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Explicit Formulas for All Scator Holomorphic Functions in the (1+2)-Dimensional Case

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Abstract: Scators form a vector space endowed with a non-distributive product, in the hyperbolic case, have physical applications related to some deformations of special relativity (breaking the Lorentz symmetry) while the elliptic case leads to new examples of hypercomplex numbers and related notions of holomorphicity. Until now, only a few particular cases of scator holomorphic functions have been found. In this paper we obtain all solutions of the generalized Cauchy–Riemann system which describes analogues of holomorphic functions in the $(1+2)$-dimensional scator space.

Keywords: scators; holomorphic functions; generalized Cauchy–Riemann equations

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1. Introduction

Scators, as defined by Manuel Fernández-Guasti and Felipe Zaldívar [1], form a linear space with a specific multiplicative structure. In fact, we have two different structures: elliptic and hyperbolic. Namely, in the elliptic case, the scator product of scators $\overset{o}{a} := (a_0; a_1, \ldots, a_n)$ and $\overset{o}{b} := (b_0; b_1, \ldots, b_n)$ is given by $\overset{o}{u} := (u_0; u_1, \ldots, u_n)$, where

$$
\begin{aligned}
u_0 &= a_0 b_0 \prod_{k=1}^{n} \left(1 - \frac{a_k b_k}{a_0 b_0}\right) , \\
u_k &= \frac{a_k b_0 + b_k a_0}{a_0 b_0 - a_k b_k}\, u_0 \qquad (k = 1, \ldots, n) ,
\end{aligned}
\tag{1}
$$

provided that $a_0 \neq 0$ and $b_0 \neq 0$ (more general case is presented and discussed in [1]). In the hyperbolic case, the formula is very similar (all minuses are replaced by pluses). In principle, one can consider mixed cases as well. The scator product is non-distributive, although a distributive approach has been proposed recently [2,3]. The so-called restricted space (defined by $a_0^2 > a_k^2$ for $k = 1, \ldots, n$) is an abelian group with respect to the scator product.

In the hyperbolic case, scators have potential physical applications related to generalizations of the special theory of relativity (breaking the Lorentz symmetry) [4,5]. The elliptic case is an interesting new example of (non-distributive) hypercomplex numbers [6].

Any hypercomplex numbers, like quaternions or Clifford numbers, lead to a natural question of defining and finding anlogues of holomorphic functions. In this paper, following [7], we focus on the most straightforward definition of holomorphicity, i.e., existence, at any point, of a direction-independent derivative. Fernández-Guasti derived a system of partial differential equations which assures scator differentiabiliy of this kind [7]. They can be considered as a generalization of Cauchy–Riemann equations of standard complex analysis:

$$
\begin{aligned}
&\frac{\partial u_0}{\partial x_0} = \frac{\partial u_j}{\partial x_j}\,, \quad \frac{\partial u_j}{\partial x_0} = -\frac{\partial u_0}{\partial x_j}\,, \\
&\frac{\partial u_0}{\partial x_j}\frac{\partial u_0}{\partial x_m} = -\frac{\partial u_j}{\partial x_j}\frac{\partial u_j}{\partial x_m}
\end{aligned}
\tag{2}
$$

for all $m \neq j$, where m and j take values from 1 to n. Note that the last (nonlinear) equations appear only for $n > 1$. The generalized Cauchy–Riemann Equation (2) consists of a set of linear equations (n copies of the Cauchy–Riemann equations, in fact) and a set of nonlinear equations (for $n > 1$). The latter is the main difference with the standard case of complex holomorphic functions (i.e., the case $n = 1$).

In this paper, we are going to solve the open problem of finding all solutions of the system (4) in the case $n = 2$. Until now only two particular solutions were reported: four-parameter family of linear affine functions [7] and components exponential function [8].

2. Generalized Cauchy–Riemann System in the Case $n = 2$

In this paper, we confine ourselves to the elliptic scator space of dimension $1 + 2$. We introduce the notation:

$$
x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad u_0 = u, \quad u_1 = v, \quad u_2 = w. \tag{3}
$$

In the new notation, the generalized Cauchy–Riemann system (2) takes the following form

$$
\begin{aligned}
&u_x = v_y, \qquad u_y = -v_x, \\
&u_x = w_z, \qquad u_z = -w_x, \\
&u_y u_z = -v_y v_z = -w_y w_z\,,
\end{aligned}
\tag{4}
$$

where here and throughout the rest of this paper the subscripts x, y, z mean partial derivative with respect to the corresponding variable.

Theorem 1. *The full set of solutions to the generalized Cauchy–Riemann Equation (4) consists of three families.*

- *Components exponential functions*

$$
\begin{aligned}
u &= q_0 + p_0 e^{k_0 x}\cos(k_0 y + k_1)\cos(k_0 z + k_2)\,, \\
v &= q_1 + p_0 e^{k_0 x}\sin(k_0 y + k_1)\cos(k_0 z + k_2)\,, \\
w &= q_2 + p_0 e^{k_0 x}\cos(k_0 y + k_1)\sin(k_0 z + k_2)\,.
\end{aligned}
\tag{5}
$$

- *Linear functions*

$$
\begin{aligned}
u &= b_0 + a_0(x + f_0 y + g_0 z)\,, \\
v &= b_1 + a_0(y - f_0 x - f_0 g_0 z)\,, \\
w &= b_2 + a_0(z - g_0 x - f_0 g_0 y)\,.
\end{aligned}
\tag{6}
$$

- *Exceptional solutions*

$$
\begin{aligned}
&u = c_0 y + c_1\,, \quad v = -c_0 x + V_1(z)\,, \quad w = W_1(z)\,, \\
&u = d_0 z + d_1\,, \qquad v = V_2(y)\,, \qquad w = -d_0 x + W_2(z)\,,
\end{aligned}
\tag{7}
$$

where $q_0, q_1, q_2, p_0, k_0, k_1, k_2, b_0, b_1, b_2, a_0, f_0, g_0, c_0, c_1, d_0$ *and* d_1 *are real constants* ($p_0 \neq 0$, $k_0 \neq 0$), *and* V_1, V_2, W_1 *and* W_2 *are arbitrary functions of one variable.*

The above set of solutions is not very rich, but one has to remember that in the case of quaternionic analysis the analogous set is much narrower and consists only of linear affine functions [9,10]. Therefore the quaternionic analysis, like the Clifford analysis, has to use other definitions of holomorphicity, see, e.g., [11].

In next sections, we present the proof of Theorem 1, by straightforward derivation of all solutions. It is convenient to divide the computation into three cases related to vanishing of the first and second x-derivatives of u.

3. Components Exponential Functions

This is the case characterized by

$$u_x \not\equiv 0 \quad \text{and} \quad u_{xx} \not\equiv 0 \,. \tag{8}$$

We are going to express the variables v and w in terms of u and its derivatives:

$$\begin{aligned} &v_x = -u_y, \quad v_y = u_x, \quad v_z = u_y u_z / u_x \,, \\ &w_x = -u_z, \quad w_z = u_x \,, \quad w_y = -u_y u_z / u_x . \end{aligned} \tag{9}$$

Necessary conditions for the existence of v and w (provided that u is known) are given by:

$$\begin{aligned} &v_{xy} = v_{yx} \,, \qquad v_{xz} = v_{zx} \,, \qquad v_{yz} = v_{zy} \,, \\ &w_{xy} = w_{yx} \,, \qquad w_{xz} = w_{zx} \,, \qquad w_{yz} = w_{zy} \,. \end{aligned} \tag{10}$$

Rewriting compatibility conditions (10) in terms of u (and its derivatives), we obtain:

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \,, \\ &u_{yz} - (u_y u_z / u_x)_x = 0 \,, \\ &u_{xz} + (u_y u_z / u_x)_y = 0 \,, \\ &u_{zy} - (u_y u_z / u_x)_x = 0 \,, \\ &u_{zz} + u_{xx} = 0 \,, \\ &u_{xy} + (u_y u_z / u_x)_z = 0 \,. \end{aligned} \tag{11}$$

In other words, removing a redundant equation, we have:

$$\begin{aligned} &u_{yy} = u_{zz} = -u_{xx} \,, \\ &u_{yz} u_x^2 = u_{xy} u_x u_z + u_{xz} u_x u_y - u_y u_z u_{xx} \,, \\ &u_{xz} u_x^2 = -u_{yy} u_x u_z - u_{yz} u_x u_y + u_y u_z u_{xy} \,, \\ &u_{xy} u_x^2 = -u_{yz} u_x u_z - u_{zz} u_x u_y + u_y u_z u_{xz} \,, \end{aligned} \tag{12}$$

or, in the matrix form:

$$\begin{pmatrix} u_x u_z & u_x u_y & -u_x^2 \\ -u_y u_z & u_x^2 & u_x u_y \\ u_x^2 & -u_y u_z & u_x u_z \end{pmatrix} \begin{pmatrix} u_{xy} \\ u_{xz} \\ u_{yz} \end{pmatrix} = u_{xx} \begin{pmatrix} u_y u_z \\ u_x u_z \\ u_x u_y \end{pmatrix} , \tag{13}$$

where u_{yy} and u_{zz} were replaced by $-u_{xx}$. This equation can be solved by inverting the matrix on the left-hand side:

$$\begin{pmatrix} u_{xy} \\ u_{xz} \\ u_{yz} \end{pmatrix} = \frac{u_{xx}}{u_x^2(u_x^2+u_y^2)(u_x^2+u_z^2)} \begin{pmatrix} u_x u_z(u_x^2+u_y^2) & 0 & u_x^2(u_x^2+u_y^2) \\ u_x u_y(u_x^2+u_z^2) & u_x^2(u_x^2+u_z^2) & 0 \\ u_y^2 u_z^2 - u_x^4 & u_x u_y(u_x^2+u_z^2) & u_x u_z(u_x^2+u_y^2) \end{pmatrix} \begin{pmatrix} u_y u_z \\ u_x u_z \\ u_x u_y \end{pmatrix}. \tag{14}$$

Now, the right-hand side turns out to be surprisingly simple and Equation (14) is equivalent to the following system of three equations:

$$u_{xy} = \frac{u_y}{u_x} u_{xx} \,, \qquad u_{xz} = \frac{u_z}{u_x} u_{xx} \,, \qquad u_{yz} = \frac{u_y u_z}{u_x^2} u_{xx} \,. \tag{15}$$

The first two equations can be expressed as conservation laws and, then, easily solved:

$$\begin{aligned} \frac{d}{dx}\left(\frac{u_y}{u_x}\right) &= 0 \quad \Longrightarrow \quad u_y = f(y,z) u_x \,, \\ \frac{d}{dx}\left(\frac{u_z}{u_x}\right) &= 0 \quad \Longrightarrow \quad u_z = g(y,z) u_x \,, \end{aligned} \tag{16}$$

where f and g are some functions of two variables. Substituting (16) into Equation (15) we obtain

$$u_{xy} = f u_{xx} \,, \qquad u_{xz} = g u_{xx} \,, \qquad u_{yz} = f g u_{xx} \,. \tag{17}$$

Differentiating (16) with respect to z and y, respectively, we get

$$u_{yz} = f_z u_x + f u_{xz} \,, \qquad u_{zy} = g_y u_x + g u_{xy} \,. \tag{18}$$

Then, using (17), we obtain $f_z = 0$ and $g_y = 0$, i.e.,

$$f = f(y) \,, \qquad g = g(z) \,. \tag{19}$$

Now, differentiating (16) with respect to y and z, respectively, we get

$$\begin{aligned} f_y u_x + f^2 u_{xx} &= -u_{xx} \\ g_z u_x + g^2 u_{xx} &= -u_{xx}, \end{aligned} \tag{20}$$

where we took into account $u_{yy} = -u_{xx}$ and $u_{zz} = -u_{xx}$. Therefore:

$$\frac{f_y}{1+f^2} = \frac{g_z}{1+g^2} = -\frac{u_{xx}}{u_x} \,. \tag{21}$$

Thus, by virtue of (19), we have

$$\frac{f_y}{1+f^2} = -k_0 \,, \qquad \frac{g_z}{1+g^2} = -k_0 \,, \qquad u_{xx} = k_0 u_x \,, \tag{22}$$

where $k_0 = \text{const}$. In this section, due to the condition (8), we confine ourselves to $k_0 \neq 0$. Then

$$\begin{aligned} &k_0 \neq 0 \quad \Longrightarrow \quad u = p(y,z) e^{k_0 x} + q(y,z) \,, \\ &f = -\tan(k_0 y + k_1) \,, \qquad g = -\tan(k_0 z + k_2) \,, \end{aligned} \tag{23}$$

where p and q are functions of two variables and k_1 and k_2 are constants. Then, Equation (16) imply

$$\begin{aligned} &q_y = q_z = 0 \quad \Longrightarrow \quad q = q_0 = \text{const}\,, \\ &p_y = k_0 p f\,, \quad p_z = k_0 p g\,. \end{aligned} \tag{24}$$

The last two equations can be solved, yielding

$$p = p_0 \cos(k_0 y + k_1) \cos(k_0 z + k_2)\,, \tag{25}$$

where $p_0 = \text{const}$. Hence, u is proved to be of the form (5). Note that now the equations from the first line of (12) are identically satisfied. Finally, the system (9) takes the form

$$\begin{aligned} &v_x = k_0 e^{k_0 x} \sin(k_0 y + k_1) \cos(k_0 z + k_2)\,, \quad v_y = w_z = k_0 e^{k_0 x} \cos(k_0 y + k_1) \cos(k_0 z + k_2)\,, \\ &w_x = k_0 e^{k_0 x} \cos(k_0 y + k_1) \sin(k_0 z + k_2)\,, \quad v_z = w_y = k_0 e^{k_0 x} \sin(k_0 y + k_1) \sin(k_0 z + k_2)\,, \end{aligned} \tag{26}$$

and its only solution is given by the last two equations of (5). Special case $k_1 = k_2 = q_0 = q_1 = q_2 = 0$ and $k_0 = p_0 = 1$, known as components exponential function, was shown to be differentiable (i.e., scator holomorphic) earlier, see [8], Lemma 2.

4. Linear Functions

Linear functions satisfying (4) can be obtained directly, by substituting a linear ansatz and computing its coefficients. However, in order to obtain all solutions to (4), we follow the pattern of the previous section, now assuming:

$$u_x \not\equiv 0\,, \quad \text{and} \quad u_{xx} \not\equiv 0\,. \tag{27}$$

Then, the third equation of (22) implies $k_0 = 0$.

$$k_0 = 0 \quad \Longrightarrow \quad u_{xx} = 0\,, u_{yy} = 0\,, u_{zz} = 0 \quad \Longrightarrow \quad u = b_0 + a_1 x + a_2 y + a_3 z\,, \tag{28}$$

where a_1, a_2, a_3 and b_0 are real constants ($a_1 \neq 0$). Moreover, due to (22), $f = f_0 = \text{const}$ and $g = g_0 = \text{const}$, and from (16) we have

$$a_1 = f_0 a_0\,, \qquad a_2 = g_0 a_0\,. \tag{29}$$

Substituting the above formula for u, we reduce the system (4) into

$$\begin{aligned} &v_x = -f_0 a_0\,, \quad v_y = a_0\,, \quad v_z = -f_0 g_0 a_0\,, \\ &w_x = -g_0 a_0\,, \quad w_y = -f_0 g_0 a_0\,, \quad w_z = a_0\,. \end{aligned} \tag{30}$$

Hence

$$\begin{aligned} &v = b_1 - f_0 a_0 x + a_0 y - f_0 g_0 a_0 z\,, \\ &w = b_2 - g_0 a_0 x - f_0 g_0 a_0 y + a_0 z\,, \end{aligned} \tag{31}$$

where b_1 and b_2 are real constants. Special case $f_0 = g_0 = 0$ (linear affine functions) was shown to be differentiable in [7], Lemma 4.1.

5. Exceptional Solutions

The last case corresponds to $u_x = 0$. Then Equation (4) yields, immediately, $v_y = w_z = 0$, and, the last equation reads $u_y u_z = 0$. Thus we have two distinct subcases: $u_y = 0$ (i.e., $u = u(z)$) and $u_z = 0$ (i.e., $u = u(y)$).

1. $u = u(y),\ v = v(x,z),\ w = w(x,y)$.

Then $w_x = 0$ and $u_y = -v_x$. Hence $u_y = c_0 = \text{const}$ and we get the following solution:

$$u = c_0 y + c_1 \,, \quad v = -c_0 x + V_1(z) \,, \quad w = W_1(y) \,, \tag{32}$$

where c_0 and c_1 are constant and $V_1 = V_1(z)$ and $W_1 = W_1(y)$ are arbitrary functions of one variable.

2. $u = u(z), \; v = v(x,z), \; w = w(x,y)$.

   Then, $w_x = -u_z$ and $v_x = 0$. Hence $u_z = d_0$ and, as a result, we get the solution:

$$u = d_0 z + d_1 \,, \quad v = V_2(y) \,, \quad w = -d_0 x + W_2(z) \,, \tag{33}$$

   where d_0 and d_1 are onstant and $V_2 = V_2(y)$ and $W_2 = W_2(z)$ are arbitrary functions of one variable.

Finally, we derived all solutions to the system (4) and thus Theorem 1 is proved.

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