symmetry

MDPI

Article

More on Hölder's Inequality and It's Reverse via the Diamond-Alpha Integral

M. Zakarya [1,2], H. A. Abd El-Hamid [3], Ghada AlNemer [4,*] and H. M. Rezk [5]

1 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia; Mohammed_Zakaria1983@azhar.edu.eg
2 Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
3 Department of Mathematics and Computer Science, Faculty of Science, Beni- Suef University, Beni-Suef, Egypt; Hoda.Ali@science.bsu.edu.eg
4 Department of Mathematical Science, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 105862, Riyadh 11656, Saudi Arabia
5 Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt; haythamrezk@azhar.edu.eg
* Correspondence: gnnemer@pnu.edu.sa

Received: 7 August 2020; Accepted: 24 September 2020; Published: 18 October 2020

Abstract: In this paper, we investigate some new generalizations and refinements for Hölder's inequality and it's reverse on time scales through the diamond-α dynamic integral, which is defined as a linear combination of the delta and nabla integrals, which are used in various problems involving symmetry. We develop a number of those symmetric inequalities to a general time scale. Our results as special cases extend some integral dynamic inequalities and Qi's inequalities achieved on time scales and also include some integral disparities as particular cases when $\mathbb{T} = \mathbb{R}$.

Keywords: Hölder's inequality; generalization; refinement; diamond-α integral; time scale

MSC: Primary 26D15; Secondary 39A13

1. Introduction

Hölder's inequality is one of the greatest inequalities in pure and applied mathematics. As is well known, Hölder's inequality plays a very important role in different branches of modern mathematics, such as linear algebra, classical real and complex analysis, probability and statistics, qualitative theory of differential equations and their applications. A large number of papers dealing with refinements, generalizations and applications of Hölder's integral inequalities and their series symmetry in different areas of mathematics have appeared (see [1–4] and the references therein).

The classical Hölder's inequality (see [5]) mentions that if $u_j \geq 0$, $v_j \geq 0$ $(j = 1, 2, \dots, n)$, $\lambda > 0$, $\mu > 0$ and $1/\lambda + 1/\mu = 1$, then

$$\sum_{j=1}^{n} u_j v_j \leq \left(\sum_{j=1}^{n} u_j^{\lambda} \right)^{\frac{1}{\lambda}} \left(\sum_{j=1}^{n} v_j^{\mu} \right)^{\frac{1}{\mu}}. \tag{1}$$

This inequality is reversed if $\lambda < 1$ $(\lambda \neq 0)$ (for $\lambda < 1$, we assume that $u_j > 0$, $v_j > 0$).

The integral version of inequality (1) is given as if $u, v \in C([r, s], \mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda > 1$ and $1/\lambda + 1/\mu = 1$, then

$$\int_r^s |u(\vartheta)v(\vartheta)|\, d\vartheta \leq \left(\int_r^s |u(\vartheta)|^{\lambda}\, d\vartheta \right)^{\frac{1}{\lambda}} \left(\int_r^s |v(\vartheta)|^{\mu}\, d\vartheta \right)^{\frac{1}{\mu}}. \tag{2}$$

This inequality is reversed if $0 < \lambda < 1$ and if $\lambda < 0$ or $\mu < 0$.

As an application of Hölder's inequality (2), Lazhar Bougoffa in [6] proved that if $u, v \in C([r, s], \mathbb{R})$ such that

$$0 < m \leq \frac{u(\vartheta)}{v(\vartheta)} \leq M < \infty,$$

then for $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$, we have

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta) v^{\frac{1}{\mu}}(\vartheta) d\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}} \int_r^s u^{\frac{1}{\mu}}(\vartheta) v^{\frac{1}{\lambda}}(\vartheta) d\vartheta, \tag{3}$$

and then

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta) v^{\frac{1}{\mu}}(\vartheta) d\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}} \left(\int_r^s u(\vartheta) d\vartheta \right)^{\frac{1}{\mu}} \left(\int_r^s v(\vartheta) d\vartheta \right)^{\frac{1}{\lambda}}. \tag{4}$$

Recently, a number of scientists ([7–9], p. 126, p. 3) have explored the reverse of Hölder's inequality, the famous ones being: Let $\lambda > 1$ and $1/\lambda + 1/\mu = 1$, if

$$0 < m \leq \frac{u(\vartheta)}{v(\vartheta)} \leq M,$$

then

$$\left(\int_r^s u(\vartheta) d\vartheta \right)^{\frac{1}{\lambda}} \left(\int_r^s v(\vartheta) d\vartheta \right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m} \right)^{\frac{1}{\lambda\mu}} \int_r^s u^{\frac{1}{\lambda}}(\vartheta) v^{\frac{1}{\mu}}(\vartheta) d\vartheta. \tag{5}$$

By replacing u by u^λ and v by v^μ, inequality (5) can be rewritten as:

$$\left(\int_r^s u^\lambda(\vartheta) d\vartheta \right)^{\frac{1}{\lambda}} \left(\int_r^s v^\mu(\vartheta) d\vartheta \right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m} \right)^{\frac{1}{\lambda\mu}} \int_r^s u(\vartheta) v(\vartheta) d\vartheta, \tag{6}$$

where $\lambda > 1$, $1/\lambda + 1/\mu = 1$ and

$$0 < m \leq \frac{u^\lambda(\vartheta)}{v^\mu(\vartheta)} \leq M.$$

See ([1,10,11], p. 9, p. 206, p. 212).

Lately, Sulaiman [12] gave the reverse Hölder inequality as follows: Let $\lambda > 0$, $\mu > 0$ and u, v be two positive functions satisfying

$$0 < m \leq \frac{u(\vartheta)}{v(\vartheta)} \leq M \quad \text{for all} \quad \vartheta \in [r,\ s],$$

then

$$\begin{aligned} & \left(\int_r^s u^\lambda(\vartheta) d\vartheta \right)^{\frac{1}{\lambda}} \left(\int_r^s v^\mu(\vartheta) d\vartheta \right)^{\frac{1}{\mu}} \\ \leq \ & \frac{M}{m} \left(\int_r^s (u(\vartheta)v(\vartheta))^{\frac{\lambda}{2}} d\vartheta \right)^{\frac{1}{\lambda}} \left(\int_r^s (u(\vartheta)v(\vartheta))^{\frac{\mu}{2}} d\vartheta \right)^{\frac{1}{\mu}}. \end{aligned} \tag{7}$$

In [13], the authors established a new inequality with a weighted function that is improvement of the reverse Hölder's inequality that, given in (7), is as follows: Let $\gamma, \beta > 0$, $\lambda > 1$, $1/\lambda + 1/\mu = 1$ and $u, v > 0$ integrable functions on $[r, s]$, ω a weight function (measurable and positive) on $[r, s]$. If

$$0 < m \leq \frac{u^\gamma(\vartheta)}{v^\beta(\vartheta)} \leq M \quad \text{for all} \ \vartheta \in [r,\ s],$$

then

$$\left(\int_r^s u^{\gamma}(\vartheta)\omega(\vartheta)d\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\beta}(\vartheta)\omega(\vartheta)d\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u^{\frac{\gamma}{\lambda}}(\vartheta)v^{\frac{\beta}{\mu}}(\vartheta)\omega(\vartheta)d\vartheta. \quad (8)$$

In [14], the authors proved the Δ-integral version of Hölder's inequalities (1) and (2) as follows: Let $r, s \in \mathbb{T}$ with $r < s$ and $u, v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$. Then

$$\int_r^s |u(\vartheta)v(\vartheta)|\,\Delta\vartheta \leq \left(\int_r^s |u(\vartheta)|^{\lambda}\,\Delta\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s |v(\vartheta)|^{\mu}\,\Delta\vartheta\right)^{\frac{1}{\mu}}, \quad (9)$$

where $\lambda > 1$ and $1/\lambda + 1/\mu = 1$. This inequality is reversed if $0 < \lambda < 1$ and if $\lambda < 0$ or $\mu < 0$.

In [15,16], the authors proved the Δ-integral version of Hölder's inequality (6) as follows: If r, $s \in \mathbb{T}$ with $r < s$ and $u, v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$, such that

$$0 < m \leq \frac{u^{\lambda}(\vartheta)}{v^{\mu}(\vartheta)} \leq M < \infty,$$

then for $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$, we have

$$\left(\int_r^s u^{\lambda}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u(\vartheta)v(\vartheta)\Delta\vartheta. \quad (10)$$

As an application of (9), the authors in [16,17] proved that if r, $s \in \mathbb{T}$ with $r < s$ and u, $v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$ such that

$$0 < m \leq \frac{u(\vartheta)}{v(\vartheta)} \leq M < \infty,$$

then for $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$, we have

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Delta\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\int_r^s u^{\frac{1}{\mu}}(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta, \quad (11)$$

and then

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Delta\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\left(\int_r^s u(\vartheta)\Delta\vartheta\right)^{\frac{1}{\mu}}\left(\int_r^s v(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}. \quad (12)$$

In [18], the researchers concluded some generalizations of the inequality (10) for time scale diamond-α calculus. Specifically, they proved that, if $r, s \in \mathbb{T}$ with $r < s$ and u, v are two positive functions satisfying

$$0 < m \leq \frac{u^{\lambda}(\vartheta)}{v^{\mu}(\vartheta)} \leq M < \infty,$$

on the set $[r,s]_{\mathbb{T}}$. If $1/\lambda + 1/\mu = 1$ with $\lambda > 1$, then

$$\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u(\vartheta)v(\vartheta)\Diamond_{\alpha}\vartheta. \quad (13)$$

For the development of dynamic inequalities on a time scale calculus, we refer the reader to the articles in [19–28]. Although there are many results for time scale calculus in the sense of delta and nabla derivative, there is not much done for diamond-α derivative. Therefore, the major contribution of this article is to extend the reverse of Hölder type inequalities for diamond-α calculus and to unify them for the arbitrary time scale. The main theorems are inspired from the papers [12,13] which presents the time scale version of inequalities (7), (8) and (12) for diamond-α calculus. By obtaining their diamond-α versions, we can show the generalizations of these inequalities for different types of time scales $\mathbb{T}$, such as real numbers and integers.

The structure of this paper is listed below. Section 2, presents the fundamental concepts of the time scale calculus in terms of delta, nabla and diamond-α calculus. Section 3, is devoted to main results, which are to generalize the inequalities (10), (12) and (13) for diamond-α time scale calculus.

2. Preliminaries

In this section, the fundamental theories of time scale delta and time scale nabla calculi will be presented. Time scale calculus whose detailed information can be found in [29,30] has been invented in order to unify continuous and discrete analysis.

A nonempty closed subset of $\mathbb{R}$ is named a time scale which is signified by $\mathbb{T}$. For $\vartheta \in \mathbb{T}$, if $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$, then the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined as $\sigma(\vartheta) = \inf(\vartheta, \infty)_{\mathbb{T}}$ and $\rho(\vartheta) = \sup(-\infty, \vartheta)_{\mathbb{T}}$, respectively. From the above two concepts, it can be mentioned that a point $\vartheta \in \mathbb{T}$ with $\inf\mathbb{T} < \vartheta < \sup\mathbb{T}$ is named right-scattered if $\sigma(\vartheta) > \vartheta$, right-dense if $\sigma(\vartheta) = \vartheta$, left-scattered if $\rho(\vartheta) < \vartheta$ and left-dense if $\rho(\vartheta) = \vartheta$.

The Δ-derivative of $\psi : \mathbb{T} \to \mathbb{R}$ at $\vartheta \in \mathbb{T}^k = \mathbb{T}/(\rho(\sup\mathbb{T}), \sup\mathbb{T}]$, indicated by $\psi^{\Delta}(\vartheta)$, is the number that enjoys the property that $\forall\varepsilon > 0$ where there is a neighborhood U of $\vartheta \in \mathbb{T}^k$, such that

$$\left|\psi(\sigma(\vartheta)) - \psi(t) - \psi^{\Delta}(\vartheta)(\sigma(\vartheta) - t)\right| \leq \varepsilon\,|\sigma(\vartheta) - t|\,, \quad \forall t \in U.$$

The ∇-derivative of $\psi : \mathbb{T} \to \mathbb{R}$ at $\vartheta \in \mathbb{T}_k = \mathbb{T}/[\inf\mathbb{T}, \sigma(\inf\mathbb{T}))$ indicated by $\psi^{\nabla}(\xi)$ is the number that enjoys the property that $\forall\varepsilon > 0$ where there is a neighborhood V of $\vartheta \in \mathbb{T}_k$, such that

$$\left|\psi(\vartheta) - \psi(\rho(t)) - \psi^{\nabla}(\vartheta)(\vartheta - \rho(t))\right| \leq \varepsilon\,|t - \rho(\vartheta)|\,, \quad \forall t \in V.$$

A function $\psi : \mathbb{T} \to \mathbb{R}$ is *rd*-continuous if it is continuous at each right-dense points in $\mathbb{T}$ and $\lim_{s\to\vartheta^-} \psi(s)$ exists as a finite number for all left-dense points in $\mathbb{T}$. The set $C_{rd}(\mathbb{T}, \mathbb{R})$ represents the class of real, *rd*-continuous functions defined on $\mathbb{T}$. If $\psi \in C_{rd}(\mathbb{T}, \mathbb{R})$, then there exists a function $\Psi(\vartheta)$ such that $\Psi^{\Delta}(\vartheta) = \psi(\vartheta)$ and the delta integral of ψ is defined by

$$\int_{x_0}^{x} \psi(\vartheta)\Delta\vartheta = \Psi(x) - \Psi(x_0).$$

A function $\psi : \mathbb{T} \to \mathbb{R}$ is *ld*-continuous if it is continuous at each left-dense points in $\mathbb{T}$ and $\lim_{s\to\vartheta^+} \psi(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T}, \mathbb{R})$ represents the class of real, *ld*-continuous functions defined on $\mathbb{T}$. If $\psi \in C_{ld}(\mathbb{T}, \mathbb{R})$, then there exists a function $\Psi(\vartheta)$ such that $\Psi^{\nabla}(\vartheta) = \psi(\vartheta)$ and the nabla integral of ψ is defined by

$$\int_{x_0}^{x} \psi(\vartheta)\nabla\vartheta = \Psi(x) - \Psi(x_0).$$

Now, we briefly introduce short introduction of diamond-α derivative and integrals [17,31].

For $\vartheta \in \mathbb{T}$, we define the diamond-α dynamic derivative $\psi^{\diamond_\alpha}(\vartheta)$ by

$$\psi^{\diamond_\alpha}(\vartheta) = \alpha\psi^{\Delta}(\vartheta) + (1 - \alpha)\psi^{\nabla}(\vartheta), \quad 0 \leq \alpha \leq 1.$$

Thus, ψ is diamond-α differentiable if and only if ψ is Δ and ∇ differentiable. The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$.

Moreover, the diamond-α derivatives offer a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $\alpha = 1/2$.

On the other hand, let $r, \vartheta \in \mathbb{T}$ and $w : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral of w is defined by

$$\int_r^{\vartheta} w(\tau)\Diamond_\alpha \tau = \alpha \int_r^{\vartheta} w(\tau)\Delta\tau + (1-\alpha)\int_r^{\vartheta} w(\tau)\nabla\tau,\ 0 \le \alpha \le 1.$$

We may note that the $\Diamond_\alpha$-integral is an integral combination of Δ and ∇. Generally speaking, we have no

$$\left(\int_r^{\vartheta} w(\tau)\Diamond_\alpha\tau\right)^{\Diamond_\alpha} = w(\vartheta),\ \vartheta \in \mathbb{T}.$$

It is clear that the diamond-α integral of w exists when w is a continuous function.

Within the following, we display some basic properties for diamond-α calculus that play a key role in inaugurating the major findings of this paper.

Theorem 1 (**[17] [Theorem 1.3.5]**). *Let $r, s, \vartheta \in \mathbb{T}$, $c \in \mathbb{R}$ and u, v be continuous functions on $[r, s]_{\mathbb{T}} = [r, s] \cap \mathbb{T}$. Then the following properties hold:*

(i) $\int_r^{\vartheta} (u(\tau) + v(\tau))\Diamond_\alpha\tau = \int_r^{\vartheta} u(\tau)\Diamond_\alpha\tau + \int_r^{\vartheta} v(\tau)\Diamond_\alpha\tau;$

(ii) $\int_r^{\vartheta} cu(\tau)\Diamond_\alpha\tau = c\int_r^{\vartheta} u(\tau)\Diamond_\alpha\tau;$

(iii) $\int_r^{\vartheta} u(\tau)\Diamond_\alpha\tau = \int_r^{s} u(\tau)\Diamond_\alpha\tau + \int_s^{\vartheta} u(\tau)\Diamond_\alpha\tau;$

(iv) *If* $u(\vartheta) \ge 0\ \forall\vartheta \in [r, s]_{\mathbb{T}}$, *then* $\int_r^{s} u(\vartheta)\Diamond_\alpha\vartheta \ge 0;$

(v) *If* $u(\vartheta) \le v(\vartheta)\ \forall\vartheta \in [r, s]_{\mathbb{T}}$, *then* $\int_r^{s} u(\vartheta)\Diamond_\alpha\vartheta \le \int_r^{s} v(\vartheta)\Diamond_\alpha\vartheta.$

Theorem 2 (**[17] [Theorem 2.3.11, Theorem 2.3.12]**). *Assume $u, v, w \in C([r, s]_{\mathbb{T}}, \mathbb{R})$ with $\int_r^{s} w(\vartheta)v^{\mu}(\vartheta)\Diamond_\alpha\vartheta > 0$. If $1/\lambda + 1/\mu = 1$ with $\lambda > 1$, then*

$$\int_r^s w(\vartheta)u(\vartheta)v(\vartheta)\Diamond_\alpha\vartheta \le \left(\int_r^s w(\vartheta)u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \left(\int_r^s w(\vartheta)v^{\mu}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}}. \tag{14}$$

This inequality is reversed if $\lambda < 0$ or $\mu < 0$.

3. Main Results

In this section, we prove new diamond-α inequalities. As particular cases we get Δ-inequalities on time scales for $\alpha = 1$ and ∇-inequalities on time scales when $\alpha = 0$. In the sequel, we will suppose that the functions (without mentioning) are non-negative continuous functions and the left hand side of the inequalities exists if the right hand side exists. In what follows, we will present the diamond α-version of Hölder's inequality (12) with a weight function by applying the diamond α-Hölder inequality (14).

Theorem 3. *Let $r, s \in \mathbb{T}$ with $r < s$ and $u, v \in C([r, s]_{\mathbb{T}}, \mathbb{R})$, w a weight function (measurable and positive) on $[r, s]_{\mathbb{T}}$, such that*

$$0 < m \le \frac{u(\vartheta)}{v(\vartheta)} \le M < \infty \quad \text{for all } \vartheta \in [r, s]_{\mathbb{T}}. \tag{15}$$

Then for $\lambda > 1$, $\mu > 1$ with $1/\lambda + 1/\mu = 1$, we have

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\int_r^s u^{\frac{1}{\mu}}(\vartheta)g^{\frac{1}{\lambda}}(\vartheta)w(\vartheta)\Diamond_\alpha\vartheta, \tag{16}$$

and hence, we get

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\left(\int_r^s w(\vartheta)u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}}\left(\int_r^s w(\vartheta)v(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}. \tag{17}$$

Proof. From Hölder's inequality (14), we obtain

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \leq \left(\int_r^s w(\vartheta)u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s w(\vartheta)v(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}}, \tag{18}$$

that is,

$$\begin{aligned}\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \quad &\leq \quad \left(\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)u^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \\ &\quad \times\left(\int_r^s w(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}}.\end{aligned}$$

Since $u^{1/\lambda}(\vartheta) \leq M^{1/\lambda}v^{1/\lambda}(\vartheta)$ and $v^{1/\mu}(t) \leq m^{-1/\mu}u^{1/\mu}(t)$, then from the above inequality it follows that

$$\begin{aligned}\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \quad &\leq \quad \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\left(\int_r^s w(\vartheta)u^{\frac{1}{\mu}}(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \\ &\quad \times\left(\int_r^s w(\vartheta)u^{\frac{1}{\mu}}(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}},\end{aligned}$$

and so,

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\int_r^s w(\vartheta)u^{\frac{1}{\mu}}(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta. \tag{19}$$

Hence, the inequality (16) is proven.

The inequality (17) follows from substituting the following

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Diamond_\alpha\vartheta \leq \left(\int_r^s w(\vartheta)u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s w(\vartheta)v(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}},$$

into (19). The evidence is complete. □

As a specific case of Theorem 3, when $\alpha = 1$ and $\alpha = 0$, we get the following findings.

Corollary 1. *Let $r, s \in \mathbb{T}$ with $r < s$ and $u, v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$, w a weight function (measurable and positive) on $[r,s]_{\mathbb{T}}$, such that*

$$0 < m \leq \frac{u(\vartheta)}{v(\vartheta)} \leq M < \infty.$$

Then for $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$, we have

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Delta\vartheta \le \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}} \int_r^s w(\vartheta)u^{\frac{1}{\mu}}(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta, \tag{20}$$

and then

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Delta\vartheta \le \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}} \left(\int_r^s w(\vartheta)u(\vartheta)\Delta\vartheta\right)^{\frac{1}{\mu}} \left(\int_r^s w(\vartheta)v(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}, \tag{21}$$

which is the delta version of (16) and (17).

Corollary 2. *Let $r, s \in \mathbb{T}$ with $r < s$ and $u, v \in C_{ld}([r,s]_{\mathbb{T}}, \mathbb{R})$, w a weight function (measurable and positive) on $[r,s]_{\mathbb{T}}$, such that*

$$0 < m \le \frac{u(\vartheta)}{v(\vartheta)} \le M < \infty.$$

Then for $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$, we have

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\nabla\vartheta \le \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}} \int_r^s w(\vartheta)u^{\frac{1}{\mu}}(\vartheta)v^{\frac{1}{\lambda}}(\vartheta)\nabla\vartheta, \tag{22}$$

and then

$$\int_r^s w(\vartheta)u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\nabla\vartheta \le \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}} \left(\int_r^s w(\vartheta)u(\vartheta)\nabla\vartheta\right)^{\frac{1}{\mu}} \left(\int_r^s w(\vartheta)v(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}}, \tag{23}$$

which is the nabla version of (16) and (17).

Remark 1. *Clearly, for the particular case $w(\vartheta) = 1$, inequalities (20) and (21) in Corollary 1 reduce to (11) and (12), respectively, in the introduction.*

Remark 2. *For the particular case $\mathbb{T} = \mathbb{R}$, inequalities (16) and (17) in Theorem 3 reduce to (3) and (4), respectively, in the introduction.*

As an application of (17) in Theorem 3, we get the next theorem.

Theorem 4. *Let $r, s \in \mathbb{T}$ such that $r < s$ and $u \in C([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \le u(\vartheta) \le M < \infty \quad \text{for all } \vartheta \in [r,\, s]_{\mathbb{T}},$$

with $M \le m^{(\lambda-1)^2}/(s-r)^{\lambda}$, then for a given positive integer $\lambda \ge 2$, we have

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta \le \left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{1-\frac{1}{\lambda}}. \tag{24}$$

Proof. Putting $v(\vartheta) = w(\vartheta) \equiv 1$ into (17) yields

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta \leq \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{\frac{1}{\mu^2}}}\left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}}\left(\int_r^s (1)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}$$

$$= \frac{(M)^{\frac{1}{\lambda^2}}}{(m)^{(1-\frac{1}{\lambda})^2}}(s-r)^{\frac{1}{\lambda}}\left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{1-\frac{1}{\lambda}},$$

that is,

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta \leq K\left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{1-\frac{1}{\lambda}},$$

where $K = \frac{M^{1/\lambda^2}(s-r)^{1/\lambda}}{m^{(1-(1/\lambda))^2}}$. Now, from $M \leq m^{(\lambda-1)^2}/(s-r)^\lambda$, we conclude that $K \leq 1$. Thus the inequality (24) is proven. □

As a specific case of Theorem 4 when $\alpha = 1$ and $\alpha = 0$, we get the following findings.

Corollary 3. *Let $r, s \in \mathbb{T}n$ such that $r < s$ and $u \in C_{rd}([r, s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \leq u(\vartheta) \leq M < \infty \quad \text{for all } \vartheta \in [r,\ s]_{\mathbb{T}},$$

with $M \leq m^{(\lambda-1)^2}/(s-r)^\lambda$, then for a given positive integer $\lambda \geq 2$, we have

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta \leq \left(\int_r^s u(\vartheta)\Delta\vartheta\right)^{1-\frac{1}{\lambda}}, \tag{25}$$

which is the delta version of (24), see [16] [Lemma 2.10].

Corollary 4. *Let $r, s \in \mathbb{T}n$ such that $r < s$ and $u \in C_{ld}([r, s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \leq u(\vartheta) \leq M < \infty \quad \text{for all } \vartheta \in [r,\ s]_{\mathbb{T}},$$

$M \leq m^{(\lambda-1)^2}/(s-r)^\lambda$, then for a given positive integer $\lambda \geq 2$, we have

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\nabla\vartheta \leq \left(\int_r^s u(\vartheta)\nabla\vartheta\right)^{1-\frac{1}{\lambda}}, \tag{26}$$

which is the nabla version of (24).

Remark 3. *In Theorem 4, if we make the substitution $u(\vartheta) = M = m$ and $s - r = 1$ with $\lambda = 2$, then the equality in (24) holds.*

Remark 4. *For the particular case $\mathbb{T} = \mathbb{R}$, inequality (24) in Theorem 4 reduces to*

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)d\vartheta \leq \left(\int_r^s u(\vartheta)d\vartheta\right)^{1-\frac{1}{\lambda}}, \tag{27}$$

which is Qi's inequality (10) [32] [Proposition 2].

The following theorems include the reverse Hölder form on time-scales.

Theorem 5. *Let $\gamma, \beta > 0$, $\lambda > 1$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C([r,s]_{\mathbb{T}}, \mathbb{R})$, w a weight function (measurable and positive) on $[r,s]_{\mathbb{T}}$. If*

$$0 < m \leq \frac{u^{\gamma}(\vartheta)}{v^{\beta}(\vartheta)} \leq M < \infty \quad \text{for all } \vartheta \in [r,s]_{\mathbb{T}}, \tag{28}$$

then

$$\left(\int_r^s u^{\gamma}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta \right)^{\frac{1}{\lambda}} \left(\int_r^s v^{\beta}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta \right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m} \right)^{\frac{1}{\lambda\mu}} \int_r^s u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta. \tag{29}$$

Proof. From the assumption (28), we have

$$m^{-\frac{1}{\mu}} \geq u^{-\frac{\gamma}{\mu}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) \geq M^{-\frac{1}{\mu}}. \tag{30}$$

Multiplying (30) by $u^{\gamma} > 0$ and using the fact that $(\mu - 1)/\mu = \lambda$, it follows that

$$\left(\frac{1}{m} \right)^{\frac{1}{\mu}} u^{\gamma}(\vartheta) \geq u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) \geq \left(\frac{1}{M} \right)^{\frac{1}{\mu}} u^{\gamma}(\vartheta),$$

and then

$$m^{\frac{1}{\mu}} u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) \leq u^{\gamma}(\vartheta) \leq M^{\frac{1}{\mu}} u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta). \tag{31}$$

By multiplying the right hand side of (31) by $w(\vartheta) > 0$ and integrating on $[r,s]_{\mathbb{T}}$ and using properties (ii) and (v) in Theorem 1, we can write that

$$\left(\int_r^s u^{\gamma}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta \right)^{\frac{1}{\lambda}} \leq M^{\frac{1}{\lambda\mu}} \left(\int_r^s u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta \right)^{\frac{1}{\lambda}}. \tag{32}$$

Similarly, from assumption (28), we get

$$m^{\frac{1}{\lambda}} \leq u^{\frac{\gamma}{\lambda}}(\vartheta) v^{-\frac{\beta}{\lambda}}(\vartheta) \leq M^{\frac{1}{\lambda}}.$$

Multiplying by $v^{\beta} > 0$ and using the fact that $(\lambda - 1)/\lambda = \mu$, we obtain

$$m^{\frac{1}{\lambda}} v^{\beta}(\vartheta) \leq u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) \leq M^{\frac{1}{\lambda}} v^{\beta}(\vartheta),$$

and then

$$\left(\frac{1}{M} \right)^{\frac{1}{\lambda}} u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) \leq v^{\beta}(\vartheta) \leq \left(\frac{1}{m} \right)^{\frac{1}{\lambda}} u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta). \tag{33}$$

Finally, we deduce that

$$\left(\int_r^s v^{\beta}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta \right)^{\frac{1}{\mu}} \leq \left(\frac{1}{m} \right)^{\frac{1}{\lambda\mu}} \left(\int_r^s u^{\frac{\gamma}{\lambda}}(\vartheta) v^{\frac{\beta}{\mu}}(\vartheta) w(\vartheta) \Diamond_{\alpha} \vartheta \right)^{\frac{1}{\mu}}. \tag{34}$$

By multiplying the inequalities (32), (34) and using the fact that $1/\lambda + 1/\mu = 1$, we have the required inequality (29). □

As a specific case of Theorem 5 when $\alpha = 1$ and $\alpha = 0$, we get the following findings.

Corollary 5. *Let $\gamma, \beta > 0$, $\lambda > 1$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$, w a weight function (measurable and positive) on $[r,s]_{\mathbb{T}}$. If*

$$0 < m \leq \frac{u^{\gamma}(\vartheta)}{v^{\beta}(\vartheta)} \leq M < \infty \quad \text{for all } \vartheta \in [r,s]_{\mathbb{T}},$$

then

$$\left(\int_r^s u^{\gamma}(\vartheta)w(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\beta}(\vartheta)w(\vartheta)\Delta\vartheta\right)^{\frac{1}{\mu}} \le \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u^{\frac{\gamma}{\lambda}}(\vartheta)v^{\frac{\beta}{\mu}}(\vartheta)w(\vartheta)\Delta\vartheta, \tag{35}$$

which is the delta version of (29).

Corollary 6. *Let $\gamma, \beta > 0$, $\lambda > 1$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C_{ld}([r, s]_{\mathbb{T}}, \mathbb{R})$, w a weight function (measurable and positive) on $[r, s]_{\mathbb{T}}$. If*

$$0 < m \le \frac{u^{\gamma}(\vartheta)}{v^{\beta}(\vartheta)} \le M < \infty \quad \textit{for all } \vartheta \in [r, s]_{\mathbb{T}},$$

$$\left(\int_r^s u^{\gamma}(\vartheta)w(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\beta}(\vartheta)w(\vartheta)\nabla\vartheta\right)^{\frac{1}{\mu}} \le \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u^{\frac{\gamma}{\lambda}}(\vartheta)v^{\frac{\beta}{\mu}}(\vartheta)w(\vartheta)\nabla\vartheta, \tag{36}$$

which is the nabla version of (29).

Remark 5. *For $\gamma = \lambda$, $\beta = \mu$ and $w(\vartheta) = 1$, inequality (35) in Corollary 5 coincides with inequality (10) in the introduction.*

Remark 6. *In Corollary 5, if $\gamma = 1$, $\beta = 1$ and $w(\vartheta) = 1$, we get the reverse Hölder type inequality*

$$\left(\int_r^s u(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v(\vartheta)\Delta\vartheta\right)^{\frac{1}{\mu}} \le \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)\Delta\vartheta, \tag{37}$$

which is [16] [Corollary 2.2].

Remark 7. *For the particular case $\mathbb{T} = \mathbb{R}$, inequality (29) in Theorem 5 reduces to (8) in the introduction.*

Remark 8. *As a particular state of Theorem 5, if $\gamma = \lambda$, $\beta = \mu$, then*

$$\left(\int_r^s u^{\lambda}(\vartheta)w(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)w(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \le \left(\frac{M}{m}\right)^{\frac{1}{\lambda\mu}}\int_r^s u(\vartheta)v(\vartheta)w(\vartheta)\Diamond_{\alpha}\vartheta. \tag{38}$$

For $w(\vartheta) = 1$, inequality (38) reduces to (13) in the introduction.

Remark 9. *In Remark 8, if we replace u^{λ} and v^{μ} by u and v, we obtain the reverse Hölder type inequality*

$$\left(\int_r^s u(\vartheta)w(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v(\vartheta)w(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \le \left(\frac{m}{M}\right)^{\frac{-1}{\lambda\mu}}\int_r^s u^{\frac{1}{\lambda}}(\vartheta)v^{\frac{1}{\mu}}(\vartheta)w(\vartheta)\Diamond_{\alpha}\vartheta. \tag{39}$$

As an application of Hölder's inequalities (38) and (39) in Remarks 8 and 9, we have the following theorems.

Theorem 6. *Let $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$ and $u \in C([r, s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \le u(\vartheta) \le M < \infty \quad \textit{for all } \vartheta \in [r, s]_{\mathbb{T}}, \tag{40}$$

then

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_{\alpha}\vartheta \ge B\left(\int_r^s u(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}-1}, \tag{41}$$

where

$$B = m(s-r)^{1+\frac{1}{\mu}}\left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}}.$$

Proof. Putting $v(\vartheta) = w(\vartheta) \equiv 1$ in Remark 9, *yields*

$$\left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \left(\int_r^s (1)^\mu \Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{m}{M}\right)^{\frac{-1}{\lambda\mu}} \int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta, \tag{42}$$

and so,

$$\left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} (s-r)^{\frac{1}{\mu}} \leq \left(\frac{m}{M}\right)^{\frac{-1}{\lambda\mu}} \int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta.$$

Therefore, we get

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta \geq \left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}} (s-r)^{\frac{1}{\mu}} \left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}-1} \left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}. \tag{43}$$

Since $0 < m \leq u(\vartheta)$, we have

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta \geq \left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}} m\,(s-r)^{\frac{1}{\mu}+1} \left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}-1}.$$

This proves (41). □

Corollary 7. *Let $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$. If*

$$m\left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}} = \frac{1}{(s-r)^{1+\frac{1}{\mu}}}$$

and $0 < m \leq u(\vartheta) \leq M < \infty$ on $\vartheta \in [r,s]_{\mathbb{T}}$, then

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta \geq \left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}-1}. \tag{44}$$

As a specific case of Theorem 6 and Corollary 7 when $\alpha = 1$ and $\alpha = 0$, we get the following findings.

Corollary 8. *If $u : [r,s]_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous and $0 < m \leq u(\vartheta) \leq M < \infty$ on $\vartheta \in [r,s]_{\mathbb{T}}$, then*

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta \geq B\left(\int_r^s u(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}-1}, \tag{45}$$

where $\lambda > 1$, $\mu > 1$ with $1/\lambda + 1/\mu = 1$ and

$$B = m(s-r)^{1+\frac{1}{\mu}}\left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}},$$

which is the delta version of (41) (see [16] [Theorem 2.6]). Additionally, if $m\,(m/M)^{1/\lambda\mu} = 1/\,(s-r)^{1+(1/\mu)}$, then

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta \geq \left(\int_r^s u(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}-1} \tag{46}$$

which is the delta version of (44) (see [16] [Corollary 2.7]).

Corollary 9. *If $u : [r,s]_{\mathbb{T}} \to \mathbb{R}$ is ld-continuous and $0 < m \leq u(\vartheta) \leq M < \infty$ on $\vartheta \in [r,s]_{\mathbb{T}}$, then*

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\nabla\vartheta \geq B\left(\int_r^s u(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}-1}, \tag{47}$$

where $\lambda > 1$, $\mu > 1$ *with* $1/\lambda + 1/\mu = 1$ *and*

$$B = m(s-r)^{1+\frac{1}{\mu}} \left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}},$$

which is the nabla version of (41). Additionally, if $m\,(m/M)^{1/\lambda\mu} = 1/\,(s-r)^{1+(1/\mu)}$, *then*

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\nabla\vartheta \geq \left(\int_r^s u(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}-1}, \tag{48}$$

which is the nabla version of (44).

Remark 10. *For the particular case* $\mathbb{T} = \mathbb{R}$, *inequality (41) in Theorem 6 reduces to*

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)d\vartheta \geq m(s-r)^{1+\frac{1}{\mu}} \left(\frac{m}{M}\right)^{\frac{1}{\lambda\mu}} \left(\int_r^s u(\vartheta)d\vartheta\right)^{\frac{1}{\lambda}-1}, \tag{49}$$

and inequality (44) in Corollary 7 reduces to

$$\int_r^s u^{\frac{1}{\lambda}}(\vartheta)d\vartheta \geq \left(\int_r^s u(\vartheta)d\vartheta\right)^{\frac{1}{\lambda}-1}, \tag{50}$$

which is Qi's inequality [32].

Theorem 7. *Let* $\lambda > 1$ *and* $\mu > 1$ *with* $1/\lambda + 1/\mu = 1$ *and* $u \in C([r,s]_{\mathbb{T}}, \mathbb{R})$. *If*

$$0 < m \leq u^{\lambda}(\vartheta) \leq M < \infty \quad \textit{for all} \quad \vartheta \in [r,s]_{\mathbb{T}}, \tag{51}$$

then

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda} \geq (s-r)^{\frac{\lambda+1}{\mu}} \left(\frac{m}{M}\right)^{\frac{\lambda+1}{\lambda\mu}} \left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}. \tag{52}$$

Proof. Putting $v(\vartheta) = w(\vartheta) \equiv 1$ into Remark 8 yields

$$\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \left(\int_r^s (1)^{\mu}\Diamond_\alpha\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{m}{M}\right)^{-\frac{1}{\lambda\mu}} \int_r^s u(\vartheta)\Diamond_\alpha\vartheta, \tag{53}$$

and so,

$$\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} (s-r)^{\frac{1}{\mu}} \leq \left(\frac{m}{M}\right)^{-\frac{1}{\lambda\mu}} \int_r^s u(\vartheta)\Diamond_\alpha\vartheta.$$

Therefore, we get

$$\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \leq (s-r)^{-\frac{1}{\mu}} \left(\frac{m}{M}\right)^{-\frac{1}{\lambda\mu}} \int_r^s u(\vartheta)\Diamond_\alpha\vartheta. \tag{54}$$

On the other hand, substituting $v(\vartheta) = w(\vartheta) \equiv 1$ in Remark 9 leads to

$$\left(\int_r^s u(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} \leq (s-r)^{-\frac{1}{\mu}} \left(\frac{m}{M}\right)^{-\frac{1}{\lambda\mu}} \int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta.$$

Further, taking the λ-th power on both sides of the above inequality yields

$$\int_r^s u(\vartheta)\Diamond_\alpha\vartheta \leq (s-r)^{-\frac{\lambda}{\mu}} \left(\frac{m}{M}\right)^{-\frac{1}{\mu}} \left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda}. \tag{55}$$

Combining (54) and (55), we have

$$\begin{aligned}\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}} &\leq (s-r)^{-\frac{1}{\mu}}\left(\frac{m}{M}\right)^{-\frac{1}{\lambda\mu}}(s-r)^{-\frac{\lambda}{\mu}}\left(\frac{m}{M}\right)^{-\frac{1}{\mu}}\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda} \\ &= (s-r)^{-\frac{1}{\mu}-\frac{\lambda}{\mu}}\left(\frac{m}{M}\right)^{-\frac{1}{\lambda\mu}-\frac{1}{\mu}}\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda} \\ &= (s-r)^{-\left(\frac{\lambda+1}{\mu}\right)}\left(\frac{m}{M}\right)^{-\left(\frac{\lambda+1}{\lambda\mu}\right)}\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda}.\end{aligned}$$

Hence,

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda} \geq (s-r)^{\frac{\lambda+1}{\mu}}\left(\frac{m}{M}\right)^{\frac{\lambda+1}{\lambda\mu}}\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}.$$

This proves (52). □

Corollary 10. *In Theorem 7, if $0 < m^{1/\lambda} \leq u \leq M^{1/\lambda} < \infty$ on $[r,s]_{\mathbb{T}}$ and $m/M = (s-r)^{-\lambda}$ for $\lambda > 1$, then*

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Diamond_\alpha\vartheta\right)^{\lambda} \geq \left(\int_r^s u^{\lambda}(\vartheta)\Diamond_\alpha\vartheta\right)^{\frac{1}{\lambda}}. \tag{56}$$

As a specific case of Theorem 7 and Corollary 10 when $\alpha = 1$ and $\alpha = 0$, we get the following findings.

Corollary 11. *Let $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$ and $u \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \leq u^{\lambda}(\vartheta) \leq M < \infty \quad \textit{for all} \quad \vartheta \in [r,\, s]_{\mathbb{T}},$$

then

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta\right)^{\lambda} \geq (s-r)^{\frac{\lambda+1}{\mu}}\left(\frac{m}{M}\right)^{\frac{\lambda+1}{\lambda\mu}}\left(\int_r^s u^{\lambda}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}, \tag{57}$$

which is the delta version of (52) (see [16,17] [Theorem 2.3], [Theorem 2.3.9]). Additionally, if $m/M = (s-r)^{-\lambda}$ for $\lambda > 1$, then

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\Delta\vartheta\right)^{\lambda} \geq \left(\int_r^s u^{\lambda}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}} \tag{58}$$

which is the delta version of (56), see [16] [Corollary 2.4].

Corollary 12. *Let $\lambda > 1$ and $\mu > 1$ with $1/\lambda + 1/\mu = 1$ and $u \in C_{ld}([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \leq u^{\lambda}(\vartheta) \leq M < \infty \quad \textit{for all} \quad \vartheta \in [r,\, s]_{\mathbb{T}},$$

then

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\nabla\vartheta\right)^{\lambda} \geq (s-r)^{\frac{\lambda+1}{\mu}}\left(\frac{m}{M}\right)^{\frac{\lambda+1}{\lambda\mu}}\left(\int_r^s u^{\lambda}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}}, \tag{59}$$

which is the nabla version of (52). Additionally, if $m/M = (s-r)^{-\lambda}$ for $\lambda > 1$, then

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)\nabla\vartheta\right)^{\lambda} \geq \left(\int_r^s u^{\lambda}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}}, \tag{60}$$

which is the nabla version of (56).

Remark 11. *For the particular case $\mathbb{T} = \mathbb{R}$, inequality (52) in Theorem 7 reduces to*

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)d\vartheta\right)^{\lambda} \geq (s-r)^{\frac{\lambda+1}{\mu}}\left(\frac{m}{M}\right)^{\frac{\lambda+1}{\lambda\mu}}\left(\int_r^s u^{\lambda}(\vartheta)d\vartheta\right)^{\frac{1}{\lambda}}.$$

and inequality (56) in Corollary 10 reduces to

$$\left(\int_r^s u^{\frac{1}{\lambda}}(\vartheta)d\vartheta\right)^{\lambda} \geq \left(\int_r^s u^{\lambda}(\vartheta)d\vartheta\right)^{\frac{1}{\lambda}}, \tag{61}$$

which is Qi's inequality [32].

Corollary 13. *Let $\lambda > 1$, $1/\lambda + 1/\mu = 1$ and $u, v \in C([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$m \leq \frac{u^{\lambda}(\vartheta)}{v^{\mu}(\vartheta)} \leq M,$$

then

$$m^{\frac{1}{\mu}} \leq \frac{u^{\frac{\lambda}{\mu}}(\vartheta)}{v(\vartheta)} \leq M^{\frac{1}{\mu}},$$

$$m^{\frac{1}{\mu}} \leq \frac{u^{\lambda-1}(\vartheta)}{v(\vartheta)} \leq M^{\frac{1}{\mu}},$$

and hence, we get

$$\left(\int_r^s u^{\lambda}(\vartheta)\diamondsuit_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\diamondsuit_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m}\right)^{\frac{1}{\lambda}}\int_r^s u(\vartheta)v(\vartheta)\diamondsuit_{\alpha}\vartheta. \tag{62}$$

Remark 12. *For the particular case $\mathbb{T} = \mathbb{R}$, Corollary 13 coincides with Corollary 2.4 in [13].*

Corollary 14. *Let $\lambda > 1$, $1/\lambda + 1/\mu = 1$ and $u, v \in C([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$m \leq \frac{u^{\lambda}(\vartheta)}{v^{\mu}(\vartheta)} \leq M,$$

then

$$m^{\frac{1}{\lambda}} \leq \frac{u(\vartheta)}{v^{\mu-1}(\vartheta)} \leq M^{\frac{1}{\lambda}},$$

and hence, we get

$$\left(\int_r^s u^{\lambda}(\vartheta)\diamondsuit_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\diamondsuit_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \leq \left(\frac{M}{m}\right)^{\frac{1}{\mu}}\int_r^s u(\vartheta)v(\vartheta)\diamondsuit_{\alpha}\vartheta. \tag{63}$$

Remark 13. *For the particular case $\mathbb{T} = \mathbb{R}$, Corollary 14 coincides with Corollary 2.5 in [13].*

Now, we present a refinement of inequality (7) on time scales.

Theorem 8. *Let $\gamma, d, \lambda, \mu, \lambda', \mu' > 0$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < d < m \leq \frac{\gamma u(\vartheta)}{v(\vartheta)} \leq M \quad \text{for all } \vartheta \in [r,\, s]_{\mathbb{T}}, \tag{64}$$

then

$$\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \tag{65}$$
$$\leq \frac{M}{\gamma}\left(\frac{\gamma}{m}\right)^{\frac{2\lambda'}{\lambda'+\mu'}}(m+d)^{\frac{\lambda'-\mu'}{\lambda'+\mu'}}(M+d)^{\frac{\mu'-\lambda'}{\lambda'+\mu'}}\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\lambda}{\lambda'+\mu'}}\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\mu}{\lambda'+\mu'}}\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}}$$

Proof. From the assumption (64), we get

$$m+d\leq\frac{\gamma u(\vartheta)+dv(\vartheta)}{v(\vartheta)}\leq M+d, \tag{66}$$

and

$$\frac{M+d}{M}\leq\frac{\gamma u(\vartheta)+dv(\vartheta)}{\gamma u(\vartheta)}\leq\frac{m+d}{m}. \tag{67}$$

Integrating the left inequalities of (66) and (67), we have

$$(m+d)\left(\int_r^s v^{\mu}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}}\leq\left(\int_r^s(\gamma u(\vartheta)+dv(\vartheta))^{\mu}\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}}, \tag{68}$$

and

$$\gamma\left(\frac{M+d}{M}\right)\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\leq\left(\int_r^s(\gamma u(\vartheta)+dv(\vartheta))^{\lambda}\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}. \tag{69}$$

By multiplying the inequalities (68) and (69), we obtain

$$\frac{\gamma}{M}(M+d)(m+d)\left(\int_r^s u^{\lambda}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}} \tag{70}$$
$$\leq\left(\int_r^s(\gamma u(\vartheta)+dv(\vartheta))^{\lambda}\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s(\gamma u(\vartheta)+dv(\vartheta))^{\mu}\Diamond_{\alpha}\vartheta\right)^{\frac{1}{\mu}}.$$

On the other hand, from the right inequalities of (66) and (67), it follows easily that

$$(\gamma u(\vartheta)+dv(\vartheta))^{\mu'}\leq(M+d)^{\mu'}v^{\mu'}(\vartheta), \tag{71}$$

and

$$(\gamma u(\vartheta)+dv(\vartheta))^{\lambda'}\leq\left(\frac{\gamma}{m}(m+d)\right)^{\lambda'}u^{\lambda'}(\vartheta). \tag{72}$$

By multiplying the inequalities (71) and (72), we get

$$(\gamma u(\vartheta)+dv(\vartheta))^{\lambda'+\mu'}\leq\left(\frac{\gamma}{m}(m+d)\right)^{\lambda'}(M+d)^{\mu'}u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta), \tag{73}$$

which, raising inequality (73) to power $1/(\lambda'+\mu')$, we obtain

$$(\gamma u(\vartheta)+dv(\vartheta))\leq\left(\frac{\gamma}{m}(m+d)\right)^{\frac{\lambda'}{\lambda'+\mu'}}(M+d)^{\frac{\mu'}{\lambda'+\mu'}}\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{1}{\lambda'+\mu'}}. \tag{74}$$

From inequality (74), we deduce that

$$\left(\int_r^s (\gamma u(\vartheta)+dv(\vartheta))^{\lambda}\lozenge_\alpha\vartheta\right)^{\frac{1}{\lambda}} \le \left(\frac{\gamma}{m}(m+d)\right)^{\frac{\lambda'}{\lambda'+\mu'}}(M+d)^{\frac{\mu'}{\lambda'+\mu'}} \times\left(\int_r^s \left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\lambda}{\lambda'+\mu'}}\lozenge_\alpha\vartheta\right)^{\frac{1}{\lambda}}, \tag{75}$$

and

$$\left(\int_r^s (\gamma u(\vartheta)+dv(\vartheta))^{\mu}\lozenge_\alpha\vartheta\right)^{\frac{1}{\mu}} \le \left(\frac{\gamma}{m}(m+d)\right)^{\frac{\lambda'}{\lambda'+\mu'}}(M+d)^{\frac{\mu'}{\lambda'+\mu'}} \times\left(\int_r^s \left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\mu}{\lambda'+\mu'}}\lozenge_\alpha\vartheta\right)^{\frac{1}{\mu}}. \tag{76}$$

By multiplying inequalities (75) and (76), we get

$$\begin{aligned}&\left(\int_r^s (\gamma u(\vartheta)+dv(\vartheta))^{\lambda}\lozenge_\alpha\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s (\gamma u(\vartheta)+dv(\vartheta))^{\mu}\lozenge_\alpha\vartheta\right)^{\frac{1}{\mu}} \\ \le\ &\left(\frac{\gamma}{m}(m+d)\right)^{\frac{2\lambda'}{\lambda'+\mu'}}(M+d)^{\frac{2\mu'}{\lambda'+\mu'}}\left(\int_r^s \left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\lambda}{\lambda'+\mu'}}\lozenge_\alpha\vartheta\right)^{\frac{1}{\lambda}} \\ &\times\left(\int_r^s \left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\mu}{\lambda'+\mu'}}\lozenge_\alpha\vartheta\right)^{\frac{1}{\mu}}.\end{aligned} \tag{77}$$

Finally, by the inequalities (70) and (77), we obtain the desired inequality (65). □

As a specific case of Theorem 8 when $\lambda' = \mu' = \gamma = 1$, we get the following findings.

Corollary 15. *Let $\lambda > 0$, $\mu > 0$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < m \le \frac{u(\vartheta)}{v(\vartheta)} \le M \quad \text{for all } \vartheta \in [r,s]_{\mathbb{T}},$$

then

$$\begin{aligned}&\left(\int_r^s u^{\lambda}(\vartheta)\lozenge_\alpha\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\lozenge_\alpha\vartheta\right)^{\frac{1}{\mu}} \\ \le\ &\frac{M}{m}\left(\int_r^s (u(\vartheta)v(\vartheta))^{\frac{\lambda}{2}}\lozenge_\alpha\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s (u(\vartheta)v(\vartheta))^{\frac{\mu}{2}}\lozenge_\alpha\vartheta\right)^{\frac{1}{\mu}},\end{aligned} \tag{78}$$

which is the diamond-α version of (7) on time scales.

As a specific case of Theorem 8 and Corollary 10 when $\alpha = 1$ and $\alpha = 0$, we get the following findings.

Corollary 16. *Let $\gamma, d, \lambda, \mu, \lambda', \mu' > 0$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < d < m \le \frac{\gamma u(\vartheta)}{v(\vartheta)} \le M \quad \text{for all } \vartheta \in [r,s]_{\mathbb{T}},$$

then

$$\left(\int_r^s u^{\lambda}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\mu}}$$
$$\le \frac{m}{\gamma}\left(\frac{\gamma}{m}\right)^{\frac{2\lambda'}{\lambda'+\mu'}}(m+d)^{\frac{\lambda'-\mu'}{\lambda'+\mu'}}(m+d)^{\frac{\mu'-\lambda'}{\lambda'+\mu'}}\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\lambda}{\lambda'+\mu'}}\Delta\vartheta\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\mu}{\lambda'+\mu'}}\Delta\vartheta\right)^{\frac{1}{\mu}}, \tag{79}$$

which is the delta version of (65).

Corollary 17. *Let $\gamma, d, \lambda, \mu, \lambda', \mu' > 0$ with $1/\lambda + 1/\mu = 1$ and $u, v \in C_{rd}([r,s]_{\mathbb{T}}, \mathbb{R})$. If*

$$0 < d < m \le \frac{\gamma u(\vartheta)}{v(\vartheta)} \le M \quad \text{for all } \vartheta \in [r,s]_{\mathbb{T}},$$

then

$$\left(\int_r^s u^{\lambda}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\mu}}$$
$$\le \frac{M}{\gamma}\left(\frac{\gamma}{m}\right)^{\frac{2\lambda'}{\lambda'+\mu'}}(m+d)^{\frac{\lambda'-\mu'}{\lambda'+\mu'}}(M+d)^{\frac{\mu'-\lambda'}{\lambda'+\mu'}}\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\lambda}{\lambda'+\mu'}}\nabla\vartheta\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\mu}{\lambda'+\mu'}}\nabla\vartheta\right)^{\frac{1}{\mu}}, \tag{80}$$

which is the nabla version of (65).

Remark 14. *For the particular case $\mathbb{T} = \mathbb{R}$, inequality (65) in Theorem 8 reduces to*

$$\left(\int_r^s u^{\lambda}(\vartheta)d\vartheta\right)^{\frac{1}{\lambda}}\left(\int_r^s v^{\mu}(\vartheta)d\vartheta\right)^{\frac{1}{\mu}}$$
$$\le \frac{M}{\gamma}\left(\frac{\gamma}{m}\right)^{\frac{2\lambda'}{\lambda'+\mu'}}(m+d)^{\frac{\lambda'-\mu'}{\lambda'+\mu'}}(M+d)^{\frac{\mu'-\lambda'}{\lambda'+\mu'}}\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\lambda}{\lambda'+\mu'}}d\vartheta\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_r^s\left(u^{\lambda'}(\vartheta)v^{\mu'}(\vartheta)\right)^{\frac{\mu}{\lambda'+\mu'}}d\vartheta\right)^{\frac{1}{\mu}}, \tag{81}$$

which is [13] [Theorem 2.6].

4. Conclusions and Future Work

The study of dynamic inequalities depends on the diamond-α integral on time scales. Hence, in the context of this article, we presented generalizations of symmetrical form for Hölder's inequality and it is reverse by means of the diamond-α integral, which is deflated as a linear combination of the delta and nabla integrals. Within this paper, we generalize certain delta and nabla-integrals inequalities on time scales to diamond-α integrals. Inequalities are considered in rather general forms and contain several special integral and discrete inequalities. The technique is based on the applications of well-known inequalities and new tools from time scale calculus. For future work, we can present such diamond-α integrals inequalities by using Riemann–Liouville type fractional integrals and fractional derivatives on time scales. It will also be very interesting to present such diamond-α integrals inequalities on quantum calculus.

Author Contributions: All authors contributed equally to the writing of this manuscript and all authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research at Princess Nourah Bint Abdulrahman University through the Fast-track Research Funding Program.

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Pecaric, J.; Pearce, C.E.M. On an inverse to the Hölder inequality. *Int. J. Math. Math. Sci.* **1997**, *20*, 205–207. [CrossRef]
2. Qiang, H.; Hu, Z. Generalizations of Hölder's and some related inequalities. *Compt. Math. Appl.* **2011**, *61*, 392–396. [CrossRef]
3. Wu, S. A new sharpened and generalized version of Hölder's inequality and its applications. *Appl. Math. Comput.* **2008**, *197*, 708–714. [CrossRef]
4. Yang, X. A generalization of Hölder inequality. *J. Math. Anal. Appl.* **2000**, *247*, 328–330. [CrossRef]
5. Hölder, O. Uber einen Mittelwerthssatz. *Nachr. Ges-Wiss. Gott.* **1889**, *1889*, 38–47.
6. Bougoffa, L. An integral inequality similar to Qi's inequality. *J. Inequal. Pure Appl. Math.* **2005**, *1*, 1–3.
7. Mitrinovic, D.S.; Pecaric, J.; Fink, A.M. *Classical and New Inequalities Inanalysis*; Springer Science and Business Media: New York, NY, USA, 2013; Volume 61, pp. 126–127.
8. Saitoh, S.; Tuan, V.K.; Yamamoto, M. Reverse convolution inequalities and applications to inverse heat source problems. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 80.
9. Xiao-Hua, L. On the inverse of Hölder inequality. *Math. Pract. Theory* **1990**, *1*, 84–88.
10. Halim, B.; Senouci, A. Some generalizations involving open problems of F. Qi. *Int. J. Open Probl. Compt. Math.* **2019**, *12*, 10–21.
11. Zhao, C.J.; Cheung, W.S. Hölder's reverse inequality and its applications. *Publ. Inst. Math.* **2016**, *99*, 211–216. [CrossRef]
12. Sulaiman, W.T. Reverses of Minkowski's, Hölder's, and Hardy's integral inequalities. *Int. J. Mod. Math. Sci.* **2012**, *1*, 14–24.
13. Benaissa, B.; Budak, H. More on reverse of Hölder's integral inequality. *Korean J. Math.* **2020**, *28*, 9–15.
14. Agarwal, R.; Bohner, M.; Peterson, A. Inequalities on time scales: A survey. *Mathl. Inequal. Appl.* **2001**, *7*, 535–557. [CrossRef]
15. Tuna, A.; Kutukcu, S. Some integral inequalities on S. time scales. *Appl. Math. Mech.* **2008**, *29*, 23–29. [CrossRef]
16. Sarikaya, M.Z.; Ozkan, U.M.; Ylldirim, H. Time scale integral inequalities similar to Qi's inequality. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 1–7.
17. Agarwal, R.; O'Regan, D.; Saker, S.H. *Dynamic Inequalities on Time Scales*; Springer International Publishing: Cham, Switzerland, 2014.
18. Ammi, M.R.S.; Torres, D.F.M. Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales. *Ann. Univ. Craiova Math. Comp. Sci. Ser.* **2010**, *37*, 1–11.
19. Ahmed, A.M.; AlNemer, G.; Zakarya, M.; Rezk, H.M. Some dynamic inequalities of hilbert's type. *J. Fun. Spaces* **2020**, 1–13. [CrossRef]
20. Abd El-Hamid, H.A.; Rezk, H.M.; Ahmed, A.M.; AlNemer, G.; Zakarya, M.; El Saify, H.A. Dynamic Inequalities in Quotients with General Kernels and Measures. *J. Fun. Spaces* **2020**, 1–12. [CrossRef]
21. Saker, S.H.; Kenawy, M.R.; AlNemer, G.; Zakarya, M. Hardys Type Via Conformable Calculus. *J. Math.* **2020**, *434*, 1–15.
22. AlNemer, G.; Zakarya, M.; Abd El-Hamid, H.A.; Agarwal, P.; Rezk, H.M. Some Dynamic Hilbert-Type Inequalities on Time Scales. *Symmetry* **2020**, *12*, 1410. [CrossRef]
23. AlNemer, G.; Zakarya, M.; Abd El-Hamid, H.A.; Kenawy, M.R.; Rezk, H.M. Dynamic Hardy-type inequalities with non-conjugate parameters. *Alex. Eng. J.* **2020**, 1–10. [CrossRef]

24. Agarwal, P.; Hyder, A.; Zakarya, M.; AlNemer, G.; Cesarano, C.; Assante, D. Exact Solutions for a Class of Wick-Type Stochastic (3+1)-Dimensional Modified Benjamin-Bona-Mahony Equations. *Axioms* **2019**, *8*, 134. [CrossRef]
25. O'Regan, D.; Rezk, H.M.; Saker, S.H. Some dynamic inequalities involving hilbert and hardy-hilbert operators with kernels. *Res. Math.* **2018**, *73*, 1–22. [CrossRef]
26. Saker, S.H.; Rezk, H.M.; O'Regan, D.; Agarwal, R.P. A variety of inverse hilbert type inequality on time scales. *Dyn. Contin. Discret. Impuls. Syst. Ser. A Math. Anal.* **2017**, *24*, 347–373.
27. Saker, S.H.; Rezk, H.M.; Krnić, M. More accurate dynamic hardy-type inequalities obtained via superquadraticity. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2019**, *113*, 2691–2713. [CrossRef]
28. Saker, S.H.; Rezk, H.M.; Abohela, I.; Baleanu, D. Refinement multidimensional dynamic inequalities with general kernels and measures. *J. Inequal. Appl.* **2019**, *306*, 1–16. [CrossRef]
29. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
30. Bohner, M.; Peterson, A. (Eds.) *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.
31. Sheng, Q.; Fadag, M.; Henderson, J.; Davis, J.M. An exploration of combined dynamic derivatives on time scales and their applications. *Nonlinear Anal. Real World Appl.* **2006**, *7*, 395–413. [CrossRef]
32. Qi, F. Several integral inequalities. *J. Inequal. Pure Appl. Math.* **2000**, *1*, 1–7.