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New Identities Dealing with Gauss Sums

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Abstract: In this article, we used the elementary methods and the properties of the classical Gauss sums to study the problem of calculating some Gauss sums. In particular, we obtain some interesting calculating formulas for the Gauss sums corresponding to the eight-order and twelve-order characters modulo p, where p be an odd prime with $p = 8k + 1$ or $p = 12k + 1$.

Keywords: Gauss sums; elementary method; identity; calculating formula

1. Introduction

For any integer $q > 1$ and any Dirichlet character χ modulo q, the famous Gauss sums $G(m, \chi; q)$ is defined as follows:

$$G(m, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma}{q}\right),$$

where m is any integer and $e(y) = e^{2\pi i y}$.

If χ is any primitive character modulo q or m co-prime to q (that is, $(m, q) = 1$), then we have the identity

$$G(m, \chi; q) = \overline{\chi}(m) G(1, \chi; q) \equiv \overline{\chi}(m) \tau(\chi).$$

If χ is a primitive character modulo q, then for any integer m, we also have the following two important identities:

$$\chi(m) = \frac{1}{\tau\left(\overline{\chi}\right)} \cdot \sum_{b=1}^{q} \overline{\chi}(b)\, e\left(\frac{mb}{q}\right) \quad \text{and} \quad |\tau(\chi)| = \sqrt{q}.$$

As it known to all, the research on the properties of Gauss sums occupies very important position in analytic number theory, many number theory problems are closely related to it. Because of this, many scholars have studied its various properties, and obtained a number of interesting results, some of them and related works can be found in [1–17]. In addition, Gauss sums are closely related to prime numbers. For example, if p is an odd prime with $p \equiv 1 \bmod 3$, then there are two integers d and b such that the identity (see [7]) holds

$$4p = d^2 + 27b^2, \tag{1}$$

where d is uniquely determined by $d \equiv 1 \bmod 3$.

Zhang, W.P. and Hu, J.Y. [1] or Berndt, B.C. and Evans, R.J. [8] studied the properties of Gauss sums of the third-order character modulo p, and proved the following result: Let p be a prime with $p \equiv 1 \bmod 3$. Then for any third-order character χ_3 mod p, one has the identity

$$\tau^3(\chi_3) + \tau^3\left(\overline{\chi_3}\right) = dp, \tag{2}$$

where d is the same as defined in (1).

Chen, Z.Y. and Zhang, W.P. [3] studied the case of the fourth-order character modulo p, and obtained the following conclusion: Let p be a prime with $p \equiv 1 \bmod 4$. Then for any fourth-order character χ_4 mod p, one has the identity

$$\tau^2(\chi_4) + \tau^2(\overline{\chi}_4) = 2\sqrt{p}\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right) = 2\sqrt{p}\cdot\alpha, \tag{3}$$

where $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre's symbol mod p and $a\overline{a} \equiv 1 \bmod p$.

And of course, α in (3) can also be represented by quadratic Gauss sum.

Chen, L. [4] studied the properties of the Gauss sums of the sixth-order character modulo p, and deduced an interesting identity (see Lemma 1 below).

Looking closely at the characteristics of these results, it is not difficult to see that the number of all such characters satisfy $\phi(3) = \phi(4) = \phi(6) = 2$, where $\phi(n)$ denotes the Euler function. So a natural thing to think about is, what if the number of the characters > 2? For example, twelfth-order character modulo p with $\phi(12) = \phi(3)\cdot\phi(4) = 2\cdot 2 = 4 > 2$.

In this paper, we shall use the properties of the classical Gauss sums, the elementary and analytic methods to study this problem, and obtain two interesting identities for them. That is, we shall prove the following two results:

Theorem 1. *Let p be an odd prime with $p \equiv 1 \bmod 8$. Then for any eighth-order character χ_8 modulo p, we have the identity*

$$\tau^4(\chi_8) + \tau^4\left(\chi_8^3\right) = \frac{2\cdot\alpha}{\sqrt{p}}\cdot\tau^2(\chi_8)\cdot\tau^2\left(\chi_8^3\right),$$

where $\alpha = \alpha(p)$ is the same as defined in (3).

Theorem 2. *Let p be an odd prime with $p \equiv 1 \bmod 12$. Then for any third-order character λ and fourth-order character χ_4 modulo p, we have the identity*

$$\left(\tau^3(\lambda\chi_4) + \tau^3(\overline{\lambda}\chi_4)\right)^2 = \frac{d^2}{p}\cdot\tau^3(\lambda\chi_4)\,\tau^3(\overline{\lambda}\chi_4),$$

where d is the same as defined in (1).

From these two theorems we may immediately deduce the following identities:

Corollary 1. *If p be an odd prime with $p \equiv 1 \bmod 8$, then for any eighth-order characters χ_8 modulo p, we have the identities*

$$\left|\tau^4(\chi_8) + \tau^4\left(\chi_8^3\right)\right| = \left|\tau^4(\overline{\chi}_8) + \tau^4\left(\overline{\chi}_8^3\right)\right| = 2p^{\frac{3}{2}}\cdot|\alpha|$$

and

$$\left|\tau^2(\chi_8) \pm \tau^2\left(\chi_8^3\right)\right| = \left|\tau^2(\overline{\chi}_8) \pm \tau^2\left(\overline{\chi}_8^3\right)\right| = p^{\frac{3}{4}}\cdot|2\alpha \pm 2\sqrt{p}|^{\frac{1}{2}}.$$

Corollary 2. *If p is an odd prime with $p \equiv 1 \bmod 12$, then for any third-order character λ and fourth-order characters χ_4 modulo p, we have the identities*

$$\left|\tau^3(\lambda\chi_4) + \tau^3(\overline{\lambda}\chi_4)\right| = \left|\tau^3(\overline{\lambda}\overline{\chi}_4) + \tau^3(\lambda\overline{\chi}_4)\right| = |d|\cdot p,$$

$$\left|\tau^6\left(\lambda\chi_4\right)+\tau^6\left(\overline{\lambda}\chi_4\right)\right|=\left|\tau^6\left(\overline{\lambda}\overline{\chi}_4\right)+\tau^6\left(\lambda\overline{\chi}_4\right)\right|=p^2\cdot\left|d^2-2p\right|$$

and

$$\left|\tau^3\left(\lambda\chi_4\right)-\tau^3\left(\overline{\lambda}\chi_4\right)\right|=\left|\tau^3\left(\overline{\lambda}\overline{\chi}_4\right)-\tau^3\left(\lambda\overline{\chi}_4\right)\right|=3\cdot\sqrt{3}\cdot|b|\cdot p.$$

Some notes: Since $\lambda\chi_4$ is a twelfth-order character modulo p in Theorem 2, so our Theorem 1 and Theorem 2 extend the results in references [1,3,4].

The constant $\alpha=\alpha(p)$ in Theorem 1 has a special meaning. In fact, if $p\equiv 1 \bmod 4$, then we have the identity (see Theorem 4–11 in [18])

$$p=\left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2+\left(\sum_{b=1}^{\frac{p-1}{2}}\left(\frac{b+r\overline{b}}{p}\right)\right)^2=\alpha^2(p)+\beta^2(p),$$

where r is any quadratic non-residue modulo p. That is, $\left(\frac{r}{p}\right)=-1$.

The value of $d=d(p)$ in Theorem 2 depends only on p, and its distribution of the values is very irregular. In order to better understand its properties, here we list the first few values of $d(p)$ as follows: $d(7)=1$, $d(13)=-5$, $d(19)=7$, $d(31)=4$, $d(37)=-11$, $d(43)=-8$, $d(61)=1$, $d(67)=-5$, $d(73)=7$, $d(79)=-17$, $d(97)=19$, $d(103)=13$, $d(109)=-2$, $d(127)=10$, $d(133)=-17,\cdots$.

2. Several Lemmas

In this section, we give several simple lemmas. Of course, the proofs of these lemmas need some knowledge of elementary and analytic number theory. They can be found in many number theory books, such as [18,19], here we do not need to list. First we have the following:

Lemma 1. *Let p be a prime with $p\equiv 1 \bmod 6$. Then for any sixth-order character $\psi \bmod p$, one has the identity*

$$\tau^3(\psi)+\tau^3\left(\overline{\psi}\right)=\begin{cases}p^{\frac{1}{2}}\cdot\left(d^2-2p\right), & \text{if } p\equiv 1 \bmod 12;\\ -i\cdot p^{\frac{1}{2}}\cdot\left(d^2-2p\right), & \text{if } p\equiv 7 \bmod 12,\end{cases}$$

where $i^2=-1$, d is defined as in (1).

Proof. This result is Lemma 3 in Chen, L. [4], so we omit the proof process. □

Lemma 2. *Let p be a prime with $p\equiv 1 \bmod 12$. Then for any third-order character λ and fourth-order character χ_4 modulo p, we have the identity*

$$\tau\left(\lambda\chi_2\right)=\frac{\lambda(2)\chi_2(2)\chi_4(-1)\sqrt{p}\,\tau\left(\overline{\lambda}\chi_4\right)}{\tau(\lambda\chi_4)},$$

where $\left(\frac{*}{p}\right)=\chi_2$ denotes the Legendre's symbol $\bmod\ p$.

Proof. Let $\psi = \lambda\chi_4$ be any twelfth-order character modulo p, where λ is a third-order character and χ_4 is a fourth-order character modulo p respectively. Then note that $\psi^2 = \lambda^2\chi_4^2 = \overline{\lambda}\chi_2$, from the properties of Gauss sums we have

$$\begin{aligned}
&\sum_{a=0}^{p-1} \psi\left(a^2-1\right) = \sum_{a=0}^{p-1} \psi\left((a+1)^2-1\right) = \sum_{a=1}^{p-1} \psi\left(a^2+2a\right) = \sum_{a=1}^{p-1} \psi(a)\psi\left(a+2\right) \\
=\ & \frac{1}{\tau\left(\overline{\psi}\right)} \sum_{a=1}^{p-1} \psi(a) \sum_{b=1}^{p-1} \overline{\psi}(b) e\left(\frac{b(a+2)}{p}\right) = \frac{1}{\tau\left(\overline{\psi}\right)} \sum_{b=1}^{p-1} \overline{\psi}(b) \sum_{a=1}^{p-1} \psi(a) e\left(\frac{b(a+2)}{p}\right) \\
=\ & \frac{\tau(\psi)}{\tau\left(\overline{\psi}\right)} \sum_{b=1}^{p-1} \overline{\psi}(b)\overline{\psi}(b) e\left(\frac{2b}{p}\right) = \frac{\tau(\psi)}{\tau\left(\overline{\psi}\right)} \sum_{b=1}^{p-1} \lambda(b)\chi_2(b) e\left(\frac{2b}{p}\right) \\
=\ & \frac{\overline{\lambda}(2)\chi_2(2)\tau(\lambda\chi_2)\tau(\psi)}{\tau\left(\overline{\psi}\right)}. \qquad (4)
\end{aligned}$$

On the other hand, note that for any integer b with $(b, p) = 1$, we have the identity

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = \chi_2(b) \cdot \tau\left(\chi_2\right) = \chi_2(b) \cdot \sqrt{p},$$

so note that $\lambda(-1) = 1$ we also have the identity

$$\begin{aligned}
&\sum_{a=0}^{p-1} \psi\left(a^2-1\right) = \frac{1}{\tau\left(\overline{\psi}\right)} \sum_{a=0}^{p-1}\sum_{b=1}^{p-1} \overline{\psi}(b) e\left(\frac{b(a^2-1)}{p}\right) \\
=\ & \frac{1}{\tau\left(\overline{\psi}\right)} \sum_{b=1}^{p-1} \overline{\psi}(b) e\left(\frac{-b}{p}\right) \sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = \frac{\tau\left(\chi_2\right)}{\tau\left(\overline{\psi}\right)} \sum_{b=1}^{p-1} \overline{\psi}(b)\chi_2(b) e\left(\frac{-b}{p}\right) \\
=\ & \frac{\tau\left(\chi_2\right)}{\tau\left(\overline{\psi}\right)} \sum_{b=1}^{p-1} \overline{\lambda}(b)\chi_4(b) e\left(\frac{-b}{p}\right) = \frac{\sqrt{p}\cdot\chi_4(-1)\cdot\tau\left(\overline{\lambda}\chi_4\right)}{\tau\left(\overline{\psi}\right)}. \qquad (5)
\end{aligned}$$

Combining (4) and (5) we have the identity

$$\tau\left(\lambda\chi_2\right) = \frac{\lambda(2)\chi_2(2)\chi_4(-1)\sqrt{p}\,\tau\left(\overline{\lambda}\chi_4\right)}{\tau(\lambda\chi_4)},$$

This proves Lemma 2. □

Lemma 3. *Let p be a prime with $p \equiv 1 \bmod 8$. Then for any eighth-order character χ_8 modulo p, we have the identity*

$$\tau\left(\overline{\chi}_4\right) = \tau\left(\overline{\chi}_8^2\right) = \frac{\overline{\chi}_4(2)\chi_8(-1)\sqrt{p}\,\tau\left(\chi_8^3\right)}{\tau\left(\chi_8\right)}.$$

Proof. Let χ_8 be an eighth-order character modulo p, then from the properties of Gauss sums we have

$$\begin{aligned}
&\sum_{a=0}^{p-1} \chi_8\left(a^2-1\right) = \sum_{a=0}^{p-1} \chi_8\left((a+1)^2-1\right) = \sum_{a=1}^{p-1} \chi_8\left(a^2+2a\right) = \sum_{a=1}^{p-1} \chi_8(a)\chi_8\left(a+2\right) \\
=\ & \frac{1}{\tau\left(\overline{\chi}_8\right)} \sum_{a=1}^{p-1} \chi_8(a) \sum_{b=1}^{p-1} \overline{\chi}_8(b) e\left(\frac{b(a+2)}{p}\right) = \frac{1}{\tau\left(\overline{\chi}_8\right)} \sum_{b=1}^{p-1} \overline{\chi}_8(b) \sum_{a=1}^{p-1} \chi_8(a) e\left(\frac{b(a+2)}{p}\right) \\
=\ & \frac{\tau(\chi_8)}{\tau\left(\overline{\chi}_8\right)} \sum_{b=1}^{p-1} \overline{\chi}_8(b)\overline{\chi}_8(b) e\left(\frac{2b}{p}\right) = \frac{\tau(\chi_8)}{\tau\left(\overline{\chi}_8\right)} \sum_{b=1}^{p-1} \overline{\chi}_4(b) e\left(\frac{2b}{p}\right) = \frac{\chi_4(2)\tau\left(\overline{\chi}_4\right)\tau(\chi_8)}{\tau\left(\overline{\chi}_8\right)}. \qquad (6)
\end{aligned}$$

On the other hand, we also have the identity

$$
\begin{aligned}
&\sum_{a=0}^{p-1}\chi_8\left(a^2-1\right)=\frac{1}{\tau\left(\overline{\chi}_8\right)}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}_8(b)e\left(\frac{b(a^2-1)}{p}\right)\\
=\;&\frac{1}{\tau\left(\overline{\chi}_8\right)}\sum_{b=1}^{p-1}\overline{\chi}_8(b)e\left(\frac{-b}{p}\right)\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right)=\frac{\sqrt{p}}{\tau\left(\overline{\chi}_8\right)}\sum_{b=1}^{p-1}\overline{\chi}_8(b)\chi_2(b)e\left(\frac{-b}{p}\right)\\
=\;&\frac{\sqrt{p}}{\tau\left(\overline{\chi}_8\right)}\sum_{b=1}^{p-1}\chi_8^3(b)e\left(\frac{-b}{p}\right)=\frac{\sqrt{p}\cdot\chi_8(-1)\cdot\tau\left(\chi_8^3\right)}{\tau\left(\overline{\chi}_8\right)}. \qquad (7)
\end{aligned}
$$

Combining (6) and (7) we have the identity

$$\tau\left(\overline{\chi}_4\right)=\frac{\overline{\chi}_4(2)\chi_8(-1)\sqrt{p}\,\tau\left(\chi_8^3\right)}{\tau\left(\chi_8\right)}.$$

This proves Lemma 3. □

3. Proof of the Theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. Since $p\equiv 1 \bmod 8$, so we have $\overline{\chi}_4^2(2)=\chi_4^2(2)=\left(\frac{2}{p}\right)=1$. Note that $\tau\left(\chi_8\right)\tau\left(\overline{\chi}_8\right)=\overline{\chi}_8(-1)\tau\left(\chi_8\right)\cdot\overline{\tau\left(\chi_8\right)}=\overline{\chi}_8(-1)\cdot p$ and $\tau^2\left(\chi_8^3\right)\tau^2\left(\overline{\chi}_8^3\right)=p^2$, from Lemma 3 we have

$$\tau^2\left(\overline{\chi}_4\right)=\frac{p\cdot\tau^2\left(\chi_8^3\right)}{\tau^2\left(\chi_8\right)} \qquad (8)$$

and

$$\tau^2\left(\chi_4\right)=\frac{p\cdot\tau^2\left(\overline{\chi}_8^3\right)}{\tau^2\left(\overline{\chi}_8\right)}=\frac{p\cdot\tau^2\left(\chi_8\right)}{\tau^2\left(\chi_8^3\right)}. \qquad (9)$$

From (3), (8) and (9) we have

$$\tau^2\left(\chi_4\right)+\tau^2\left(\overline{\chi}_4\right)=2\sqrt{p}\cdot\alpha=\frac{p\cdot\tau^2\left(\chi_8\right)}{\tau^2\left(\chi_8^3\right)}+\frac{p\cdot\tau^2\left(\chi_8^3\right)}{\tau^2\left(\chi_8\right)}$$

or

$$\tau^4\left(\chi_8\right)+\tau^4\left(\chi_8^3\right)=\frac{2\cdot\alpha}{\sqrt{p}}\cdot\tau^2\left(\chi_8\right)\cdot\tau^2\left(\chi_8^3\right).$$

This proves Theorem 1.

Now we prove Theorem 2. Note that $\lambda(-1)=1$, $\chi_2(2)=(-1)^{\frac{p-1}{4}}$ and $\tau(\lambda\chi_4)\tau\left(\overline{\lambda}\overline{\chi}_4\right)=\overline{\chi}_4(-1)\cdot\tau(\lambda\chi_4)\cdot\overline{\tau\left(\lambda\chi_4\right)}=\overline{\chi}_4(-1)\cdot p$, from Lemma 2 we have

$$\tau^3(\lambda\chi_2)=\frac{\lambda^3(2)\chi_2^3(2)\chi_4^3(-1)p^{\frac{3}{2}}\,\tau^3\left(\overline{\lambda}\chi_4\right)}{\tau^3(\lambda\chi_4)}=p^{\frac{3}{2}}\cdot\frac{\tau^3\left(\overline{\lambda}\chi_4\right)}{\tau^3(\lambda\chi_4)} \qquad (10)$$

and

$$\tau^3\left(\overline{\lambda}\chi_2\right)=p^{\frac{3}{2}}\cdot\frac{\tau^3\left(\lambda\chi_4\right)}{\tau^3\left(\overline{\lambda}\chi_4\right)}. \qquad (11)$$

Then from (10), (11) and Lemma 1 we have the identity

$$\tau^3(\lambda\chi_2) + \tau^3\left(\overline{\lambda}\chi_2\right) = p^{\frac{1}{2}} \cdot \left(d^2 - 2p\right) = p^{\frac{3}{2}} \cdot \frac{\tau^3\left(\overline{\lambda}\chi_4\right)}{\tau^3(\lambda\chi_4)} + p^{\frac{3}{2}} \cdot \frac{\tau^3\left(\lambda\chi_4\right)}{\tau^3\left(\overline{\lambda}\chi_4\right)}$$

or

$$\frac{\tau^3\left(\overline{\lambda}\chi_4\right)}{\tau^3(\lambda\chi_4)} + \frac{\tau^3\left(\lambda\chi_4\right)}{\tau^3\left(\overline{\lambda}\chi_4\right)} = \frac{d^2 - 2p}{p}. \tag{12}$$

From (12) we have the identity

$$\tau^6\left(\lambda\chi_4\right) + \tau^6\left(\overline{\lambda}\chi_4\right) + 2\tau^3\left(\lambda\chi_4\right)\tau^3\left(\overline{\lambda}\chi_4\right) = \frac{d^2}{p} \cdot \tau^3\left(\lambda\chi_4\right)\tau^3\left(\overline{\lambda}\chi_4\right)$$

or

$$\left(\tau^3\left(\lambda\chi_4\right) + \tau^3\left(\overline{\lambda}\chi_4\right)\right)^2 = \frac{d^2}{p} \cdot \tau^3\left(\lambda\chi_4\right)\tau^3\left(\overline{\lambda}\chi_4\right).$$

This completes the proof of Theorem 2.

4. Conclusions

The main results of this paper are two identities involving some special Gauss sums. Theorem 1 obtained an identity for the Gauss sums involving the eighth-order character modulo p. Theorem 2 proved an identity for the Gauss sums involving the twelfth-order character sums modulo p. As some corollaries of these theorems, the results in the references [1,3,4] are generalized and extended. These results not only give the exact values of some special Gauss sums, and they are also a new contribution to research in related fields.

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