H} \) and \( \mathcal{R}, \mathcal{N} \) be the sets of real numbers and natural numbers, respectively. Assume that \( f \) is a bifunction \( f : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \) and \( EP(f, C) \) denotes the solution set of an equilibrium problem over the set \( C \). Now, consider the following definitions of a bifunction monotonicity (see [1,2] for more details). A function \( f : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \) on \( C \) for \( \gamma > 0 \) is said to be:

1. \( \gamma \)-strongly monotone if
   \[ f(z_1, z_2) + f(z_2, z_1) \leq -\gamma \| z_1 - z_2 \|^2, \quad \forall z_1, z_2 \in C; \]
(2) **monotone if**
\[
f(z_1, z_2) + f(z_2, z_1) \leq 0, \forall z_1, z_2 \in C;
\]

(3) **γ-strongly pseudomonotone if**
\[
f(z_1, z_2) \geq 0 \implies f(z_2, z_1) \leq -\gamma \|z_1 - z_2\|^2, \forall z_1, z_2 \in C;
\]

(4) **pseudomonotone if**
\[
f(z_1, z_2) \geq 0 \implies f(z_2, z_1) \leq 0, \forall z_1, z_2 \in C.
\]

It is clear from the definitions mentioned above that they have the following consequences:
\[
(1) \implies (2) \implies (4) \text{ and } (1) \implies (3) \implies (4).
\]

In general, the converses are not true. A bifunction \( f : H \times H \to \mathbb{R} \) is said to be Lipschitz-type continuous on \( C \) if there exist two positive constants \( c_1, c_2 \) such that
\[
f(z_1, z_3) \leq f(z_1, z_2) + f(z_2, z_3) + c_1 \|z_1 - z_2\|^2 + c_2 \|z_2 - z_3\|^2, \forall z_1, z_2, z_3 \in C.
\]

Let \( C \) be a nonempty closed convex subset of \( H \) and \( f : H \times H \to \mathbb{R} \) be a bifunction with \( f(z_1, z_1) = 0 \), for all \( z_1 \in C \). An equilibrium problem \([1,3]\) for \( f \) on the set \( C \) is to
\[
\text{find } u^* \in C \text{ such that } f(u^*, z_1) \geq 0, \forall z_1 \in C. \tag{1}
\]

An equilibrium problem \((1)\) had many mathematical problems as a particular case, i.e., the variational inequality problems (VIP), optimization problems, fixed point problems, complementarity problems, the Nash equilibrium of non-cooperative games, saddle point problems and the vector optimization problem (for details see [1,4,5]). The equilibrium problem is also known as the famous Ky Fan inequality [3]. However, the particular format of an equilibrium problem \((1)\) was initiated by Muu and Oettli [6] in 1992 and further investigation on its theoretical properties were provided by Blum and Oettli [1].

The construction of new iterative schemes and the modification of existing methods, as well as the study their convergence analysis, constitute an important research direction in equilibrium problem theory. Several methods have been developed in the past few years to approximate the solution of an equilibrium problem in finite and infinite dimensional real Hilbert spaces, i.e., extragradient methods [7–16], subgradient methods [17–22], inertial methods [23–25] and methods for particular classes of equilibrium problems [26–35].

In particular, a proximal method [36] was used to solve equilibrium problems based on solving minimization problems. This approach was also known as the two-step extragradient-like method in [7] due to the early contribution of the Korpelevich [37] extragradient method to solve the saddle point problems. More precisely, Tran et al. introduced a method in [7], and an iterative sequence \( \{u_n\} \) was generated as follows:

\[
\begin{cases}
u_n = \arg \min \{\lambda f(u_n, v) + \frac{1}{2}\|u_n - v\|^2 : v \in C\}, \\
u_{n+1} = \arg \min \{\lambda f(v_n, v) + \frac{1}{2}\|u_n - v\|^2 : v \in C\},
\end{cases}
\]

where \( 0 < \lambda < \min \left\{ \frac{1}{c_1}, \frac{1}{c_2} \right\} \). The iterative sequence generated from the above-mentioned method provides a weak convergent iterative sequence and in order to operate it, prior information regarding the Lipschitz-type constants is required. These Lipschitz-type constants are mostly unknown or hard to compute. To overcome this situation, Hieu et al. [14] introduced an extension of the method in [38].
for solving the equilibrium problem as follows: Let $[t]_+ := \max\{t, 0\}$ and choose $u_0 \in C$, $\mu \in (0, 1)$ with $\lambda_0 > 0$ such that

$$\begin{align*}
v_n &= \arg\min\{\lambda_n f(u_n, v) + \frac{1}{2}\|u_n - v\|^2 : v \in C\}, \\
u_{n+1} &= \arg\min\{\lambda_n f(v_n, v) + \frac{1}{2}\|u_n - v\|^2 : v \in C\},
\end{align*}$$

where the stepsize sequence $\{\lambda_n\}$ is updated in the following way:

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu(\|u_n - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2(f(u_{n+1}, u_{n+1}) - f(u_n, v_n) - f(v_{n+1}, v_{n+1}))}\right\}.$$ 

Recently, Vinh and Muu proposed an inertial iterative algorithm in [39] to solve a pseudomonotone equilibrium problem. Their main contribution is the availability of an inertial effect in the algorithm that is used to improve the convergence rate of the iterative sequence. The iterative sequence $\{u_n\}$ has been generated in the following manner:

(i) Choose $u_{-1}, u_0 \in C$, $\theta \in [0, 1)$, $0 < \lambda < \min\left\{\frac{1}{2\|\cdot\|}, \frac{1}{2\|\cdot\|}\right\}$ while a sequence $\{\varepsilon_n\} \subset [0, +\infty)$ is satisfying the following condition:

$$\sum_{n=0}^{+\infty} \varepsilon_n < +\infty.$$ 

(ii) Choose $\vartheta_n$ such that $0 \leq \vartheta_n \leq \bar{\vartheta}_n$ where

$$\bar{\vartheta}_n = \begin{cases} 
\min\{\vartheta : \|u_n - u_{n-1}\|\} & \text{if } u_n \neq u_{n-1}, \\
\vartheta & \text{otherwise}.
\end{cases}$$

(iii) Determine

$$\begin{align*}
\eta_n &= u_n + \bar{\vartheta}_n (u_n - u_{n-1}), \\
v_n &= \arg\min\{\lambda f(\eta_n, v) + \frac{1}{2}\|\eta_n - v\|^2 : v \in C\}, \\
u_{n+1} &= \arg\min\{\lambda f(v_n, v) + \frac{1}{2}\|\eta_n - v\|^2 : v \in C\}.
\end{align*}$$

This article focuses on projection methods that are well-known and easy to execute due to their efficient and straightforward mathematical computation. Motivated by the works of [14,40], we formulate an inertial explicit subgradient extragradient algorithm to solve the pseudomonotone equilibrium problem. The proposed algorithm can be seen as the modification of the methods that appear in [7,14,39]. Under certain mild conditions, a weak convergence result has been proven to correspond to the iterative sequence of the algorithm. Moreover, experimental studies have shown that the proposed method tends to be more efficient compared to the existing method [39].

The remainder of this paper is arranged as follows: Section 2 contains some definitions and basic results used in the paper. Section 3 contains our main algorithm and proves its convergence. Sections 4 and 5 incorporate the implementation of our results. Section 6 carries out the numerical results that demonstrates the computational effectiveness of our proposed algorithm.

2. Background

Let $h : C \to \mathcal{R}$ be a convex function on a nonempty, closed and convex subset $C$ of a real Hilbert space $\mathcal{H}$, and the subdifferential of a function $h$ at $z_1 \in C$ is defined as:

$$\partial h(z_1) = \{z_3 \in \mathcal{H} : h(z_2) - h(z_1) \geq \langle z_3, z_2 - z_1 \rangle, \forall z_2 \in C\}.$$
Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and a normal cone of $C$ at $z_1 \in C$ is defined by:

$$N_C(z_1) = \{ z_3 \in H : \langle z_3, z_2 - z_1 \rangle \leq 0, \forall z_2 \in C \}.$$  

The metric projection $P_C(z_1)$ for $z_1 \in H$ onto a closed and convex subset $C$ of $H$ is defined by:

$$P_C(z_1) = \text{arg min}\{ \| z_2 - z_1 \| : z_2 \in C \}.$$  

**Lemma 1** ([41]). Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $P_C : H \to C$ be a metric projection from $H$ onto $C$.

(i) Let $z_1 \in C$ and $z_2 \in H$; we have

$$\| z_1 - P_C(z_2) \|^2 + \| P_C(z_2) - z_2 \|^2 \leq \| z_1 - z_2 \|^2.$$  

(ii) $z_3 = P_C(z_1)$ if and only if

$$\langle z_1 - z_3, z_2 - z_3 \rangle \leq 0, \forall z_2 \in C.$$  

(iii) For $z_2 \in C$ and $z_1 \in H$

$$\| z_1 - P_C(z_1) \| \leq \| z_1 - z_2 \|.$$  

**Lemma 2** ([42]). Let $h : C \to \mathbb{R}$ be a convex, subdifferentiable and lower semicontinuous function on $C$, where $C$ is a nonempty, convex and closed subset of a real Hilbert space $H$. Then, an element $z_1 \in C$ is a minimizer of a function $h$ if and only if $0 \in \partial h(z_1) + N_C(z_1)$, where $\partial h(z_1)$ and $N_C(z_1)$ represent the subdifferential of $h$ at $z_1 \in C$ and normal cone of $C$ at $z_1$, respectively.

**Lemma 3** ([43]). Let $\{ u_n \}$ be a sequence in $H$ and $C \subset H$ such that the following conditions hold:

(i) For each $u \in C$, the $\lim_{n \to \infty} \| u_n - u \|$ exists;

(ii) Each sequentially weak cluster limit point of the sequence $\{ u_n \}$ belongs to $C$.

Then, the sequence $\{ u_n \}$ weakly converges to some element in $C$.

**Lemma 4.** [44] Let $\{ q_n \}$ and $\{ p_n \}$ be sequences of non-negative real numbers satisfying $q_{n+1} \leq q_n + p_n$, for each $n \in \mathbb{N}$. If $\sum p_n < \infty$, then $\lim_{n \to \infty} q_n$ exists.

Assume that a bifunction $f$ satisfies the following conditions:

(f1) $f(z_2, z_2) = 0$, for all $z_2 \in C$ and $f$ is pseudomonotone on $C$;

(f2) $f$ satisfies the Lipschitz-type condition on $H$ through $c_1 > 0$ and $c_2 > 0$;

(f3) $\limsup_{n \to \infty} f(z_n, v) \leq f(z^*, v)$ for every $v \in C$ and $\{ z_n \} \subset C$ satisfying $z_n \to z^*$;

(f4) $f(z_1, \cdot)$ needs to be convex and subdifferentiable on $H$ for each $z_1 \in H$. 

3. Convergence Analysis for an Algorithm

We provide a method consisting of two strongly convex minimization problems through an inertial factor and an explicit stepsize formula, which are being used to improve the convergence rate of the iterative sequence and to make the method independent of the Lipschitz constants. The detailed method is provided below Algorithm 1:

**Algorithm 1** (Inertial methods for pseudomonotone equilibrium problems)

**Initialization:** Choose $u_{-1}, u_0 \in C$, $\mu \in (0, 1)$, $\lambda_0 > 0$, $\vartheta \in [0, 1)$ and a sequence $\{\varepsilon_n\} \subset [0, +\infty)$ satisfying

\[
\sum_{n=0}^{\infty} \varepsilon_n < +\infty.
\]  

(4)

**Iterative steps:** Choose $\vartheta_n$ satisfying $0 \leq \vartheta_n \leq \vartheta$ and

\[
\vartheta_n = \begin{cases} 
\min \left\{ \frac{\varepsilon_n}{\|u_n - u_{n-1}\|} \right\} & \text{if } u_n \neq u_{n-1}, \\
\vartheta & \text{otherwise.}
\end{cases}
\]  

(5)

**Step 1:** Determine

\[
v_n = \arg \min_{y \in C} \{ \lambda_n f(\eta_n, y) + \frac{1}{2} \| \eta_n - y \|^2 \},
\]

where $\eta_n = u_n + \vartheta_n(u_n - u_{n-1})$. If $\eta_n = v_n$, STOP. Otherwise, go to next step.

**Step 2:** Determine a half-space

\[
\mathcal{H}_n = \{ z \in \mathcal{H} : \langle \eta_n - \lambda_n \omega_n - v_n, z - v_n \rangle \leq 0 \},
\]

where $\omega_n \in \partial_2 f(\eta_n, v_n)$ and evaluate

\[
u_{n+1} = \arg \min_{y \in \mathcal{H}_n} \{ \lambda_n f(v_n, y) + \frac{1}{2} \| \eta_n - y \|^2 \}.
\]

**Step 3:** Set $d_1 = f(\eta_n, u_{n+1}) - f(\eta_n, v_n) - f(v_n, u_{n+1})$ and evaluate

\[
\lambda_{n+1} = \begin{cases} 
\min \left\{ \lambda_n, \frac{\mu}{2c_1} \| \eta_n - v_n \|^2 + \frac{\mu}{2c_2} \| u_{n+1} - v_n \|^2 \right\} & \text{if } d_1 > 0, \\
\lambda_n & \text{otherwise.}
\end{cases}
\]  

Set $n := n + 1$ and go back to **Iterative steps**.

**Lemma 5.** The sequence $\{\lambda_n\}$ is decreasing monotonically with a lower bound $\min \left\{ \frac{\mu}{2\max\{c_1, c_2\}}, \lambda_0 \right\}$ and converges to $\lambda > 0$.

**Proof.** From the definition of $\{\lambda_n\}$, we see that this sequence is monotone and non-increasing. It is given that $f$ satisfies the Lipschitz-type condition with constants $c_1$ and $c_2$. Let $f(\eta_n, u_{n+1}) - f(\eta_n, v_n) - f(v_n, u_{n+1}) > 0$, such that

\[
\frac{\mu(\| \eta_n - v_n \|^2 + \| u_{n+1} - v_n \|^2)}{2[f(\eta_n, u_{n+1}) - f(\eta_n, v_n) - f(v_n, u_{n+1})]} \geq \frac{\mu(\| \eta_n - v_n \|^2 + \| u_{n+1} - v_n \|^2)}{2(c_1 \| \eta_n - v_n \|^2 + c_2 \| u_{n+1} - v_n \|^2)} \geq \frac{\mu}{2\max\{c_1, c_2\}}.
\]

(6)

The above implies that the sequence $\{\lambda_n\}$ has a lower bound $\min \left\{ \frac{\mu}{2\max\{c_1, c_2\}}, \lambda_0 \right\}$. Moreover, there exists a real number $\lambda > 0$, such that $\lim_{n \to \infty} \lambda_n = \lambda$. \qed
Remark 1. Due to the summability of \( \sum_{n=0}^{+\infty} \varepsilon_n \), Expression (5) implies that:

\[
\sum_{n=1}^{+\infty} \partial_n \| u_n - u_{n-1} \| \leq \sum_{n=1}^{+\infty} \partial_n \| u_n - u_{n-1} \| \leq \sum_{n=1}^{+\infty} \partial \| u_n - u_{n-1} \| < \infty, \tag{7}
\]

which implies that:

\[
\lim_{n \to \infty} \partial \| u_n - u_{n-1} \| = 0. \tag{8}
\]

Lemma 6. Assume that a bifunction \( f : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \) satisfies the conditions (f1)–(f4). For each \( u^* \in \text{EP}(f, \mathcal{C}) \neq \emptyset \), we have

\[
\| u_{n+1} - u^* \|^2 \leq \| \eta_n - u^* \|^2 - \left( 1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) \| \eta_n - v_n \|^2 - \left( 1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) \| u_{n+1} - v_n \|^2. \tag{9}
\]

Proof. From the value of \( u_{n+1} \), we have

\[
0 \in \partial_2 \left\{ \lambda_n f(v_n, y) + \frac{1}{2} \| \eta_n - y \|^2 \right\}(u_{n+1}) + N_{\mathcal{H}_n}(u_{n+1}).
\]

For some \( \omega \in \partial f(v_n, u_{n+1}) \) there exists \( \overline{\omega} \in N_{\mathcal{H}_n}(u_{n+1}) \) such that

\[
\lambda_n \omega + u_{n+1} - \eta_n + \overline{\omega} = 0.
\]

The above equality implies that

\[
\langle \eta_n - u_{n+1}, y - u_{n+1} \rangle = \lambda_n \langle \omega, y - u_{n+1} \rangle + \langle \overline{\omega}, y - u_{n+1} \rangle, \quad \forall y \in \mathcal{H}_n.
\]

Since \( \overline{\omega} \in N_{\mathcal{H}_n}(u_{n+1}) \), it follows that \( \langle \overline{\omega}, y - u_{n+1} \rangle \leq 0 \), for all \( y \in \mathcal{H}_n \). Thus, we have

\[
\langle \eta_n - u_{n+1}, y - u_{n+1} \rangle \leq \lambda_n \langle \omega, y - u_{n+1} \rangle, \quad \forall y \in \mathcal{H}_n. \tag{10}
\]

Further, \( \omega \in \partial f(v_n, u_{n+1}) \) and due to the definition of subdifferential, we have

\[
f(v_n, y) - f(v_n, u_{n+1}) \geq \langle \omega, y - u_{n+1} \rangle, \quad \forall y \in \mathcal{H}. \tag{10}
\]

Combining Expressions (9) and (10), we obtain

\[
\lambda_n f(v_n, y) - \lambda_n f(v_n, u_{n+1}) \geq \langle \eta_n - u_{n+1}, y - u_{n+1} \rangle, \quad \forall y \in \mathcal{H}_n. \tag{11}
\]

From the definition of \( \mathcal{H}_n \), we can write

\[
\lambda_n \langle \omega_n, u_{n+1} - v_n \rangle \geq \langle \eta_n - v_n, u_{n+1} - v_n \rangle. \tag{12}
\]

Due to \( \omega_n \in \partial f(\eta_n, v_n) \), we have

\[
f(\eta_n, y) - f(\eta_n, v_n) \geq \langle \omega_n, y - v_n \rangle, \quad \forall y \in \mathcal{H}.
\]

By substituting \( y = u_{n+1} \) in the above expression, we have

\[
f(\eta_n, u_{n+1}) - f(\eta_n, v_n) \geq \langle \omega_n, u_{n+1} - v_n \rangle, \quad \forall y \in \mathcal{H}. \tag{13}
\]

Combining Expressions (12) and (13), we obtain

\[
\lambda_n f(\eta_n, u_{n+1}) - f(\eta_n, v_n) \geq \langle \eta_n - v_n, u_{n+1} - v_n \rangle. \tag{14}
\]
By substituting \( y = u^* \) in Expression (11), we have
\[
\lambda_n f(v_n, u^*) - \lambda_n f(v_n, u_{n+1}) \geq \langle \eta_n - u_{n+1}, u^* - u_{n+1} \rangle. \tag{15}
\]

Since \( u^* \in EP(f, C) \), we have \( f(u^*, v_n) \geq 0 \). From the pseudomonotonicity of bifunction \( f \), we obtain \( f(v_n, u^*) \leq 0 \). Hence, it follows from Expression (15) that
\[
\langle \eta_n - u_{n+1}, u_{n+1} - u^* \rangle \geq \lambda_n f(v_n, u_{n+1}). \tag{16}
\]

From the definition of \( \lambda_{n+1} \), we obtain
\[
f(\eta_n, u_{n+1}) - f(\eta_n, v_n) - f(v_n, u_{n+1}) \leq \frac{\mu \|\eta_n - v_n\|^2 + \mu \|u_{n+1} - v_n\|^2}{2\lambda_{n+1}}. \tag{17}
\]

From Expressions (16) and (17), we have
\[
\langle \eta_n - u_{n+1}, u_{n+1} - u^* \rangle \geq \lambda_n \{ f(\eta_n, u_{n+1}) - f(\eta_n, v_n) \}
\]
\[
- \frac{\mu \lambda_n}{2\lambda_{n+1}} \|\eta_n - v_n\|^2 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|u_{n+1} - v_n\|^2. \tag{18}
\]

Combining Expressions (14) and (18), we obtain
\[
\langle \eta_n - u_{n+1}, u_{n+1} - u^* \rangle \geq \langle \eta_n - v_n, u_{n+1} - v_n \rangle
\]
\[
- \frac{\mu \lambda_n}{2\lambda_{n+1}} \|\eta_n - v_n\|^2 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|u_{n+1} - v_n\|^2. \tag{19}
\]

We have the following formulas:
\[
- 2\langle \eta_n - u_{n+1}, u_{n+1} - u^* \rangle = -\|\eta_n - u^*\|^2 + \|u_{n+1} - \eta_n\|^2 + \|u_{n+1} - u^*\|^2. \tag{20}
\]
\[
2\langle v_n - \eta_n, v_n - u_{n+1} \rangle = \|\eta_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|\eta_n - u_{n+1}\|^2. \tag{21}
\]

Combining the relations (19)–(21), we get
\[
\|u_{n+1} - u^*\|^2 \leq \|\eta_n - u^*\|^2 - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) \|\eta_n - v_n\|^2 - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) \|u_{n+1} - v_n\|^2.
\]

\( \square \)

**Theorem 1.** Assume that a bifunction \( f : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \) satisfies the conditions (f1)–(f4) and \( u^* \) belongs to solution set \( EP(f, C) \). Then, the sequences \( \{\eta_n\}, \{u_n\} \) and \( \{v_n\} \) generated by Algorithm 1 converge weakly to the \( u^* \) solution of the problem (1). In addition, \( \lim_{n \to \infty} P_{EP(f, C)}(u_n) = u^* \).

**Proof.** Since \( \lambda_n \to \lambda \), there exists a fixed number \( \epsilon \in (0, 1 - \mu) \) such that
\[
\lim_{n \to \infty} \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \epsilon > 0.
\]

Thus, there is a finite number \( n_1 \in \mathcal{N} \) such that
\[
\left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) > \epsilon > 0, \quad \forall n \geq n_1. \tag{22}
\]
By Lemma 6, we obtain
\[ \|u_{n+1} - u^*\|^2 \leq \|\eta_n - u^*\|^2, \forall n \geq n_1. \] (23)

From the definition of \(\eta_n\) in Algorithm 1, we have
\[ \|\eta_n - u^*\|^2 = \|u_n + \theta_n(u_n - u_{n-1}) - u^*\|^2 \\
= \|(1 + \theta_n)(u_n - u^*) - \theta_n(u_{n-1} - u^*)\|^2 \\
= (1 + \theta_n)\|u_n - u^*\|^2 - \theta_n\|u_{n-1} - u^*\|^2 + \theta_n(1 + \theta_n)\|u_n - u_{n-1}\|^2 \\
\leq (1 + \theta_n)\|u_n - u^*\|^2 - \theta_n\|u_{n-1} - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2. \] (24)

Expression (25) can be written as
\[ \|u_{n+1} - u^*\|^2 \leq (1 + \theta_n)\|u_n - u^*\|^2 - \theta_n\|u_{n-1} - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2, \forall n \geq n_1. \] (26)

From the definition of the \(\eta_n\), we also have
\[ \|\eta_n - u^*\| = \|u_n + \theta_n(u_n - u_{n-1}) - u^*\| \leq \|u_n - u^*\| + \theta_n\|u_n - u_{n-1}\|. \] (27)

Combining relations (23) and (27), we obtain
\[ \|u_{n+1} - u^*\| \leq \|u_n - u^*\| + \theta\|u_n - u_{n-1}\|, \forall n \geq n_1. \] (28)

By using Lemma 4 with (7) and (28), we have
\[ \lim_{n \to \infty} \|u_n - u^*\| = l, \text{ for some finite } l \geq 0. \] (29)

From Equality (8), we have
\[ \lim_{n \to \infty} \|u_n - u_{n-1}\| = 0. \] (30)

By letting \(n \to \infty\) in Expression (24), we obtain
\[ \lim_{n \to \infty} \|\eta_n - u^*\| = l. \] (31)

From Lemma 6 and Expression (25), we have
\[ \|u_{n+1} - u^*\|^2 \\
\leq (1 + \theta_n)\|u_n - u^*\|^2 - \theta_n\|u_{n-1} - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2 \\
- \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right)\|\eta_n - v_n\|^2 - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right)\|u_n - v_n\|^2, \] (32)

which further implies that (for \(n \geq n_1\))
\[ \epsilon\|\eta_n - v_n\|^2 + \epsilon\|v_n - u_{n+1}\|^2 \\
\leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + \theta_n\left(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2\right) + 2\theta\|u_n - u_{n-1}\|^2. \] (33)

By letting \(n \to \infty\) in (33), we obtain
\[ \lim_{n \to \infty} \|\eta_n - v_n\| = \lim_{n \to \infty} \|v_n - u_{n+1}\| = 0. \] (34)
By using the Cauchy inequality and Expression (34), we obtain

$$\lim_{n \to \infty} \|y_n - u_{n+1}\| \leq \lim_{n \to \infty} \|y_n - v_n\| + \lim_{n \to \infty} \|u_{n+1} - v_n\| = 0.$$  \hspace{1cm} (35)

From Expressions (31) and (34), we also obtain

$$\lim_{n \to \infty} \|v_n - u^*\| = l.$$  \hspace{1cm} (36)

It follows from Expressions (29), (31), and (36) that the sequences \(\{\eta_n\}, \{u_n\}, \text{ and } \{v_n\}\) are bounded. Next, we need to use Lemma 3, for it is compulsory to prove that all sequential weak cluster limit points of the sequence \(\{u_n\}\) belong to the solution set \(EP(f, C)\). Assume that \(z\) is any weak cluster limit point of the sequence \(\{u_n\}\), i.e., there exists a subsequence \(\{u_{n_k}\}\) of \(\{u_n\}\) such that \(u_{n_k} \to z\). Since \(\|u_n - v_n\| \to 0\), it follows that \(\{v_{n_k}\}\) also weakly converges to \(z\) and so \(z \in C\). Now, it remains to prove that \(z \in EP(f, C)\). By Expression (11), the definition of \(\lambda_{n+1}\), and (14), we have

$$\lambda_{n_k} f(v_{n_k}, y) \geq \lambda_{n_k} f(v_{n_k}, u_{n_k+1}) + \langle \eta_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle$$

$$\geq \lambda_{n_k} f(\eta_{n_k}, u_{n_k+1}) - \lambda_{n_k} f(\eta_{n_k}, v_{n_k}) - \frac{\mu \lambda_{n_k}}{2 \lambda_{n_k+1}} \|\eta_{n_k} - v_{n_k}\|^2$$

$$- \frac{\mu \lambda_{n_k}}{2 \lambda_{n_k+1}} \|v_{n_k} - u_{n_k+1}\|^2 + \langle \eta_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle$$

$$\geq \langle \eta_{n_k} - v_{n_k}, u_{n_k+1} - v_{n_k} \rangle - \frac{\mu \lambda_{n_k}}{2 \lambda_{n_k+1}} \|\eta_{n_k} - v_{n_k}\|^2$$

$$- \frac{\mu \lambda_{n_k}}{2 \lambda_{n_k+1}} \|v_{n_k} - u_{n_k+1}\|^2 + \langle \eta_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle,$$  \hspace{1cm} (37)

where \(y \in H\). It follows from (30), (34), (35), and the boundedness of \(\{u_n\}\) that the right hand side tends to zero. Due to \(\lambda_{n_k} > 0\), condition (f3), and \(v_{n_k} \to z\), we have

$$0 \leq \limsup_{k \to \infty} f(v_{n_k}, y) \leq f(z, y), \forall y \in H.$$

Since \(C \subset H\), it follows that \(f(z, y) \geq 0, \forall y \in C\). This implies that \(z \in EP(f, C)\). Finally, from Lemma 3, the sequences \(\{\eta_n\}, \{u_n\}, \text{ and } \{v_n\}\) converge weakly to \(u^*\) as \(n \to \infty\).

Moreover, the renaming part consists of proving that \(\lim_{n \to \infty} P_{EP(f, C)}(u_n) = u^*\). Let \(q_n := P_{EP(f, C)}(u_n), \forall n \in \mathcal{N}\). For any \(u^* \in EP(f, C)\), we have

$$\|q_n\| \leq \|q_n - u_n\| + \|u_n\| \leq \|u^* - u_n\| + \|u_n\|.$$  \hspace{1cm} (38)

The above expression implies that the sequence \(\{q_n\}\) is bounded. Next, we prove that \(\{q_n\}\) is a Cauchy sequence. By Lemma 1(iii) and (27), we have

$$\|u_{n+1} - q_{n+1}\| \leq \|u_{n+1} - q_n\| + \|q_n - q_{n}\| + \|u_n - u_{n-1}\|, \forall n \geq n_1.$$  \hspace{1cm} (39)

Lemma 4 provides the existence of \(\lim_{n \to \infty} \|u_n - q_n\|\). From Expression (27) for all \(m > n \geq n_1\), we have

$$\|q_n - u_m\| \leq \|q_n - u_{m-1}\| + \|u_m - u_{m-1}\|$$

$$\leq \cdots \leq \|q_n - u_n\| + \|u_n - u_{n-1}\| + \sum_{k=n}^{m-1} \|u_k - u_{k+1}\|.$$  \hspace{1cm} (40)
Suppose that \( q_m, q_n \in EP(f, C) \) for \( m > n \geq n_1 \). By using Lemma 1(i) and Expression (40), we have

\[
\|q_n - q_m\|^2 \\
\leq \|q_n - u_m\|^2 - \|q_m - u_m\|^2 \\
\leq \|q_n - u_n\|^2 + (\theta \sum_{k=n}^{m-1} \|u_n - u_{n-1}\|) + 2\theta \|q_n - u_n\| \sum_{k=n}^{m-1} \|u_n - u_{n-1}\| - \|q_m - u_m\|^2.
\]

The existence of \( \lim_{n \to \infty} \|u_n - q_n\| \) and the summability of the series \( \sum_{n} \|u_n - u_{n-1}\| \) imply that \( \lim_{n \to \infty} \|q_n - q_m\| = 0 \) for all \( m > n \). As a result, \( \{q_n\} \) is a Cauchy sequence and due to the closeness of a solution set \( EP(f, C) \) the sequence \( \{q_n\} \) strongly converges to \( q^* \in EP(f, C) \). Next, we show that \( q^* = u^* \). Due to Lemma 1(ii) and \( u^*, q^* \in EP(f, C) \), we can write

\[
\langle u_n - q_n, u^* - q_n \rangle \leq 0.
\]

Due to \( q_n \to q^* \) and \( u_n \to u^* \), we obtain

\[
\langle u^* - q^*, u^* - q^* \rangle \leq 0
\]

which gives that \( u^* = q^* = \lim_{n \to \infty} P_{EP(f,C)}(u_n) \). \( \square \)

4. Applications to Solve Fixed Point Problems

Now, consider the applications of our results from Section 3 to solve fixed-point problems involving \( \kappa \)-strict pseudo-contraction. A mapping \( T : C \to C \) is said to be

(i) \( \kappa \)-strict pseudo-contraction [45] on \( C \) if

\[
\|Tz_1 - Tz_2\|^2 \leq \|z_1 - z_2\|^2 + \kappa \|z_1 - Tz_1 - (z_2 - Tz_2)\|^2, \forall z_1, z_2 \in C,
\]

which is equivalent to

\[
\langle Tz_1 - Tz_2, z_1 - z_2 \rangle \leq \|z_1 - z_2\|^2 - \frac{1 - \kappa}{2} \|z_1 - Tz_1 - (z_2 - Tz_2)\|^2, \forall z_1, z_2 \in C;
\]

(ii) sequentially weakly continuous on \( C \) if

\[
T(u_n) \to T(p) \text{ for every sequence in } C \text{ satisfying } u_n \to p \text{ (weakly converges)}.
\]

The fixed point problem for a mapping \( T : C \to C \) is formulated in the following way:

Find \( u^* \in C \) such that \( T(u^*) = u^* \).

Note: If we define bifunction \( f(x, y) = \langle x - Tx, y - x \rangle, \forall x, y \in C. \) Then, the equilibrium problem (1) converts into the fixed point problem with \( 2c_1 = 2c_2 = \frac{3 - 2\kappa}{1 - \kappa} \). From the value of \( v_n \) in Algorithm 1, we have

\[
\]
Choose Symmetry and consequently $0$

Let Corollary 1.

As a consequence of the results in Section 3, we have the following fixed point theorem:

$$\omega_n \in \partial_2 f(\eta_n, v_n),$$

it follows from the definition of the subdifferential that we have

$$\langle \omega_n, y - v_n \rangle \leq \langle \eta_n - T(\eta_n), y - \eta_n \rangle - \langle \eta_n - T(\eta_n), v_n - \eta_n \rangle, \forall y \in \mathcal{H}$$

and consequently $0 \leq \langle \eta_n - T(\eta_n) - \omega_n, y - v_n \rangle$. This implies that

$$\langle (1 - \lambda_n)\eta_n + \lambda_n T(\eta_n) - v_n, y - v_n \rangle$$

$$\leq (1 - \lambda_n)\eta_n + \lambda_n T(\eta_n) - v_n, y - v_n + \lambda_n (\eta_n - T(\eta_n) - \omega_n, y - v_n)$$

Similarly to Expression (45), we obtain

$$u_{n+1} = P_{\mathcal{H}_n} [\eta_n - \lambda_n (v_n - T(v_n))].$$

As a consequence of the results in Section 3, we have the following fixed point theorem:

**Corollary 1.** Let $C$ be a subset of a Hilbert space $\mathcal{H}$ and $T : C \to C$ be a $\kappa$-strict pseudocontraction and weakly continuous with $\text{Fix}(T) \neq \emptyset$. The sequences $\eta_n, u_n, \text{and } v_n$ are generated in the following way:

(i) Fix $u_{-1}, u_0 \in C, \lambda_0 > 0, \mu \in (0,1)$ and $\theta \in [0,1]$ with a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that

$$\sum_{n=0}^{+\infty} \epsilon_n < +\infty.$$  \hspace{1cm} (49)

(ii) Choose $\theta_n$ such that $0 \leq \theta_n \leq \theta$ and

$$\theta_n = \begin{cases} \min \left\{ \theta \mid \frac{\epsilon_n}{\|u_n - u_{n-1}\|} \right\} & \text{if } u_n \neq u_{n-1}, \\ \theta & \text{otherwise.} \end{cases}$$  \hspace{1cm} (50)

(iii) Evaluate

$$\eta_n = u_{n} + \theta_n (u_n - u_{n-1}),$$

$$v_n = P_C [\eta_n - \lambda_n (\eta_n - T(\eta_n))],$$

$$u_{n+1} = P_{\mathcal{H}_n} [\eta_n - \lambda_n (v_n - T(v_n))],$$

where $\mathcal{H}_n = \{ z \in \mathcal{H} : ((1 - \lambda_n)\eta_n + \lambda_n T(\eta_n) - v_n, z - v_n) \leq 0 \}$.  \hspace{1cm} (51)

(iv) Set $d_2 = ((\eta_n - v_n) - (T(\eta_n) - T(v_n)), u_{n+1} - v_n)$ and revise the stepsize $\lambda_{n+1}$ in the following way:

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n \frac{\mu \|\eta_n - v_n\|^2 + \|u_{n+1} - v_n\|^2}{2d_2} \right\} & \text{if } d_2 > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$
Then, sequences \( \{\eta_n\} \), \( \{u_n\} \), and \( \{v_n\} \) weakly converge to \( u^* \in \text{Fix}(T) \).

5. Application to Solve Variational Inequality Problems

Now, consider the applications of our results from in Section 3 to solve variational inequality problems involving a pseudomonotone and Lipschitz-type continuous operator. An operator \( K : \mathcal{H} \to \mathcal{H} \) is said to be

(i) L-Lipschitz continuous on \( C \) if

\[
\|K(z_1) - K(z_2)\| \leq L\|z_1 - z_2\|, \forall z_1, z_2 \in C;
\]

(ii) pseudomonotone on \( C \) if

\[
\langle K(z_1), z_2 - z_1 \rangle \geq 0 \implies \langle K(z_2), z_1 - z_2 \rangle \leq 0, \forall z_1, z_2 \in C.
\]

The variational inequality problem for a operator \( K : \mathcal{H} \to \mathcal{H} \) is formulated in the following way:

Find \( u^* \in C \) such that \( \langle K(u^*), y - u^* \rangle \geq 0, \forall y \in C \).

Note: If we define a bifunction \( f(x, y) := \langle K(x), y - x \rangle, \forall x, y \in C \). Thus, the equilibrium problem (1) translates into a variational inequality problem with \( L = 2c_1 = 2c_2 \). From the value of \( v_n \), we have

\[
v_n = \arg\min_{y \in C} \left\{ \lambda_n f(\eta_n, y) + \frac{1}{2}\|\eta_n - y\|^2 \right\}
\]

\[
= \arg\min_{y \in C} \left\{ \lambda_n \langle K(\eta_n), y - \eta_n \rangle + \frac{1}{2}\|\eta_n - y\|^2 + \frac{\lambda_n^2}{2}\|K(\eta_n)\|^2 - \frac{\lambda_n^2}{2}\|K(\eta_n)\|^2 \right\}
\]

\[
= \arg\min_{y \in C} \left\{ \frac{1}{2}\|y - (\eta_n - \lambda_n K(\eta_n))\|^2 \right\}
\]

\[
= P_C[\eta_n - \lambda_n K(\eta_n)].
\]

Since \( \omega_n \in \partial_2 f(\eta_n, v_n) \), it follows from the subdifferential definition that we have

\[
\langle \omega_n, y - v_n \rangle \leq \langle K(\eta_n), y - \eta_n \rangle - \langle K(\eta_n), v_n - \eta_n \rangle, \forall y \in \mathcal{H}
\]

\[
= \langle K(\eta_n), y - v_n \rangle, \forall y \in \mathcal{H},
\]

and consequently \( 0 \leq \langle K(\eta_n) - \omega_n, y - v_n \rangle \). This implies that

\[
\langle \eta_n - \lambda_n K(\eta_n) - v_n, y - v_n \rangle
\]

\[
\leq \langle \eta_n - \lambda_n K(\eta_n) - v_n, y - v_n \rangle + \lambda_n \langle K(\eta_n) - \omega_n, y - v_n \rangle
\]

\[
\leq \langle \eta_n - \lambda_n \omega_n - v_n, y - v_n \rangle.
\]

In similar way to Expression (52), we have

\[
u_{n+1} = P_{\text{Fix}}[\eta_n - \lambda_n K(\eta_n)].
\]

Suppose that \( K \) satisfies the following conditions:

(K1) \( K \) is pseudomonotone on \( C \) with \( \text{VI}(K, \mathcal{C}) \neq \emptyset \);

(K2) \( K \) is L-Lipschitz continuous on \( C \) with \( L > 0 \);

(K3) \( \limsup_{n \to \infty} \langle K(u_n), y - u_n \rangle \leq \langle K(p), y - p \rangle, \forall y \in \mathcal{C} \) and \( \{u_n\} \subset \mathcal{C} \) satisfying \( u_n \to p \).
Corollary 2. Assume that an operator \( K : C \to H \) satisfies the conditions (K1)-(K3) and that the sequences \( \{ \eta_n \}, \{ u_n \}, \) and \( \{ v_n \} \) are generated in the following way:

(i) Choose \( u_{-1}, u_0 \in C, \lambda_0 > 0, \mu \in (0,1) \) and \( \eta \in [0,1) \) with \( \{ \epsilon_n \} \subset [0, +\infty) \) such that

\[
\sum_{n=0}^{+\infty} \epsilon_n < +\infty. \tag{55}
\]

(ii) Choose \( \theta_n \) satisfying \( 0 \leq \theta_n \leq \bar{\theta}_n \) such that

\[
\bar{\theta}_n = \begin{cases} 
\min \left\{ \theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|} \right\} & \text{if } u_n \neq u_{n-1}, \\
\theta & \text{otherwise.}\end{cases} \tag{56}
\]

(iii) Set \( \eta_n = u_n + \bar{\theta}_n(u_n - u_{n-1}) \) and compute

\[
\begin{align*}
\{ v_n & = P_C[\eta_n - \lambda_n K(\eta_n)], \\
u_{n+1} & = P_H[\eta_n - \lambda_n K(v_n)],
\end{align*} \tag{57}
\]

where \( H_n = \{ z \in H : \langle \eta_n - \lambda_n K(\eta_n) - v_n, z - v_n \rangle \leq 0 \} \).

(iv) Set \( d_3 = \langle K(\eta_n) - K(v_n), u_{n+1} - v_n \rangle \) and stepsize \( \lambda_{n+1} \) is revised in the following way:

\[
\lambda_{n+1} = \begin{cases} 
\min \left\{ \lambda_n, \frac{\mu \|u_n - v_n\|^2 + \mu \|u_{n+1} - v_n\|^2}{2d_3} \right\} & \text{if } d_3 > 0, \\
\lambda_n & \text{otherwise.}\end{cases} \tag{58}
\]

Then, the sequences \( \{ \eta_n \}, \{ u_n \}, \) and \( \{ v_n \} \) weakly converge to \( u^* \in VI(K, C) \).

6. Numerical Experiments

The computational results are presented in this section to illustrate the effectiveness of our proposed Algorithm 1 (EiEGM) compared to Algorithm 1 (iEGM) in [39]. The MATLAB program was operated on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70GHz 1.70GHz, RAM 4.00 GB) in MATLAB version 9.5 (R2018b). We used the built-in MATLAB fmincon function to solve the minimization problems.

Example 1. Let \( f : C \times C \to \mathcal{R} \) be defined by

\[
f(u, v) = \sum_{i=2}^{5} (v_i - u_i)\|u\|, \quad \forall u, v \in \mathcal{R}^5,
\]

where \( C = \{ (u_1, \cdots, u_5) : u_1 \geq -1, u_i \geq 1, i = 2, \cdots, 5 \} \). The bifunction \( f \) is Lipschitz-type continuous operator with constants \( c_1 = c_2 = 2 \), and it satisfies conditions (f1)-(f4). To evaluate the best possible value of the control parameters \( \lambda, \lambda_0 \) and inertial factor \( \eta \). The numerical results are shown in the Tables 1 and 2 by choosing \( u_{-1} = u_0 = (2,3,2,5,5), \mu = 0.33 \) and \( D_n = \|\eta_n - v_n\| \leq \varepsilon = 10^{-4} \).
Table 1. Example 1: Algorithm 1 numerical comparison with Algorithm 1 in [39].

<table>
<thead>
<tr>
<th>$\vartheta$</th>
<th>$\lambda$</th>
<th>$\lambda_0$</th>
<th>Number of Iterations</th>
<th>Execution Time in Seconds</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.22</td>
<td>1.00</td>
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<td>7</td>
</tr>
<tr>
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<td>7</td>
</tr>
<tr>
<td>0.45</td>
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<td>0.40</td>
<td>21</td>
<td>8</td>
</tr>
<tr>
<td>0.45</td>
<td>0.01</td>
<td>0.20</td>
<td>25</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 2. Example 1: Algorithm 1 numerical comparison with Algorithm 1 in [39].

<table>
<thead>
<tr>
<th>$\vartheta$</th>
<th>$\lambda$</th>
<th>$\lambda_0$</th>
<th>Number of Iterations</th>
<th>Execution Time in Seconds</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>0.50</td>
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<td>7</td>
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<tr>
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<td>0.50</td>
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<td>7</td>
</tr>
<tr>
<td>0.35</td>
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<td>0.50</td>
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<tr>
<td>0.15</td>
<td>0.20</td>
<td>0.50</td>
<td>17</td>
<td>9</td>
</tr>
</tbody>
</table>

Example 2. Consider the Nash–Cournot equilibrium model that appeared in the paper [7]. The bifunction $f$ has been defined in the following way:

$$f(u,v) = (Au + Bv + c, v - u)$$

where $c \in \mathbb{R}^5$ and matrices $A$, $B$ are

$$A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

while Lipschitz constants $c_1 = c_2 = \frac{1}{2} \| A - B \|$ (see for more details [7,46,47]). The set $C \subset \mathbb{R}^5$ is $C := \{u \in \mathbb{R}^5 : -5 \leq u_i \leq 5\}$. Figures 1 and 2 and Table 3 report the numerical results by choosing $u_{-1} = u_0 = (1, \cdots, 1)$, $\mu = 0.33$ and $\epsilon = 10^{-6}$.

Table 3. Figures 1 and 2: Algorithm 1 numerical comparison with Algorithm 1 in [39].

<table>
<thead>
<tr>
<th>$\vartheta$</th>
<th>$\lambda$</th>
<th>$\lambda_0$</th>
<th>Number of Iterations</th>
<th>Execution Time in Seconds</th>
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<td>0.50</td>
<td>0.25</td>
<td>0.95</td>
<td>45</td>
<td>38</td>
</tr>
</tbody>
</table>
Example 2. Let $f(\hat{p}, \hat{q}) = \langle F(\hat{p}), \hat{q} - \hat{p} \rangle$ and $F(\hat{p}) = G(\hat{p}) + H(\hat{p})$, where

$$G(\hat{p}) = (g_1(\hat{p}), g_2(\hat{p}), \ldots, g_n(\hat{p})),$$

$$H(\hat{p}) = E\hat{p} + c, \quad c = (-1, -1, \ldots, -1)$$

and

$$g_i(\hat{p}) = \hat{p}_{i-1}^2 + \hat{p}_i^2 + \hat{p}_{i-1}\hat{p}_i + \hat{p}_i\hat{p}_{i+1}, \quad i = 1, 2, \ldots, n, \quad \hat{p}_0 = \hat{p}_{n+1} = 0.$$
The entries of a square matrix \( E \) are taken in the following way:

\[
e_{ij} = \begin{cases} 
4 & j = i \\
1 & i - j = 1 \\
-2 & i - j = -1 \\
0 & \text{otherwise},
\end{cases}
\]

where \( \mathcal{C} = \{(u_1, \ldots, u_n) \in \mathbb{R}^n : u_i \geq 1, i = 2, \ldots, n \} \). To see the optimum values of the control parameters, some experiments were carried out taking into account the variation of the control parameters \( \lambda_0 \) and the inertial factor \( \vartheta \). Figures 3–8 and Tables 4 and 5 report the numerical results by choosing \( u_{-1} = u_0 = (1, \ldots, 1) \), \( \mu = 0.33 \) and \( \epsilon = 10^{-6} \).

![Figure 3. Numerical conduct of Algorithm 1 in \( \mathbb{R}^{50} \) by choosing different values of \( \lambda_0 \).](image1)

![Figure 4. Numerical conduct of Algorithm 1 in \( \mathbb{R}^{50} \) by choosing different values of \( \lambda_0 \).](image2)
Figure 5. Numerical conduct of Algorithm 1 in $\mathcal{R}^{200}$ by choosing different values of $\lambda_0$.

Figure 6. Numerical conduct of Algorithm 1 in $\mathcal{R}^{200}$ by choosing different values of $\lambda_0$. 
Figure 7. Numerical conduct of Algorithm 1 in $\mathbb{R}^{50}$ by choosing different values of $\vartheta$.

Figure 8. Numerical conduct of Algorithm 1 in $\mathbb{R}^{50}$ by choosing different values of $\vartheta$.

Table 4. Numerical results for Algorithm 1 in $\mathbb{R}^{50}$ by choosing different values of $\lambda_0$ and $\vartheta$.

<table>
<thead>
<tr>
<th>$\lambda_0$</th>
<th>Number of Iterations</th>
<th>CPU Time</th>
<th>$\vartheta$</th>
<th>Number of Iterations</th>
<th>Execution Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>64</td>
<td>0.9276</td>
<td>0.90</td>
<td>116</td>
<td>1.8066</td>
</tr>
<tr>
<td>0.80</td>
<td>74</td>
<td>1.0975</td>
<td>0.70</td>
<td>80</td>
<td>1.2464</td>
</tr>
<tr>
<td>0.60</td>
<td>84</td>
<td>1.2849</td>
<td>0.50</td>
<td>73</td>
<td>1.1262</td>
</tr>
<tr>
<td>0.40</td>
<td>91</td>
<td>1.3740</td>
<td>0.30</td>
<td>88</td>
<td>1.3788</td>
</tr>
<tr>
<td>0.20</td>
<td>97</td>
<td>1.6298</td>
<td>0.10</td>
<td>102</td>
<td>1.6159</td>
</tr>
</tbody>
</table>
Table 5. Numerical results for Algorithm 1 in $\mathbb{R}^{200}$ by choosing different values of $\lambda_0$ and $\vartheta$.

<table>
<thead>
<tr>
<th>$\lambda_0$</th>
<th>Number of Iterations</th>
<th>CPU Time</th>
<th>$\vartheta$</th>
<th>Number of Iterations</th>
<th>Execution Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>67</td>
<td>13.3967</td>
<td>0.90</td>
<td>105</td>
<td>20.6972</td>
</tr>
<tr>
<td>0.80</td>
<td>78</td>
<td>15.3566</td>
<td>0.70</td>
<td>80</td>
<td>15.5770</td>
</tr>
<tr>
<td>0.60</td>
<td>88</td>
<td>17.3471</td>
<td>0.50</td>
<td>80</td>
<td>15.4838</td>
</tr>
<tr>
<td>0.40</td>
<td>96</td>
<td>18.894</td>
<td>0.30</td>
<td>94</td>
<td>19.4532</td>
</tr>
<tr>
<td>0.20</td>
<td>102</td>
<td>19.9705</td>
<td>0.10</td>
<td>108</td>
<td>21.9745</td>
</tr>
</tbody>
</table>

Example 4. Suppose that $\mathcal{H} = L^2([0, 1])$ is a Hilbert space with an inner product $\langle u, v \rangle = \int_0^1 u(t)v(t)dt$, $\forall u, v \in \mathcal{H}$ and the induced norm

$$||u|| = \sqrt{\int_0^1 |u(t)|^2 dt}.$$

Assume that $C := \{u \in L^2([0, 1]): ||u|| \leq 1\}$ is the unit ball. Let $K : C \to \mathcal{H}$ be

$$K(u)(t) = \int_0^1 (u(t) - H(t,s)f(u(s)))ds + g(t),$$

where

$$H(t, s) = \frac{2ts e^{(1+s)}}{e^{\sqrt{2^2 - 1}}}, \quad f(u) = \cos x, \quad g(t) = \frac{2te^t}{e^{\sqrt{2^2 - 1}}}.$$

We can see in [48] that $K$ is monotone and Lipschitz-continuous with a Lipschitz constant of $L = 2$. Figures 9 and 10 and Table 6 show the numerical results by choosing different values of $u_0$ and $\epsilon = 10^{-6}$.

![Figure 9](image-url)

Figure 9. Algorithm 1 comparison with Algorithm 1 in [39] by choosing values of $u_{-1} = u_0 = 1$. 
Figure 10. Algorithm 1 comparison with Algorithm 1 in [39] by choosing values of $u_{-1} = u_0 = t$.

Table 6. Example 4: numerical comparison of Algorithm 1 with Algorithm 1 in [39].

<table>
<thead>
<tr>
<th>$u_{-1} = u_0$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$\lambda_0$</th>
<th>iEGM</th>
<th>EiEGM</th>
<th>iEGM</th>
<th>EiEGM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.50</td>
<td>0.20</td>
<td>0.50</td>
<td>31</td>
<td>25</td>
<td>0.0158</td>
<td>0.0158</td>
</tr>
<tr>
<td>$t$</td>
<td>0.50</td>
<td>0.20</td>
<td>0.50</td>
<td>31</td>
<td>27</td>
<td>0.0158</td>
<td>0.0158</td>
</tr>
<tr>
<td>$2t^2$</td>
<td>0.50</td>
<td>0.20</td>
<td>0.50</td>
<td>33</td>
<td>30</td>
<td>0.0158</td>
<td>0.0158</td>
</tr>
<tr>
<td>$\sin(t)$</td>
<td>0.50</td>
<td>0.20</td>
<td>0.50</td>
<td>37</td>
<td>30</td>
<td>0.0158</td>
<td>0.0158</td>
</tr>
<tr>
<td>$\exp(t)$</td>
<td>0.50</td>
<td>0.20</td>
<td>0.50</td>
<td>42</td>
<td>32</td>
<td>0.0158</td>
<td>0.0158</td>
</tr>
</tbody>
</table>

Discussion on the Numerical Experiments: We have the following findings about the above-mentioned experiments:

1. The proposed Algorithm 1 does not depend on the Lipschitz constants, unlike Algorithm 1 in [39]. Algorithm 1 uses a variable stepsize that is updated for each iteration and depends on some of the previous iterations. The key advantage of Algorithm 1 is that it works without prior knowledge of the Lipschitz-type constants $c_1$ and $c_2$, unlike Algorithm 1 in [39].

2. Four examples were discussed to compare our proposed method with Algorithm 1 in [39]. In particular, information on Lipschitz constants is missing in Example 3. Due to the missing information of the Lipschitz constants we cannot run Algorithm 1 in [39] because the stepsize $\lambda$ is dependent on Lipschitz constants, i.e., $0 < \lambda < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$. However, we can use the proposed Algorithm 1 to solve Example 3.

3. It is noted that the selection of the $\theta$ value is always important, and precisely the value $\theta \in (3, 6)$ is better than most other values.

4. Choosing of the $\lambda_0$ value is critical and the proposed algorithm performs better when $\lambda_0$ is closer to 1.

5. It can also be acknowledged that the efficiency of an algorithms significantly depends on the nature of the problem and tolerance. More time and a considerable number of iterations are needed for large-scale problems. In this case, we could see that a certain stepsize value improves the efficiency of the algorithm and improves the convergence rate.
Figures 9 and 10 and Table 6 suggest that the choice of initial points and the complexity of the bifunction have an effect on the performance of algorithms in terms of number of iterations and time elapsed.


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