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$\mathcal{PT}$-Symmetric Qubit-System States in the Probability Representation of Quantum Mechanics

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Abstract: $\mathcal{PT}$-symmetric qubit-system states are considered in the probability representation of quantum mechanics. The new energy eigenvalue equation for probability distributions identified with qubit and qutrit states is presented in an explicit form. A possibility to test $\mathcal{PT}$-symmetry and its violation by measuring the probabilities of spin projections for qubits in three perpendicular directions is discussed.

Keywords: probability representation; qubit; energy levels; superposition principle; Born's rule; non-Hermitian Hamiltonian

1. Introduction

In the conventional formulation of quantum mechanics, the pure states of physical systems are identified with state vectors $|\psi\rangle$ belonging to a Hilbert space H [1] and with the wave functions [2,3] $\psi(x) = \langle x|\psi\rangle$ satisfying the Schrödinger equation [2] determined by the Hamiltonian operator $\hat{H}$ acting in the Hilbert space and being the Hermitian operator, i.e., $\hat{H}^\dagger = \hat{H}$. The eigenfunctions of the Hamiltonian describe the energy levels of quantum systems. The eigenfunctions of the Hamiltonian are associated with the stationary states of the quantum systems. For qubits (two-level atoms or spin-1/2 systems), the Hilbert space H is a two-dimensional space, and the Hamiltonian operator $\hat{H}$ is represented by the Hermitian 2×2-matrix with real eigenvalues and orthogonal eigenvectors. Additionally, in conventional quantum mechanics, the superposition principle means that, for any two arbitrary state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, there exists the vector $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$, with c_1 and c_2 being the complex numbers; this vector $|\psi\rangle$ describes the state of the system which does exist [1].

Recently, the problem of non-Hermitian Hamiltonian operators with real eigenvalues was discussed in [4–8] in connection with $\mathcal{PT}$-symmetry (parity–time) of the physical problems, where the eigenvectors and eigenvalues of non-Hermitian Hamiltonians with such symmetries are associated with specific properties of the physical systems (see, for example, [9,10]), including the dynamical Casimir effect reviewed in [11,12] where the non-Hermiticity of the Hamiltonian is related to accounting for the dissipation processes [13]. The problems of non-Hermitian quantum mechanics were studied in [14–18]. The differences between conventional quantum mechanics and non-Hermitian quantum mechanics in the Hilbert-space representation of pure and mixed quantum states were analyzed in [19]. The applications of the $\mathcal{PT}$-symmetry approach in quantum optics and physics of oscillators were discussed in [20]. The anti-$\mathcal{PT}$-symmetry approach in qubit states was presented in [21]. An exactly solvable pseudo-Hermitian system with $SU(1,1)$ symmetry was studied in [22].

The system of oscillators with $\mathcal{PT}$-symmetries was considered in [23]. The Heisenberg representation for the non-Hermitian systems was given in [24]. The $\mathcal{PT}$-symmetric states of different systems, including time-dependent oscillators, were investigated in [25,26]. The experimental study of a single dissipative qubit connected with quantum-state tomography related to the system behavior near exceptional points associated with $\mathcal{PT}$-symmetry 0f the non-Hermitian Hamiltonian was presented in [27].

In conventional quantum mechanics, in addition to the wave functions [2] and density matrices [28,29], different representations of quantum states were introduced. The Wigner function $W(q,p)$ [30], Husimi–Kano function $Q(q,p)$ [31,32], and Glauber–Sudarshan $P(q,p)$-function [33,34] are important examples of the phase–space representations of the states of quantum systems with continuous variables, e.g., oscillator systems.

On the other hand, the probability representation of quantum states for such systems was suggested in [35], where the states were identified with fair tomographic-probability distributions—symplectic tomograms $w(X|\mu,\nu)$ of the oscillator position X depending on real parameters μ and ν determining the reference frames in the phase space where the position X is measured. A recent review of the tomographic-probability representation of quantum states can be found in [36]. The tomogram is connected with the density matrix and Wigner function by the invertible Radon integral transform [37]. The relation of the tomographic-probability representation with the phase–space representation [38] of quantum state was discussed in [39]. The probability representation was found for such systems as oscillators and photon states, and others. In the case of two-level atoms (qubit, spin-1/2 system), the quantum-system states were identified with fair probability distributions of three dichotomic random variables [40–46]. It was shown that the states of quantum systems, such as oscillators and qubits, can completely be described by fair probability distributions. This approach provided the possibility to study new entropic-information inequalities for quantum states associated with the descriptions of the states by probability distributions determining the entropy in conventional probability theory [47,48].

The Schrödinger equations determining the energy levels of quantum systems with Hermitian Hamiltonians and the evolution of the wave functions in the probability representation take the form of equations for the probability distributions determining quantum states [49,50]. Till now the systems with non-Hermitian Hamiltonians were not considered in the probability representations of quantum states. The aim of our work was to construct, for an example of qubit states, the representation of these states by the probability distributions of dichotomic random variables. We present the Schrödinger equation for eigenvectors of non-Hermitian Hamiltonians with real eigenvalues in the form of linear equations for dichotomic probability distributions of random variables determined by matrix elements of the non-Hermitian matrices with $\mathcal{PT}$-symmetry properties. In fact, we show the way to construct an invertible map of the problem of solving the Schrödinger eigenvalue equation for non-Hermitian Hamiltonians with real eigenvalues, considering the problem of solving the corresponding linear equation for the probabilities associated with some sets in simplex, analogously to the case of quantum-mechanical states with Hermitian Hamiltonians studied in [48]. Additionally, we constructed the probability representation for eigenvectors of arbitrary complex matrices with complex eigenvalues.

2. Two-Level Atom States

We start with the example of qubit state. We consider the generic non-Hermitian Hamiltonian operator for the two-level atom

$$\hat{H} = \alpha|1\rangle\langle 1| + \beta|2\rangle\langle 2| + \gamma|1\rangle\langle 2| + \delta|2\rangle\langle 1|, \qquad \langle j|k\rangle = \delta_{jk}, \tag{1}$$

where α, β, γ, and δ are complex numbers. The matrix H of the Hamiltonian on the basis of two vectors $|1\rangle$ and $|2\rangle$, such that $\langle 1|1\rangle = \langle 2|2\rangle = 1$ and $\langle 1|2\rangle = 0$, has the form

$$H = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}; \tag{2}$$

complex eigenvalues of this matrix read

$$\lambda_1 = \frac{\alpha+\beta}{2} + \sqrt{\left(\frac{\alpha-\beta}{2}\right)^2 + \gamma\delta}, \qquad \lambda_2 = \frac{\alpha+\beta}{2} - \sqrt{\left(\frac{\alpha-\beta}{2}\right)^2 + \gamma\delta}. \tag{3}$$

If $\mathrm{Im}\,(\alpha+\beta) = 0$ and the number $\left([(\alpha-\beta)/2]^2 + \gamma\delta\right)$ is a real number $s^2 \geq 0$, the eigenvalues λ_1 and λ_2 of the matrix H and its eigenvectors are expressed in terms of the matrix elements of the matrix H as follows:

$$|\psi_{1,2}\rangle = N_{1,2} \begin{pmatrix} 1 \\ (\lambda_{1,2}-\alpha)/\gamma \end{pmatrix}. \tag{4}$$

Two nonnegative normalization constants $N_{1,2}$ satisfy the equation

$$|N_{1,2}|^2 \left[1 + \left|\frac{\lambda_{1,2}-\alpha}{\gamma}\right|^2\right] = 1, \tag{5}$$

where eigenvalues $\lambda_{1,2}$ are determined by Equation (3).

To construct the probability representation of non-Hermitian Hamiltonian eigenvectors $|\psi_1\rangle$ and $|\psi_2\rangle$, we review the probability representation of qubit states (spin-1/2 states, two-level atom states) [40–46]. For Hermitian Hamiltonians $\hat{H}^\dagger = \hat{H}$, the matrix elements of matrix H in Equation (2) satisfy the conditions $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta = 0$ and $\delta = \gamma^*$. In this case, the eigenvalues of matrix H satisfy the equalities $\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = 0$. The energy levels $E_1 = \lambda_1$ and $E_2 = \lambda_2$ of two-level atom are expressed in terms of the matrix elements of Hamiltonian (2), i.e., real numbers α and β and complex numbers γ and $\delta = \gamma^*$, as follows:

$$E_{1,2} = \frac{\alpha+\beta}{2} \pm \sqrt{\left(\frac{\alpha-\beta}{2}\right)^2 + |\gamma|^2}. \tag{6}$$

In the case of Hermitian matrix H, the eigenvectors (4) are expressed in terms of the matrix elements of matrix H: they read

$$|\psi_{E_{1,2}}\rangle = \begin{pmatrix} N_{1,2} \\ N_{1,2}\,(E_{1,2}-\alpha)/\gamma \end{pmatrix}, \tag{7}$$

where nonnegative normalization constants $N_{1,2}$ are expressed in terms of the matrix elements of Hamiltonian matrix H, i.e.,

$$N_{1,2} = |\gamma| \left[|\gamma|^2 + (E_{1,2}-\alpha)^2\right]^{-1/2}. \tag{8}$$

The eigenvectors (7) satisfy the orthogonality condition $\langle\psi_{E_1}|\psi_{E_2}\rangle = 0$ and the normalization conditions $\langle\psi_{E_1}|\psi_{E_1}\rangle = \langle\psi_{E_2}|\psi_{E_2}\rangle = 1$. The density 2×2-matrices of pure states of the two-level atom with Hermitian Hamiltonian has the form determined by the density operator

$$\hat{\rho}_{1,2} = |\psi_{E_{1,2}}\rangle\langle\psi_{E_{1,2}}| = \rho_{11}^{E_{1,2}}|1\rangle\langle 1| + \rho_{22}^{E_{1,2}}|2\rangle\langle 2| + \rho_{12}^{E_{1,2}}|1\rangle\langle 2| + \rho_{21}^{E_{1,2}}|2\rangle\langle 1|, \tag{9}$$

with the density matrix $\rho^{E_{1,2}}$.

As it was shown in [40–46], the physical meanings of the matrix elements of density operator (9) can be associated with three dichotomic probability distributions $(p_1,\ 1-p_1)$, $(p_2,\ 1-p_2)$,

and $(p_3,\, 1-p_3)$ of spin-1/2 projections $m = \pm 1/2$ in three perpendicular directions in the space, i.e., x, y, and z directions. One can check that the traces of the product of $\rho^{E_{1,2}}$ with three density matrices
$\frac{1}{2}1+\frac{1}{2}\sigma_x = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$; $\frac{1}{2}1+\frac{1}{2}\sigma_y = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}$; $\frac{1}{2}1+\frac{1}{2}\sigma_x = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}+\frac{1}{2}1$,
expressed in terms of three Pauli matrices $\frac{1}{2}+\frac{1}{2}\,\mathrm{Tr}\left(\rho^{E_{1,2}}\,\sigma_{x,y,z}\right)$, are equal to three probabilities

$$0 \leq p_1^{E_{1,2}},\ p_2^{E_{1,2}},\ p_3^{E_{1,2}} \leq 1.$$

According to Born's rule, the trace of the density matrix $\rho^{E_{1,2}}$ with the other density matrix $\rho_j = |\psi_j\rangle\langle\psi_i|$, i.e., $\mathrm{Tr}\left(\rho^{E_{1,2}}\rho_j\right)$, is equal to the probability $p_j^{E_{1,2}}$ to obtain the properties of the state $|\psi_j\rangle$ in the state $\rho^{E_{1,2}}$. For the pure state, this probability is $p_j^{E_{1,2}} = |\langle\psi_{E_{1,2}}|\psi_j\rangle|^2$.

For mixed states with density matrices ρ_A and ρ_B, Born's rule also provides the expression of analogous probabilities P_B^A. The probabilities P_B^A are the probabilities of the results obtained following the procedure: (i) one needs to obtain the properties of the system in the state ρ_A; (ii) for that, one performs the measurement for this system prepared in the state ρ_B; (iii) finally, one employs Born's rule, which provides the equality $P_B^A = \mathrm{Tr}\,(\rho_A\rho_B)$.

In view of this method, it was found [40–46] that any density 2×2-matrix of the qubit state can be mapped onto three probability distributions; i.e., the qubit-state density matrix reads

$$\rho = \begin{pmatrix} p_3 & (p_1-1/2)-i\,(p_2-1/2) \\ (p_1-1/2)+i\,(p_2-1/2) & 1-p_3 \end{pmatrix}. \tag{10}$$

Thus, an arbitrary qubit state is determined by three probabilities $0 \leq p_1,\ p_2,\ p_3 \leq 1$ satisfying the nonnegativity condition of the density operator

$$(p_1-1/2)^2+(p_2-1/2)^2+(p_3-1/2)^2 \leq 1/4. \tag{11}$$

Here, numbers p_1, p_2, and p_3 are the probabilities of the result of the experiment with qubit state, where one measures spin projections $m = +1/2$ in three perpendicular directions x, y, and z. The form of the density matrix (10) demonstrates that any qubit state is identified with the probability distributions of dichotomic random variables, which we call the probability representation of quantum state.

3. Schrödinger's Equation for Energy Levels in the Probability Representation

The energy levels of quantum system satisfy the eigenvalue equation for a Hamiltonian operator $\hat{H}$ acting in a Hilbert space

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle. \tag{12}$$

We consider the operator $\hat{H}$ to be non-Hermitian one, i.e., $\hat{H}^\dagger \neq \hat{H}$. Thus, we have the equality

$$\langle\psi_E|\hat{H}^\dagger = E^*\langle\psi_E|. \tag{13}$$

In the case of $\mathcal{PT}$-symmetry, one has the property $E = E^*$. Then, in view of Equation (13), for the operator $\hat{\rho}_E = |\psi_E\rangle\langle\psi_E|$, we obtain the following equations:

$$\frac{1}{2}\left[\hat{H}\hat{\rho}_E+\hat{\rho}_E\hat{H}^\dagger\right] = E\hat{\rho}_E, \qquad \left[\hat{H}\hat{\rho}_E-\hat{\rho}_E\hat{H}^\dagger\right] = 0. \tag{14}$$

Vectors $|\psi_E\rangle$ can be chosen to satisfy the normalization condition $\mathrm{Tr}\,\hat{\rho}_E = \langle\psi_E|\psi_E\rangle = 1$.

In [40,41], the qudit-state density matrices were introduced in the probability representation, i.e., as Hermitian trace-one matrices $|\psi_E\rangle\langle\psi_E|$ with nonnegative eigenvalues. The matrix elements of these Hermitian matrices are $\rho_{jk} = \left(p_1^{(jk)} - 1/2\right) - i\left(p_2^{(jk)} - 1/2\right)$ for $j < k$ and $j;k = 1,2,\ldots,N$, $\rho_{jj} = p_3^{(jj)}$ for $j = 1,2,\ldots,N-1$, and $\rho_{NN} = 1 - \sum_{j=1}^{N-1} p_3^{(jj)}$. The matrix elements ρ_{jk} satisfy the Silvester criterion of the nonnegativity of the eigenvalues of matrix ρ_{jk}; the nonnegative numbers $p_{1,2,3}^{(jk)}$ can be interpreted as the probability distributions of dichotomic random variables. For the states $|\psi_E\rangle$, the eigenvalues of matrix ρ_{jk} satisfy this condition.

Now we consider the example of pure qubit state in the case of Hermitian Hamiltonian $\hat{H}$. Since the density matrix ρ_{jk} of qubit state (10) is determined by three probability distributions of dichotomic random variables $(p_1, 1-p_1)$, $(p_2, 1-p_2)$, and $(p_3, 1-p_3)$, namely, $\rho_{11} = p_3$, $\rho_{12} = \rho_{21}^* = (p_1 - 1/2) - i(p_2 - 1/2)$, and $\rho_{22} = (1 - p_3)$, we can obtain the state vectors of the qubit.

For pure states, $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the vector $|\psi\rangle$ can be written in the form [48,51]

$$|\psi\rangle = \begin{pmatrix} \sqrt{p_3} \\ \sqrt{1-p_3}\,e^{i\varphi} \end{pmatrix}, \qquad \cos\varphi = \frac{p_1 - 1/2}{\sqrt{p_3(1-p_3)}}, \qquad \sin\varphi = \frac{p_2 - 1/2}{\sqrt{p_3(1-p_3)}}, \tag{15}$$

where the probabilities p_j; $j = 1,2,3$ satisfy the equality

$$(p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 = 1/4. \tag{16}$$

We choose the phase of ψ_1 to be equal to zero due to the gauge symmetry property of the wave functions of quantum systems [52].

The probability parametrization of an arbitrary normalized vector $|\psi\rangle$ can be used, including the case of eigenvectors of non-Hermitian Hamiltonian $\hat{H}$; this means that one can provide the invertible map of the solutions of Equation (12) onto a set of three probability distributions of dichotomic random variables, using Equation (15). In the case of $E = E^*$, after introducing the Hamiltonian matrix H of the Hamiltonian operator $\hat{H}$, relation (14) takes the form of the system of equations for the probabilities

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} p_3 & p^* \\ p & 1-p_3 \end{pmatrix} + \begin{pmatrix} p_3 & p^* \\ p & 1-p_3 \end{pmatrix} \begin{pmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{pmatrix} = 2\,E \begin{pmatrix} p_3 & p^* \\ p & 1-p_3 \end{pmatrix}, \tag{17}$$

where $p = (p_1 - 1/2) - i(p_2 - 1/2)$ and inequality (16) is valid.

Equation (12) can be written in the form of a system of linear equations for the four-vector $|\rho_E\rangle$, with components ρ_{E_1}, ρ_{E_2}, ρ_{E_3}, and ρ_{E_4}, expressed in terms of probabilities of dichotomic random variables, as follows:

$$\rho_{E_1} = p_3, \qquad \rho_{E_2} = (p_1 - 1/2) - i\,(p_2 - 1/2)\,, \qquad \rho_{E3} = \rho_{E_2}^*, \qquad \rho_{E_4} = 1 - p_3.$$

Since $\hat{\rho}_E = |\psi_E\rangle\langle\psi_E\rangle$, one can check that there exists the system of four equations given in vector form; it reads

$$(H \times 1)\,|\rho_E\rangle = E|\rho_E\rangle, \qquad (1 \times H^*)\,|\rho_E\rangle = E^*|\rho_E\rangle. \tag{18}$$

In these equations given in vector forms, vectors $|\rho_E\rangle$ satisfy Equation (14) written for $\mathcal{PT}$-symmetric systems with Hamiltonians having real eigenvalues.

4. Probability Representation of the Eigenvalue Equations for Generic Non-Hermitian Hamiltonians of Qubit Systems

Our aim now is to extend the probability representation of the eigenvalue equation for the $\mathcal{PT}$-symmetric qubit system (17) and derive a system of equations for generic non-Hermitian

Hamiltonian with 2×2-matrix H, employing the vector representation of operator $\hat{\rho}_E$, where the matrix of this operator is presented as vector $|\rho_E\rangle$. We consider an example of the density matrix $\rho_{jk};\ j,k = 1,2$ written in the form of four-vector $|\rho_E\rangle = \begin{pmatrix} p_3 \\ p^* \\ p \\ 1-p_3 \end{pmatrix}$. Vector $\langle\rho_E|$ has the form $\langle\rho_E| = (p_3, p, p^*, 1-p_3)$, where p_1, p_2, and p_3 are interpreted as probabilities of spin projections $m = +1/2$ in three perpendicular directions.

Equation (18) take the matrix form

$$\begin{pmatrix} H_{11} & 0 & H_{12} & 0 \\ 0 & H_{11} & 0 & H_{12} \\ H_{21} & 0 & H_{22} & 0 \\ 0 & H_{21} & 0 & H_{22} \end{pmatrix} \begin{pmatrix} p_3 \\ p^* \\ p \\ 1-p_3 \end{pmatrix} = E \begin{pmatrix} p_3 \\ p^* \\ p \\ 1-p_3 \end{pmatrix}, \tag{19}$$

$$\begin{pmatrix} H_{11}^* & H_{12}^* & 0 & 0 \\ H_{21}^* & H_{22}^* & 0 & 0 \\ 0 & 0 & H_{11}^* & H_{12}^* \\ 0 & 0 & H_{21}^* & H_{22}^* \end{pmatrix} \begin{pmatrix} p_3 \\ p \\ p^* \\ 1-p_3 \end{pmatrix} = E^* \begin{pmatrix} p_3 \\ p \\ p^* \\ 1-p_3 \end{pmatrix}, \tag{20}$$

where the complex number E is the eigenvalue of non-Hermitian Hamiltonian matrix H. In the case of $\mathcal{PT}$-symmetry $E = E^*$, and Equations (19) and (20) for the eigenvector of non-Hermitian Hamiltonians with real eigenvalues are given by (17). Analogous forms of Equations (19) and (20) in the probability representation of eigenstates of generic Hamiltonian matrix can be written for N-dimensional states $|\psi_E\rangle$.

5. The Schrödinger Equation for States with Eigenvalues of Energy as Equations for Eigenvectors with Components—Probabilities Determining Qubit States

Employing Equations (19) and (20), we obtain a new form of the Schrödinger equation for probabilities p_1, p_2, p_3, $(1-p_1)$, $(1-p_2)$, and $(1-p_3)$ determining stationary states of the spin-1/2 particle. First, we consider the Hermitian Hamiltonian for which in (19) and (20) $E = E^*$, $\mathrm{Im}\, H_{11} = \mathrm{Im}\, H_{22} = 0$, and $H_{12}^* = H_{21}$. Equations (19) and (20) are written for four-vectors. For matrices

$$\mathcal{H}_1 = \begin{pmatrix} H_{11} & 0 & H_{12} & 0 \\ 0 & H_{11} & 0 & H_{12} \\ H_{21} & 0 & H_{12} & 0 \\ 0 & H_{21} & 0 & H_{22} \end{pmatrix}, \qquad \mathcal{H}_2 = \begin{pmatrix} H_{11} & H_{21} & 0 & 0 \\ H_{12} & H_{22} & 0 & 0 \\ 0 & 0 & H_{11} & H_{21} \\ 0 & 0 & H_{12} & H_{22} \end{pmatrix} \tag{21}$$

and vector $|\rho_E\rangle$, the condition

$$\det\left(\mathcal{H}_{1,2} - E\mathbf{1_4}\right) = 0 \tag{22}$$

means that the following relation:

$$\left[\left(H_{11} - E\right)\left(H_{22} - E\right) - H_{12}H_{21}\right]^2 = 0 \tag{23}$$

is valid. For generic complex Hamiltonian matrix H, the relation

$$\left[\left(H_{11}^* - E^*\right)\left(H_{22}^* - E^*\right) - H_{12}^*H_{21}^*\right]^2 = 0 \tag{24}$$

corresponds to Equation (20).

One can see that for an arbitrary complex 2×2-matrix $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{12} \end{pmatrix}$, the Schrödinger Equations (12) and (13) in the form of eigenvalue equations for the four-vector $|\rho_E\rangle$ yield the system of complex conjugate eigenvalue equations for probabilities p_1, p_2, and p_3.

Thus, we obtained the system of linear equations for the spectrum of complex 2×2-matrix H in the form of equations for the four-vector $|\rho_E\rangle$, with components expressed in terms of three probability distributions $(p_1, 1-p_1)$, $(p_2, 1-p_2)$, and $(p_3, 1-p_3)$ of three dichotomic random variables. This result can be extended to the case of arbitrary qudit states. In the next section, we consider the qutrit state.

6. A Probability Representation for the Non-Hermitian Hamiltonian Eigenvalue Equation of the Qutrit State

Our goal in this section is to demonstrate the extension of the approach to other states; for this, we construct the probability representation of eigenvectors for qudit system on the example of the qutrit state. As it was shown in [51,53], an arbitrary $N \times N$ matrix ρ, such that $\rho = \rho^\dagger$, $\mathrm{Tr}\,\rho = 1$, with nonzero eigenvalues, has matrix elements which can be parameterized as follows: $\rho_{jj} = p_3^{(jj)}$; $j = 1, 2, \ldots, N$, $\rho_{NN} = 1 = \sum_{j=1}^{N-1} p_3^{(jj)}$, and $\rho_{jk} = \left(p_1^{(jk)} - 1/2\right) - i\left(p_2^{(jk)} - 1/2\right)$; $j < k$, where numbers $p_{1,2,3}^{(jk)}$ are probabilities of dichotomic random variables satisfying the Silvester criterion for the $N \times N$ matrix ρ.

The density matrix ρ for the qutrit state can be mapped onto nine-vector $\langle\rho|$ with components $\rho_1 = p_3^{(11)}$, $\rho_2 = \left(p_1^{(12)} - 1/2\right) + i\left(p_2^{(12)} - 1/2\right)$, $\rho_3 = \left(p_1^{(13)} - 1/2\right) + i\left(p_2^{(12)} - 1/2\right)$, $\rho_4 = \rho_2^*$, $\rho_5 = \rho_3^{(22)}$, $\rho_6 = \left(p_1^{(23)} - 1/2\right) + i\left(p_2^{(23)} - 1/2\right)$, $\rho_7 = \rho_3^*$, $\rho_8 = \rho_6^*$, and $\rho_9 = 1 - p_3^{(11)} - p_3^{(22)}$. Numbers $\rho_{1,2,3}^{(jk)}$ are probabilities of dichotomic random variables; these probabilities satisfy the Silvester criterion for 3×3-matrix ρ. The spectrum for the 3×3-matrix ρ of the generic Hamiltonian matrix H satisfies the system of Equation (18). These equations have the form of a system of equations for the probability distributions of dichotomic random variables, where numbers $\rho_{1,2,3}^{(jk)}$ can be interpreted as probabilities to have artificial spin projections $m = +1/2$ in eight directions in the space.

The presented probability-representation construction is valid for an arbitrary 3×3-matrix ρ, such that $\rho = \rho^\dagger$, $\mathrm{Tr}\,\rho = 1$, with nonnegative eigenvalues. The tool we used is the application of the density matrix representation $|\psi\rangle\langle\psi|$ of eigenvectors $|\psi\rangle$ in the case of the Hamiltonian 3×3-matrix.

The eigenvalue equations for non-Hermitian Hamiltonian 3×3-matrix of the three-level atom $H \neq H^\dagger$ in the vector form for the nine-vector $|\rho_{Ej}\rangle$; $j = 1, 2, \ldots, 9$ read

$$\sum_{k=1}^{9} \left\{ (H \times 1)_{jk} \right\} |\rho_{Ek}\rangle = E|\rho_{Ej}\rangle, \tag{25}$$

$$\sum_{k=1}^{9} \left\{ (1 \times H^*)_{jk} \right\} |\rho_{Ek}\rangle = E^*|\rho_{Ej}\rangle. \tag{26}$$

Since the vector components ρ_{Ej} are expressed in terms of dichotomic probabilities $p_{1,2,3}^{Ej}$; $j, k = 1, 2, 3$, linear Equations (25) and (26) have the form of a system of equations for the set of probability distributions $p_{1,2,3}^{Ej}$ determining the qutrit Hamiltonian eigenstates. The explicit probabilistic form of Equation (25) reads

$$\begin{pmatrix} H_{11} & 0 & 0 & H_{12} & 0 & 0 & H_{13} & 0 & 0 \\ 0 & H_{11} & 0 & 0 & H_{12} & 0 & 0 & H_{13} & 0 \\ 0 & 0 & H_{11} & 0 & 0 & H_{12} & 0 & 0 & H_{13} \\ H_{21} & 0 & 0 & H_{22} & 0 & 0 & H_{23} & 0 & 0 \\ 0 & H_{21} & 0 & 0 & H_{22} & 0 & 0 & H_{23} & 0 \\ 0 & 0 & H_{21} & 0 & 0 & H_{22} & 0 & 0 & H_{23} \\ H_{31} & 0 & 0 & H_{32} & 0 & 0 & H_{33} & 0 & 0 \\ 0 & H_{31} & 0 & 0 & H_{32} & 0 & 0 & H_{33} & 0 \\ 0 & 0 & H_{31} & 0 & 0 & H_{32} & 0 & 0 & H_{33} \end{pmatrix} \cdot \begin{pmatrix} p_3^{(11)} \\ p^{*(12)} \\ p^{*(13)} \\ p^{(12)} \\ p_3^{(22)} \\ p^{*(23)} \\ p^{(13)} \\ p^{(22)} \\ p_3^{(33)} \end{pmatrix} = E \begin{pmatrix} p_3^{(11)} \\ p^{*(12)} \\ p^{*(13)} \\ p^{(12)} \\ p_3^{(22)} \\ p^{*(23)} \\ p^{(13)} \\ p^{(22)} \\ p_3^{(33)} \end{pmatrix}, \tag{27}$$

and explicit probabilistic form of Equation (26) is

$$\begin{pmatrix} H^*_{11} & H^*_{12} & H^*_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ H^*_{21} & H^*_{22} & H^*_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ H^*_{31} & H^*_{32} & H^*_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H^*_{11} & H^*_{12} & H^*_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & H^*_{21} & H^*_{22} & H^*_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & H^*_{31} & H^*_{32} & H^*_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H^*_{11} & H^*_{12} & H^*_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & H^*_{21} & H^*_{22} & H^*_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & H^*_{31} & H^*_{32} & H^*_{33} \end{pmatrix} \cdot \begin{pmatrix} p_3^{(11)} \\ p^{*(12)} \\ p^{*(13)} \\ p^{(12)} \\ p_3^{(22)} \\ p^{*(23)} \\ p^{(13)} \\ p^{(22)} \\ p_3^{(33)} \end{pmatrix} = E^* \begin{pmatrix} p_3^{(11)} \\ p^{*(12)} \\ p^{*(13)} \\ p^{(12)} \\ p_3^{(22)} \\ p^{*(23)} \\ p^{(13)} \\ p^{(22)} \\ p_3^{(33)} \end{pmatrix}, \tag{28}$$

where $p_3^{(33)} = 1 - p_3^{(11)} - p_3^{(22)}$.

Thus, for the three-level atom, we can obtain the energy spectrum for an arbitrary non-Hermitian Hamiltonian with 3×3-density matrix H by solving Equations (27) and (28) written for dichotomic probabilities $p_3^{(11)}$ and $p_3^{(22)}$ and complex numbers $p^{(13)}$, $p^{(13)}$, and $p^{(23)}$ expressed as linear combinations of dichotomic probabilities $p_{1,2}^{(jk)}$, namely,

$$p^{(jk)} = \left(p_1^{(jk)} - 1/2\right) + i\left(p_2^{(jk)} - 1/2\right); \quad j < k, \quad j,k = 1,2,3.$$

The result obtained demonstrates that for $\mathcal{PT}$-symmetric systems the probability description of Hamiltonian spectrum is described by Equations (27) and (28), with $E = E^*$. It is worth pointing out that specific behavior of the three-level atom with a non-Hermitian Hamiltonian was experimentally studied in [27].

7. An Example of a Non-Hermitian Hamiltonian with $\mathcal{PT}$-Symmetry

The particular case of non-Hermitian Hamiltonian

$$H = \begin{pmatrix} z & s \\ s & z^* \end{pmatrix}, \qquad s = s^*, \qquad s^2 \geq (\operatorname{Im} z)^2 \tag{29}$$

was considered in [4,5,19]. Here, we consider a particular example of the Hamiltonian with $z = 1 + i$; i.e., we have the energy levels $E_1 = 1 + \sqrt{s^2 - 1}$ and $E_2 = 1 - \sqrt{s^2 - 1}$, where $s > 1$, and there exists the matrix a,

$$a = \begin{pmatrix} 1 & s^{-1}\left(\sqrt{s^2 - 1} - i\right) \\ 1 & -s^{-1}\left(\sqrt{s^2 - 1} - i\right) \end{pmatrix}, \tag{30}$$

which provides the matrix equality

$$a\begin{pmatrix} 1+i & s \\ s & 1-i \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} a. \tag{31}$$

The normalized eigenvectors $|1\rangle$ and $|2\rangle$ of the Hamiltonian H in Equation (29) read

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \dfrac{\sqrt{s^2-1}-i}{s} \end{pmatrix}, \qquad |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \dfrac{-\sqrt{s^2-1}-i}{s} \end{pmatrix}. \tag{32}$$

For $s \to \infty$, $\quad |1\rangle \to \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|2\rangle \to \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
The density matrices of pure states $|1\rangle$ and $|2\rangle$ read

$$\rho_1 = \begin{pmatrix} \dfrac{1}{2} & \dfrac{\sqrt{s^2-1}+i}{2\,s} \\ \dfrac{\sqrt{s^2-1}-i}{2\,s} & \dfrac{1}{2} \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \dfrac{1}{2} & \dfrac{-\sqrt{s^2-1}+i}{2\,s} \\ \dfrac{-\sqrt{s^2-1}-i}{2\,s} & \dfrac{1}{2} \end{pmatrix}. \tag{33}$$

The probability parameters $p_1(s)$, $p_2(s)$, and $p_3(s)$, describing the qubit states ρ_1 and ρ_2, are

for pure state $|1\rangle$, $\quad p_3^{(1)}(s) = \dfrac{1}{2}, \quad p_1^{(1)}(s) = \dfrac{1}{2} + \dfrac{1}{2s}\sqrt{s^2-1}, \quad p_2^{(1)}(s) = \dfrac{1}{2} - \dfrac{1}{2s},$

for pure state $|2\rangle$, $\quad p_3^{(2)}(s) = \dfrac{1}{2}, \quad p_1^{(2)}(s) = \dfrac{1}{2} - \dfrac{1}{2s}\sqrt{s^2-1}, \quad p_2^{(2)}(s) = \dfrac{1}{2} - \dfrac{1}{2s}.$

One can see that for the spin-1/2 state (qubit state), we have the following interpretation of the written states $|1\rangle$ and $|2\rangle$: in these states, the probability to have the projection $m = +1/2$ in the z direction is equal to $1/2$. However, there are different probabilities to have the spin projection equal to $m = +1/2$ in the x and y directions. In the state ρ_1, we have the probability $p_1(s)$ as the probability of the spin projection $m = +1/2$ on the x axis and $p_2(s)$ as the probability of the spin projection $m = +1/2$ on the y axis. Thus, we have both states with specific symmetry properties with respect to the two directions determined by the x and y axes. Born's rule provides the dependence of the probability $\mathrm{Tr}\,(\rho_1\,\rho_2) = s^{-2}$.

If one has spin-1/2 states, which are eigenstates $|\rho^{E_{1,2}}\rangle$ of the $\mathcal{PT}$-symmetric Hamiltonian (32), and measures the probabilities to get spin projections $m = +1/2$ on three perpendicular directions $p_1^{E_1}$, $p_2^{E_1}$, $p_3^{E_1}$, $p_1^{E_2}$, $p_2^{E_2}$, and $p_3^{E_2}$ in these states, there exists a difference in the behavior of spin-1/2 states for eigenstates of any Hermitian Hamiltonian and the non-Hermitian Hamiltonian we are now discussing. For spin-1/2 states defined as eigenstates of any Hermitian Hamiltonian, the probabilities $\mathcal{P}_1^{E_1}$, $\mathcal{P}_2^{E_1}$, $\mathcal{P}_3^{E_1}$, $\mathcal{P}_1^{E_2}$, $\mathcal{P}_2^{E_2}$, and $\mathcal{P}_3^{E_2}$ of spin projections $m = +1/2$ on three perpendicular directions must satisfy the equality

$$\mathrm{Tr}\left[\begin{pmatrix} \mathcal{P}_3^{E_1} & (\mathcal{P}_1^{E_1}-1/2) - i\,(\mathcal{P}_2^{E_1}-1/2) \\ (\mathcal{P}_1^{E_1}-1/2) + i\,(\mathcal{P}_2^{E_1}-1/2) & 1-\mathcal{P}_3^{E_1} \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} \mathcal{P}_3^{E_2} & (\mathcal{P}_1^{E_2}-1/2) - i\,(\mathcal{P}_2^{E_1}-1/2) \\ (\mathcal{P}_1^{E_2}-1/2) + i\,(\mathcal{P}_2^{E_2}-1/2) & 1-\mathcal{P}_3^{E_2} \end{pmatrix}\right] = 0. \tag{34}$$

In view of Born's rule, $\mathrm{Tr}\left(\rho^{E_1}\rho^{E_2}\right) = W_{E_1}^{E_2}$ is the probability given by the scalar product of the wave functions $|\langle\psi_{E_1}|\psi_{E_2}\rangle|^2 = W_{E_1}^{E_2}$.

For two density matrices of the form (10), one has

$$W_{E_1}^{E_2} = 2\left[1 + \mathcal{P}_1^{E_1}\mathcal{P}_1^{E_2} + \mathcal{P}_2^{E_1} + \mathcal{P}_{\in}^{\mathcal{E}_{\in}} + \mathcal{P}_{\ni}^{\mathcal{E}_{\infty}}\mathcal{P}_3^{E_2} - \mathcal{P}_1^{E_1} - \mathcal{P}_2^{E_1} - \mathcal{P}_3^{E_1} - \mathcal{P}_1^{E_2} - \mathcal{P}_2^{E_2} - \mathcal{P}_3^{E_2}\right]. \tag{35}$$

For $\mathcal{PT}$-symmetric Hamiltonian, the written trace of the product of these two density matrices, expressed in terms of probabilities $p_1^{E_1}(s)$, $p_2^{E_1}(s)$, $p_3^{E_1}(s)$, $p_1^{E_2}(s)$, $p_2^{E_2}(s)$, and $p_3^{E_2}(s)$, is different from zero. For considered example, $\text{Tr}\left(\rho^{E_1}\rho^{E_2}\right) = W_{E_1}^{E_2}$ depends on the parameter s and is equal to zero only at $s \to \infty$. This difference can be a tool for detecting the $\mathcal{PT}$-symmetric behavior of qubit (spin or two-level atom) systems.

In the limit $s \to 1$ (exceptional point), $W_{E_1}^{E_2} \to 1$.

Thus, the $\mathcal{PT}$-symmetric behavior can be detected, if one measures the probabilities of spin projections $m = +1/2$ in three directions, x, y, and z, in the two states with the given qubit state vectors $|\psi_{E_1}\rangle$ and $|\psi_{E_2}\rangle$.

8. The Superposition Principle and $\mathcal{PT}$-Symmetric States

According to the superposition principle in conventional quantum mechanics, the state $|\chi\rangle$, being a superposition of existing states $|\psi_0\rangle$ and $|\varphi_0\rangle$ of the form

$$|\chi\rangle = c_1|\psi_0\rangle + c_2|\varphi_0\rangle, \tag{36}$$

where c_1 and c_2 are complex numbers, does exist. Let us consider two qubit states $|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\varphi_0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The states $|\psi_{E_1}\rangle$ and $|\psi_{E_2}\rangle$ are normalized superposition states

$$|\psi_{E_1}\rangle = \frac{1}{\sqrt{2}}|\psi_0\rangle + \frac{1}{s\sqrt{2}}\left(\sqrt{s^2-1} - i\right)|\varphi_0\rangle, \quad |\psi_{E_2}\rangle = \frac{1}{\sqrt{2}}|\psi_0\rangle - \frac{1}{s\sqrt{2}}\left(\sqrt{s^2-1} + i\right)|\varphi_0\rangle. \tag{37}$$

We see that these states are superpositions of two states with spin projections on the z axis $m = +1/2$ for the state $|\psi_0\rangle$ and $m = -1/2$ for the state $|\varphi_0\rangle$. According to the superposition principle of quantum mechanics, these states exist. On the other hand, these states are the eigenstates of the $\mathcal{PT}$-symmetric Hamiltonian $H = \begin{pmatrix} 1+i & s \\ s & 1-i \end{pmatrix}$. Consequently, one can produce a measurement of spin projections in the z direction and other directions in the states $|\psi_{E_1}\rangle$ and $|\psi_{E_2}\rangle$ given by (37) and experimentally obtain the probabilities to get positive values of the projections. The experimental value of the probability $W_{E_1}^{E_2}$ given by Born's rule characterizes the presence of the $\mathcal{PT}$-symmetry in the qubit states depending on the parameter s of the non-Hermitian matrix H.

9. Qubit States with Broken $\mathcal{PT}$-Symmetry

The discussed approach provides the possibility to understand how to detect the states with broken $\mathcal{PT}$-symmetry Hamiltonians, using the probability representation of these states. We address the question: How can one find experimental characteristics of qubit states, which are the eigenstates of non-Hermitian Hamiltonians with complex eigenvalues? To consider the case of system with broken $\mathcal{PT}$-symmetry Hamiltonian, we formulate the following problem.

Take two qubit vectors $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $|\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$. Let us construct the matrices $\rho_\psi = |\psi\rangle\langle\psi|$ and $\rho_\varphi = |\varphi\rangle\langle\varphi|$ and assume that the vector $|\psi\rangle$ is the eigenvector of a complex matrix H with a complex eigenvector E_1 along with the vector $|\varphi\rangle$, which is the eigenvector of a complex matrix H with another complex eigenvector E_2. Then we have the relations

$$\begin{aligned} H|\psi\rangle\langle\psi| &= E_1|\psi\rangle\langle\psi|\,, & \qquad H|\varphi\rangle\langle\varphi| &= E_2|\varphi\rangle\langle\varphi|\,, \\ |\psi\rangle\langle\psi|H^\dagger &= E_1^*|\psi\rangle\langle\psi|\,, & \qquad |\varphi\rangle\langle\varphi|H^\dagger &= E_2^*|\varphi\rangle\langle\varphi|\,. \end{aligned} \tag{38}$$

It is easy to have the situation where $E_2 = E_1^*$, if the eigenvalues are not real.

Now we consider the example of the real parameter s in (29) in the case of inequality $0 < s < 1$. In this case, one has two complex eigenvalues of the matrix H, which are $E_1 = 1 + i\sqrt{1-s^2}$ and $E_2 = 1 - i\sqrt{1-s^2} = E_1^*$. In this case, the eigenvectors are

$$\begin{aligned} |\psi\rangle &= N_1 \begin{pmatrix} 1 \\ \frac{i}{s}\left(\sqrt{1-s^2}-1\right) \end{pmatrix}, & \qquad N_1 &= \left[1 + \frac{(\sqrt{1-s^2}-1)^2}{s^2}\right]^{-1/2}, \\ |\varphi\rangle &= N_2 \begin{pmatrix} 1 \\ -\frac{i}{s}\left(\sqrt{1-s^2}+1\right) \end{pmatrix}, & \qquad N_2 &= \left[1 + \frac{(\sqrt{1-s^2}+1)^2}{s^2}\right]^{-1/2}. \end{aligned} \tag{39}$$

For any state $|\psi\rangle$, the density matrix $\rho = |\psi\rangle\langle\psi|$, and for states (39) we have two density matrices

$$\begin{aligned} \rho_1 &= N_1^2 \begin{pmatrix} 1 & -\frac{i}{s}\left(\sqrt{1-s^2}-1\right) \\ \frac{i}{s}\left(\sqrt{1-s^2}-1\right) & N_1^{-2}-1 \end{pmatrix}, \\ \rho_2 &= N_2^2 \begin{pmatrix} 1 & \frac{i}{s}\left(\sqrt{1-s^2}+1\right) \\ -\frac{i}{s}\left(\sqrt{1-s^2}+1\right) & N_2^{-2}-1 \end{pmatrix}. \end{aligned} \tag{40}$$

Then, in view of (10), we obtain the probability representations of two states, which are eigenstates of non-Hermitian Hamiltonian with broken $\mathcal{PT}$-symmetry. The probabilities $p_{1,2,3}^{(1,2)}$ determining the density matrices ρ_1 and ρ_2 (40) are expressed as follows:

$$\begin{aligned} p_1^{(1)}(s) &= \frac{1}{2}, & \quad p_2^{(1)}(s) &= \frac{1}{2} + \frac{N_1^2}{s}\left(\sqrt{1-s^2}-1\right), & \quad p_3^{(1)}(s) &= N_1^2, \\ p_1^{(2)}(s) &= \frac{1}{2}, & \quad p_2^{(2)}(s) &= \frac{1}{2} - \frac{N_2^2}{s}\left(\sqrt{1-s^2}+1\right), & \quad p_3^{(2)}(s) &= N_2^2. \end{aligned} \tag{41}$$

The specific properties of the states with density matrices (33) are that $\mathrm{Tr}\,(\rho_1\rho_2) \neq 0$ and one of the probability distributions $p_2^{(1,2)}(s)$ corresponds to the completely chaotic distribution $(1/2, 1/2)$. Thus, such states can be detected as existing superposition states by measuring the spin-1/2 positive projections; i.e., in the case of experimental obtaining the probabilities of the form (41), the latter ones can witness the presence of non-Hermitian Hamiltonian violating the $\mathcal{PT}$-symmetry.

In the generic case of non-Hermitian Hamiltonian H with complex eigenvalues E_1 and E_2, which are given by four real parameters and violate the $\mathcal{PT}$-symmetry, one can also construct the probability representation of two eigenvectors of such a Hamiltonian. One can check that, in this case, the density matrices of the eigenstates of such Hamiltonians violating the $\mathcal{PT}$-symmetry are not orthogonal, and Born's probability given by scalar product of normalized eigenvectors $\left|\langle\psi_{E_1}|\psi_{E_2}\rangle\right|^2 = W_1^2$ depends on the Hamilton parameters and is not equal to zero. This case can be also detected while measuring the probabilities of spin projections, because the results for $W_{E_1}^{E_2}$, where E_1 and E_2 are complex numbers, are different for the situations with the $\mathcal{PT}$-symmetry and with Hermitian Hamiltonians.

10. Conclusions

To conclude, we point out the main results of our work.

In Hermitian and non-Hermitian Hamiltonian systems with corresponding real eigenvalues, the pure states describing the eigenvectors of such Hamiltonians can be expressed in terms of probabilities of dichotomic random variables. We demonstrated this result on the example of qubit systems. We wrote the Schrödinger equation for such a system for the eigenvector of Hamiltonian H in a new form of the eigenvalue equation for the Hamiltonian $\mathcal{H} = H \otimes 1$, where components of the eigenvector are expressed in the form of probabilities of classical-like dichotomic variables.

For qubit states (spin-1/2 states), these probabilities are the probabilities to have the spin projection $m = +1/2$ in three perpendicular directions x, y, and z. For systems with Hermitian Hamiltonians, these probabilities satisfies the conditions of orthogonalities of two-dimensional eigenvectors of the 2×2 matrix H. For $\mathcal{PT}$-symmetric Hamiltonians, which are non-Hermitian ones, the probabilities are such that the trace of pure-state density matrices $\rho_1 = |\psi_{E_1}\rangle\langle\psi_{E_1}|$ and $\rho_2 = |\psi_{E_2}\rangle\langle\psi_{E_2}|$, i.e., $k_{12} = \mathrm{Tr}\,(\rho_1\rho_2)$, where E_1 and E_2 are real eigenvalues of the matrix H, is not equal to zero, and the value k_{12} characterizes properties of the $\mathcal{PT}$-symmetric system. The nonorthogonality of the non-Hermitian Hamiltonian eigenvectors associated with $\mathcal{PT}$-symmetry properties of quantum systems was mentioned, e.g., in [19,27].

The systems with broken $\mathcal{PT}$-symmetry, for which the Hamiltonian has complex eigenvalues, we also expressed the complex state vectors of the Hamiltonian in terms of probabilities of dichotomic random variables given in terms of the probabilities to have spin projection $m = +1/2$ in three perpendicular directions x, y, and z. In principle, these probabilities and corresponding means of the spin projections can be measured experimentally, and qubit systems with $\mathcal{PT}$-symmetry can be compared with systems characterized by Hermitian Hamiltonians. Additionally, qubit systems with broken $\mathcal{PT}$-symmetry can be compared with $\mathcal{PT}$-symmetric ones, in view of values of the probabilities and corresponding parameters determined by Born's rule. The result can be extended to the case of $\mathcal{PT}$-symmetric qudit systems, and the probability representation of qudit systems with generic complex Hamiltonian can be obtained by employing the approach demonstrated in this paper.

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