New Oscillation Results for Second-Order Neutral Differential Equations with Deviating Arguments

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Abstract: In this paper, we focus on the second-order neutral differential equations with deviating arguments which are under the canonical condition. New oscillation criteria are established, which are based on a first-order delay differential equation and generalized Riccati transformations. The idea of symmetry is a useful tool, not only guiding us in the right way to study this function but also simplifies our proof. Our results are generalizations of some previous results and we provide an example to illustrate the main results.

Keywords: neutral delay differential equations; oscillation; Riccati transformations

MSC: 34C10; 34K11

1. Introduction

In this paper, we consider a second-order neutral differential equations with deviating arguments of the form

\[
(r(t)\psi(x(t)) (y'(t))^2)' + \sum_{i=1}^{n} q_i(t) f(x(\sigma_i(t))) = 0,
\]

for \( t > t_0 \), where \( t_0 > 0 \) is a constant and \( y(t) = x(t) + \sum_{j=1}^{m} p_j(t) x(\tau_j(t)) \), \( \gamma \) is a quotient of odd positive integers.

Throughout the paper, it is assumed that the following hypothesis hold true

\[(A_1)\quad r(t) \in C^1([t_0, \infty), (0, \infty)), \text{ and } \int_{t_0}^{\infty} r^{-1}(s)ds = \infty.\]
\[(A_2)\quad q_i(t) \in C^1([t_0, \infty), (0, \infty)), p_j(t) \in C^1([t_0, \infty), (0, \infty)), \text{ and } \sum_{j=1}^{m} p_j(t) < 1, \text{ for } i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}.\]
\[(A_3)\quad \sigma_i(t), \tau_i(t) \in C([t_0, \infty), (0, \infty)), \text{ and there exists a function } \sigma(t), \text{ such that } \sigma(t) \leq \min \{\sigma_i(t)\}, \text{ where } \sigma(t) \leq t, \tau_i(t) \leq t, \lim_{t \to \infty} t \tau_i(t) = \lim_{t \to \infty} \sigma_i(t) = \infty, \text{ for } i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}.\]
\[(A_4)\quad \psi(x) \in C([0, \infty), 0 < \psi(x) \leq M \text{ for } M \text{ is a positive constant.}\]
\[(A_5)\quad f(x) \in C([0, \infty), \mathbb{R}), \text{ and there exists a constant } k > 0 \text{ such that } f(x) \geq k \text{ for } x \neq 0.\]

We restrict our attention to those solutions \( x(t) \) of Equation (1), which means a function \( x \in C^2([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) (y'(t))^2 \in C^1([L_x, \infty), L_x \geq t_0, \text{ and has the property } r(t)\psi(x(t)) \]
The qualitative properties of ordinary differential equations could be regarded as an age old theme. Such equations are often widely used in physics, population dynamics, economic problems, mechanical control, and other fields. Although it is much more difficult to study differential equations with delay than those without delay, with delay or multiple delay differential equations can better reflect the practical problems. For example, one of the problems is to describe the vibrating masses attached to an elastic bar. In the study of these equations, the solutions of the oscillation are the most popular topic and still receives much attention. The existed researches about delay differential equations can refer to the book [1] and article [2]. Different kinds of deviating arguments equations have been studied extensively, since Fite [3] first studied the oscillatory properties of the deviating arguments equation in 1921. For examples, Zhao and Chatzarakis studied the second-order equations in [4] and [5] respectively. Sun considered the third-order nonlinear differential equations [6]. Li studied the fourth-order differential equations [7]. Moaaz studied the even-order differential equations [8].

It is well-known that the study of the equations are based on the canonical case or non-canonical case, and the oscillation results obtained in these two cases are also different [9–12]. In this article, we only consider the canonical case, and a large number of articles provided different oscillation criteria for the canonical second-order differential equations (CSODE). For examples, Irena [9] gave us the Kneser-type oscillation criteria for CSODE with delay. Xu [13] studies the Kamenev-type oscillation criteria for CSODE with neutral and delay. Gai [14] studies the Philos-type oscillation results for second-order neutral nonlinear differential equations.

In the following, Xu [13] considered the second-order neutral equations

\[ [r(t)(x(t) + p(t)y(t))]' + \sum_{i=1}^{n} q_i(t)f(x(\sigma_i(t))) = 0. \]

Meng [15] considered the even-order neutral equations

\[ [x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t))]^{(n)} + \sum_{j=1}^{l} q_j(t)f(x(\sigma_j(t))) = 0. \]

Ye [16] studied the second-order quasilinear neutral equations

\[ (r(t)\psi(x(t))Z'(t))^{n-1}Z'(t)' + q(t)f[x(\sigma(t))] = 0. \]

Zhao [4] studied the second-order neutral equations with distributed deviating argument

\[ (r(t)\psi(x(t))Z'(t))' + \int_{a}^{b} p(t, \xi)f[x(g(t, \xi))]d\sigma(\xi) = 0. \]

In addition, in [17], Bazighifan used the oscillation of second-order equation to give the criterion of oscillation of higher-order equation. In [18], Moaaz used the oscillation of the first-order equation to give the criterion of oscillation of second-order equation. In [19], we get a new method to estimate the fraction function \( \frac{\psi(x(t))}{y(t)} \). Ref. [2] gives us a new method to set Kamenev-type oscillation criteria. The background outlined above motivates our present research. We also want to study some new methods of oscillation results for second-order delay differential equation, which have more than one neutral delay under the canonical condition. We objective to study the Equation (1) by applying the known oscillation results of first-order differential equations or the generalized Riccati-type function.

2. Preliminaries

In this part, we considered the function \( x(t) \) as the eventually-positive solution. In fact, for the case of the eventually negative solution, the proof process is similar to the eventually positive solution. We will not prove it in detail. In the next sections of our paper, we need the following lemmas.
Lemma 1 ([20]). For any $t \in (-\infty, +\infty)$, set $B(v) = b_1(t)v - b_2(t)v^{\gamma+1}$ where the function $b_1(t)$ is arbitrary, function $b_2(t)$ and function $v$ are always positive, $\gamma$ is a positive constant. Then the function $B(v)$ has the maximum value $B_{\max}$ at $v_0$ on $(-\infty, +\infty)$, such that

$$B(v) \leq B_{\max} = \frac{\gamma^\gamma b_1^{\gamma+1}(t)}{(\gamma + 1)^{\gamma+1} b_2^{\gamma+1}(t)},$$

where $v_0 = \left(\frac{\gamma b_1(t)}{\gamma+1 b_2(t)}\right)^{\gamma}$.

Lemma 2. Suppose that $(A_1)$ to $(A_5)$ hold. If $x(t)$ is a positive solution of Equation (1), then there exists a $t_1 \geq t_0$, such that

$$y(t) > 0, \quad y'(t) > 0, \quad \left(r(t)\psi(x(t))\left(y'(t)\right)^\gamma\right)' < 0,$$

for $t \geq t_1$.

Proof. Suppose $x(t)$ is the positive solution of Equation (1) in $[t_0, \infty)$. There exists sufficiently large $t_1 > t_0$, such that $x(t_1) > 0, x(\tau_j(t)) > 0, x(\sigma_i(t)) > 0$, for $t > t_1$ and $i \in \{1, 2, ..., n\}, j \in \{1, 2, ..., m\}$.

From $y(t) = x(t) + \sum_{j=1}^n p_j(t)x(\tau_j(t))$, then we get $y(t) > x(t) > 0$.

Based on Equation (1), we obtain

$$\left(r(t)\psi(x(t))\left(y'(t)\right)^\gamma\right)' = -\sum_{i=1}^n q_i(t)f(x(\sigma_i(t))) < 0.$$ (4)

That means $r(t)\psi(x(t))\left(y'(t)\right)^\gamma$ is eventually one sign. Due to $(A_1)$ and $(A_4)$, we have $y'(t) > 0$ or $y'(t) < 0$ hold. We assert $y'(t) > 0$. If $y'(t) < 0$, from (4), there exists $t_2 > t_1$, such that when $t > t_2$, we have

$$r(t)\psi(x(t))\left(y'(t)\right)^\gamma < r(t_2)\psi(x(t_2))\left(y'(t_2)\right)^\gamma =: -N,$$

where $N$ is a positive constant. Then integrate $y'(t)$ in (5) from $t_2$ to $t$, we have

$$y(t) - y(t_2) < -N^{\frac{1}{\gamma}} \int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\psi^{-\frac{1}{\gamma}}(x(s))ds.$$ (5)

From $(A_5)$, we get $\psi^{-\frac{1}{\gamma}}(x) \geq M^{-\frac{1}{\gamma}}$, that is to say

$$y(t_2) > \left(\frac{N}{M}\right)^{\frac{1}{\gamma}} \int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)ds.$$ (6)

Let $t \to \infty$, (6) is contradicts assumption $(A_1)$. This completes the proof. □

3. Oscillation Results

In this section, we established some new oscillation criteria for Equation (1). We use the following notations for the simplicity:

$$P(t) = 1 - \sum_{j=1}^n p_j(t), \quad Q(t) = \sum_{i=1}^n q_i(t)P^{\gamma}(\sigma_i(t)),$$

$$\Psi(t) = \int_{t_1}^{t} r^{-\frac{1}{\gamma}}(s)ds + \frac{k}{\gamma M} \int_{t_1}^{t} Q(u) \int_{t_1}^{u} r^{-\frac{1}{\gamma}}(s)ds \left(\int_{t_1}^{u} r^{-\frac{1}{\gamma}}(s)ds\right)^{\gamma} du,$$
Theorem 1. Suppose that \((A_1) - (A_5)\) hold. If the first-order delay differential equation
\[
z'(t) + \frac{k}{M} Q(t) \Psi(\sigma(t)) z(\sigma(t)) = 0, \quad (7)
\]
is oscillatory, then every solutions \(x(t)\) of Equation (1) are oscillatory.

Proof. Suppose to contrary that there exists a nonoscillatory solution of Equation (1). Without loss of
generality, we can assume that \(x(t)\) is an eventually positive solution of Equation (1). From Lemma 2,
we have \(x(t) > 0, x(\sigma_i(t)) > 0, x(\tau_j(t)) > 0, y(t) > x(t) > 0\) and (3) hold, where \(i \in \{1, 2, ..., n\},\)
\(j \in \{1, 2, ..., m\}\) and \(t \geq t_1 \geq t_0\). From \(\sigma_i(t) \leq t\) in \((A_3)\), we get
\[
\begin{align*}
x(t) &= y(t) - \sum_{j=1}^{m} p_j(t) x(\sigma_j(t)) \\
 &\geq y(t) - \sum_{j=1}^{m} p_j(t) y(\sigma_j(t)) \\
 &\geq y(t) - \sum_{j=1}^{m} p_j(t) y(t) \\
 &\geq y(t) \left( 1 - \sum_{j=1}^{m} p_j(t) \right) := y(t) P(t). \quad (8)
\end{align*}
\]

Using Equation (1) and \((A_5)\), note \(\sigma_i(t)\) has a minimum delay \(\sigma(t) \leq \min\{\sigma_i(t)\}\), we have
\[
\begin{align*}
\left( r(t) \psi(x(t)) (y'(t))^\gamma \right)' &= -\sum_{i=1}^{n} q_i(t) f(x(\sigma_i(t))) \\
 &\leq -k \sum_{i=1}^{n} q_i(t) x^\gamma(\sigma_i(t)) \\
 &\leq -k \sum_{i=1}^{n} q_i(t) y(\sigma_i(t)) P(\sigma_i(t)) y^\gamma \\
 &\leq -k y^\gamma(\sigma(t)) \sum_{i=1}^{n} q_i(t) P^\gamma(\sigma_i(t)) := -k Q(t) y^\gamma(\sigma(t)). \quad (9)
\end{align*}
\]

Next, we give a useful function
\[
\Phi(t) = \int_{t_1}^{t} r^{-\frac{1}{\gamma}}(s) \psi^{-\frac{1}{\gamma}}(x(s)) ds.
\]
We obtain \(\Phi(t) > M^{-\frac{1}{\gamma}} \int_{t_1}^{t} r^{-\frac{1}{\gamma}}(s) ds > 0\) and \(\Phi'(t) = r^{-\frac{1}{\gamma}}(t) \psi^{-\frac{1}{\gamma}}(x(t)) > 0\) easily. Applying the
chain rule, we have
\[
\Phi(t) \left( r(t) \psi(x(t)) \left( y'(t) \right) \right) \left( y'(t) \right) = \Phi(t) \left[ \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) \right] \left( y'(t) \right)
\]
\[
= \gamma \Phi(t) \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) \left( y'(t) \right)
\]
\[
= -\gamma \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) \left( y'(t) \right)
\]
\[
\times \left[ -\Phi(t) \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) + y'(t) - \Phi(t) r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right]
\]
\[
= -\gamma \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) \left( y'(t) \right)
\]

It is easy to see that
\[
\left[ y(t) - \Phi(t) r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right] \left( y'(t) \right) = \frac{1}{\gamma} \Phi(t) \left( r(t) \psi(x(t)) \left( y'(t) \right) \right) \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) 1^{-\gamma}. (10)
\]

From (9) and (10), we have
\[
\left[ y(t) - \Phi(t) r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right] \left( y'(t) \right) \geq \frac{k}{\gamma} \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) 1^{-\gamma} \Phi(t) Q(t) y^\gamma(\sigma(t)).
\]

Integrating the above inequality form \( t_1 \) to \( t \), we obtain
\[
y(t) - \Phi(t) r^\gamma(t) \psi^\gamma(x(t)) y'(t) - \left[ y(t_1) - \Phi(t_1) r^\gamma(t_1) \psi^\gamma(x(t_1)) y'(t_1) \right] \geq \int_{t_1}^{t} \frac{k}{\gamma} \left( r^\gamma(s) \psi^\gamma(x(s)) y'(s) \right) 1^{-\gamma} \Phi(s) Q(s) y^\gamma(\sigma(s)) ds.
\]

Note \( y(t_1) > 0 \) and \( \Phi(t_1) = 0 \), so we get
\[
y(t) \geq \Phi(t) r^\gamma(t) \psi^\gamma(x(t)) y'(t)
\]
\[
+ \frac{k}{\gamma} \int_{t_1}^{t} \left( r^\gamma(s) \psi^\gamma(x(s)) y'(s) \right) 1^{-\gamma} \Phi(s) Q(s) y^\gamma(\sigma(s)) ds. (11)
\]

By simple computation, we have
\[
sgn \left[ \left( r^\gamma(t) \psi^\gamma(x(t)) y'(t) \right) \left( y'(t) \right) \right] = sgn \left[ \left( r(t) \psi(x(t)) \left( y'(t) \right) \right) \left( y'(t) \right) \right].
\]

Then, from \( r(t) \psi(x(t)) \left( y'(t) \right) < 0 \), we get
\[
y(t) = y(t_1) + \int_{t_1}^{t} y'(s) ds
\]
\[
= y(t_1) + \int_{t_1}^{t} r^{-\gamma} \left( s \right) \psi^{-\gamma} \left( x(s) \right) \left( r^\gamma \left( s \right) \psi^\gamma \left( x(s) \right) y'(s) \right) ds
\]
\[
\geq r^\gamma \left( t \right) \psi^\gamma \left( x(t) \right) y'(t) \Phi(t). (12)
\]
Combining (11) and (12), using \((r(t)\psi(x(t)) (y'(t))')' < 0\), we have

\[
y(t) \geq \Phi(t) r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t) + \frac{k}{\tau} \int_{t_1}^{t} \left( r^\frac{1}{\gamma} (s) \psi^\frac{1}{\gamma} (x(s)) y'(s) \right)^{1-\gamma} \Phi(s) Q(s) \nonumber
\]

\[
\times r(\sigma(s)) \psi(\sigma(s)) (y'(\sigma(s)))^{\gamma} \Phi(\sigma(s)) ds \nonumber
\]

\[
\geq \Phi(t) r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t) + \frac{k}{\tau} \int_{t_1}^{t} \left( r^\frac{1}{\gamma} (s) \psi^\frac{1}{\gamma} (x(s)) y'(s) \right)^{1-\gamma} \Phi(s) Q(s) \nonumber
\]

\[
\times r(s) \psi(s) (y'(s))^\gamma \Phi(s) ds \nonumber
\]

\[
\geq \Phi(t) r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t) + \frac{k}{\tau} \int_{t_1}^{t} \left( r^\frac{1}{\gamma} (s) \psi^\frac{1}{\gamma} (x(s)) y'(s) \right)^{1-\gamma} \Phi(s) Q(s) \Phi^\gamma (\sigma(s)) ds \nonumber
\]

\[
\geq r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t) \left( \Phi(t) + \frac{k}{\gamma} \int_{t_1}^{t} \Phi(s) Q(s) \Phi^\gamma (\sigma(s)) ds \right). \nonumber
\]

From the fact that \(\Phi(t) > \left(\frac{1}{M}\right)^\frac{1}{\gamma} \int_{t_1}^{t} r^{-\frac{1}{\gamma}} (s) ds\), the above inequality becomes

\[
y(t) \geq \left(\frac{1}{M}\right)^\frac{1}{\gamma} r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t) \nonumber
\]

\[
\times \left( \int_{t_1}^{t} r^{-\frac{1}{\gamma}} (s) ds + \frac{k}{\gamma} \int_{t_1}^{t} Q(s) \int_{t_1}^{s} r^{-\frac{1}{\gamma}} (\tau) ds \left( \int_{t_1}^{\tau} Q(\tau) \right) \right)^\gamma du \nonumber
\]

\[
\geq \left(\frac{1}{M}\right)^\frac{1}{\gamma} \Psi(t) r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t). \nonumber
\]

(13)

Put \(z(t) = r(t)\psi(x(t)) (y'(t))',\) we have \(z(t) > 0\). Then combining (9) and (13), we have

\[
z'(t) \leq -\frac{k}{M} Q(t) \Psi^\gamma (\sigma(t)) z(\sigma(t)). \nonumber
\]

Compute \(\Psi^\gamma (\sigma(t))\), there exist \(t_2 \geq t_1\), we have \(Q(t) \Psi^\gamma (\sigma(t))\) is positive \(t \geq t_2\), that is to say the inequality

\[
z'(t) + \frac{k}{M} Q(t) \Psi^\gamma (\sigma(t)) z(\sigma(t)) \leq 0, \nonumber
\]

(14)

has a positive solution \(z(t)\). Using Philos’s study of nonoscillatory solutions about the first-order delay differential equation [21], we have Equation (7) has a positive solution, which is a contradiction. The proof is complete. \(\Box\)

**Remark 1.** The eventually negative solution in the proof is similar, we don’t give the proof and the corresponding lemma. However, the application of symmetry can unify the proof. If we assume that \(\tilde{x}(t)\) is an eventually negative solution of Equation (1). Because \(\gamma\) is a quotient of odd positive integers, we get \((-1)^\gamma = -1\). Multiplying both sides of Equation (1) with \(-1\), we obtain \((r(t)\psi(\tilde{x}(t)) (\tilde{y}'(t))')' + \sum_{i=1}^{n} q_i(t) [-f(\tilde{x}(\sigma(t)))] = 0\), where \(\tilde{y}(t) = -\tilde{x}(t) + \sum_{i=1}^{m} p_i(t) [-\tilde{x}(\tau_i(t))]\). From (A4) and (A5), we get \(0 < \psi(-x) \leq M\) and \(-f(\tilde{x}) \geq k(-\tilde{x})^\gamma\). Set \(x(t) = -\tilde{x}(t)\), then the conclusion will go back to the eventually positive solution case.

**Corollary 1.** If \(\sigma'(t) \geq 0\) and

\[
\lim \inf_{t \to \infty} \int_{\sigma(t)}^{t} Q(s) \Psi^\gamma (\sigma(s)) ds > \frac{M}{ke}, \nonumber
\]

(15)

then all solutions of Equation (1) are oscillatory.
**Proof.** In view of Ladde et al. [22] (Theorem 2.1.1), if we have
\[
\liminf_{t \to \infty} \int_{\sigma(t)}^{t} \frac{1}{M} Q(s) \Psi'(\sigma(s)) ds > \frac{1}{c'},
\]
then we get Equation (7) is oscillatory. From Theorem 1, we get Equation (1) is oscillatory. The proof is complete. □

**Corollary 2.** If \( \lim_{t \to \infty} \int_{\sigma(t)}^{t} Q(s) \Psi'(\sigma(s)) ds \) does not exist, \( \sigma'(t) \geq 0 \) and
\[
\limsup_{t \to \infty} \int_{\sigma(t)}^{t} Q(s) \Psi'(\sigma(s)) ds > \frac{M}{k}
\]
hold, then Equation (1) is oscillatory.

**Proof.** The proof is similar to that Ladde et al. [22] (Theorem 2.1.3). So we ignore. The proof is complete. □

**Remark 2.** The conclusion of the Theorem 1 is based on the method of Mozza’s paper [18]. The advantage of this method is that the existing oscillation criterion of the first-order delay equation can be directly applied to Equation (7), and then the oscillation results of CSODE can be given. However, from page 440 to 461 in Book [1], we noted most of the conclusions of the first-order delay differential equation must be calculated the function \( \frac{k}{M} Q(t) \Psi'(\sigma(t)) \). That is to say, the calculation of this method is more complex.

Put \( \sigma'(t) > 0 \) in \((A_3)\), using the generalized Riccati transformations, we obtain the next Theorem.

**Theorem 2.** Suppose that \((A_1) - (A_3)\) and \( \sigma'(t) > 0 \) hold. If there exist a function \( \rho(t) \in C^1([t_0, \infty), (0, \infty)) \) and a constant \( \alpha \in (1, \infty) \), such that
\[
\limsup_{t \to \infty} \frac{1}{\rho(t)} \int_{t_0}^{t} (t-s) \left[ k \rho(s) Q(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\rho'(s))^{\gamma+1} r(\sigma(s))}{(\rho(s) \sigma'(s))^{\gamma}} \right] ds = \infty,
\]
then Equation (1) is oscillatory.

**Proof.** Suppose to contrary that there exists a nonoscillatory solution of Equation (1). Without loss of generality, we can assume that \( x(t) \) is an eventually positive solution of Equation (1). From Lemma 2 and Theorem 1, we also have (3) and (9) hold.

Define the generalized Riccati-type function \( \omega(t) \) by
\[
\omega(t) = \rho(t) \frac{r(t) \psi(x(t)) (y'(t))^\gamma}{y^\gamma(\sigma(t))}.
\]

Clearly \( \omega(t) > 0 \) for \( t \geq t_1 \). Differentiating \( \omega(t) \), we obtain
\[
\omega'(t) = \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \left[ \frac{r(t) \psi(x(t)) (y'(t))^\gamma}{y^\gamma(\sigma(t))} \right]' - \gamma \rho(t) \frac{r(t) \psi(x(t)) (y'(t))^\gamma y'(\sigma(t)) \sigma'(t)}{y^{\gamma+1}(\sigma(t))}.
\]

Since \( \left( r^\frac{1}{\gamma} (\sigma(t)) \psi^\frac{1}{\gamma} (x(\sigma(t))) y'(t) \right)' < 0 \) and \( \sigma(t) < t \), then there exist \( t_2 \geq t_1 \) such that when \( t > t_2 \),
\[
r^\frac{1}{\gamma} (\sigma(t)) \psi^\frac{1}{\gamma} (x(\sigma(t))) y'(\sigma(t)) > r^\frac{1}{\gamma} (t) \psi^\frac{1}{\gamma} (x(t)) y'(t).
\]
Then from \((A_1), (A_4), y'(t) > 0\) and \((9)\), for all \(t > t_2\), we have
\[
\omega'(t) = \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \left( \frac{r(t)\psi(x(t)) (y'(t))^{\gamma'}}{y^*(\sigma(t))} - \gamma \frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))} \omega(t) \right) \\
\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - k\rho(t)Q(t) - \frac{\gamma\sigma'(t)r^\frac{1}{\gamma} (\sigma(t))^{\frac{1}{\gamma}} (x(\sigma(t))) y'(\sigma(t))}{r^\frac{1}{\gamma} (\sigma(t))^{\frac{1}{\gamma}} (x(\sigma(t))) y(\sigma(t))} \omega(t) \\
\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - k\rho(t)Q(t) - \frac{\gamma\sigma'(t)r^\frac{1}{\gamma} (\sigma(t))^{\frac{1}{\gamma}} (x(\sigma(t))) y'(\sigma(t))}{(\rho(t)r(\sigma(t)))^{\frac{1}{\gamma}} (\sigma(t))^{\frac{1}{\gamma}}} \omega(t) \\
\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - k\rho(t)Q(t) - \frac{\gamma\sigma'(t)}{(M\rho(t)r(\sigma(t)))^{\frac{1}{\gamma}}} \omega^{\frac{1}{\gamma}}(t).
\]

(19)

Set
\[
b_1(t) = \frac{\rho'(t)}{\rho(t)}, \quad b_2(t) = \frac{\gamma\sigma'(t)}{(M\rho(t)r(\sigma(t)))^{\frac{1}{\gamma}}}, \quad v = \omega(t).
\]

By Lemma 1 and \((19)\), we obtain
\[
\omega'(t) \leq -k\rho(t)Q(t) + \frac{M}{(\gamma + 1)^{\gamma + 1}} \frac{\rho'(t)^{\gamma + 1} r(\sigma(t))}{(\rho(t)\sigma'(t))^{\gamma}}.
\]

Multiplying both sides of the above inequality with \((t - s)^{\alpha}\) for \(s \leq t\), then integrating it from \(t_2\) to \(t\), we get
\[
- \int_{t_2}^{t} (t - s)^{\alpha} \omega'(s) ds \geq \int_{t_2}^{t} (t - s)^{\alpha} \left[ k\rho(s)Q(s) - \frac{M}{(\gamma + 1)^{\gamma + 1}} \frac{(\rho'(s))^{\gamma + 1} r(\sigma(s))}{(\rho(s)\sigma'(s))^{\gamma}} \right] ds.
\]

Since
\[
\int_{t_2}^{t} (t - s)^{\alpha} \omega'(s) ds = \frac{\omega_2^\alpha}{\Gamma(\alpha + 1)} (t - s)^{\alpha - 1} \omega(s) ds - (t - t_2)^{\alpha} \omega(t_2).
\]

(20)

Thus, we have
\[
\left( \frac{t - t_2}{t} \right)^{\alpha} \omega(t_2) - \frac{\omega_2}{\Gamma(\alpha + 1)} \int_{t_2}^{t} (t - s)^{\alpha - 1} \omega(s) ds \\
\geq \frac{1}{\Gamma(\alpha + 1)} \int_{t_2}^{t} (t - s)^{\alpha} \left[ k\rho(s)Q(s) - \frac{M}{(\gamma + 1)^{\gamma + 1}} \frac{(\rho'(s))^{\gamma + 1} r(\sigma(s))}{(\rho(s)\sigma'(s))^{\gamma}} \right] ds,
\]

which implies that
\[
\frac{1}{\Gamma(\alpha + 1)} \int_{t_2}^{t} (t - s)^{\alpha} \left[ k\rho(s)Q(s) - \frac{M}{(\gamma + 1)^{\gamma + 1}} \frac{(\rho'(s))^{\gamma + 1} r(\sigma(s))}{(\rho(s)\sigma'(s))^{\gamma}} \right] ds \leq \left( \frac{t - t_2}{t} \right)^{\alpha} \omega(t_2).
\]

Since \(\limsup_{t \to \infty} \left( \frac{t - t_2}{t} \right)^{\alpha} \omega(t_2) = \omega(t_2)\), we have
\[
\limsup_{t \to \infty} \frac{1}{\Gamma(\alpha + 1)} \int_{t_2}^{t} (t - s)^{\alpha} \left[ k\rho(s)Q(s) - \frac{M}{(\gamma + 1)^{\gamma + 1}} \frac{(\rho'(s))^{\gamma + 1} r(\sigma(s))}{(\rho(s)\sigma'(s))^{\gamma}} \right] ds \leq \omega(t_2).
\]
which contradicts (17). The proof is complete. □

Put $\psi(x) > D > 0$ for all $x \in (-\infty, +\infty)$ hold, where $D$ is a positive constant, we obtain the next Theorem.

**Theorem 3.** Suppose that $(A_1) - (A_5)$ hold and $\psi(x) > D > 0$. If there exists a function $\delta(t) \in C^1([t_0, \infty), (0, \infty))$, such that

$$\limsup_{t \to \infty} \int_{t_0}^{t} k\delta(s)Q(s)E^\gamma(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} ds = \infty,$$  \hspace{1cm} (21)

where $E(s) = \exp \left( - \left( \frac{M}{D^{\gamma}} \right) \frac{1}{\gamma} \int_{\sigma(t)}^{s} \frac{1}{\Psi(u)r^\gamma(u)} du \right)$, then Equation (1) is oscillatory.

**Proof.** Suppose to contrary that there exists a nonoscillatory solution of Equation (1). Without loss of generality, we can assume that $x(t)$ is an eventually positive solution of Equation (1). From Lemma 2 and Theorem 1, we also have (3) and (13) hold.

Then, from (13) and $\psi(x) > D > 0$, we have

$$\frac{y'(t)}{y(t)} \leq \frac{M^\gamma}{D^\gamma \Psi(t)r^\gamma(t)}. \hspace{1cm} (22)$$

Integrating (22) from $\sigma(t)$ to $t$, we obtain

$$\frac{y(\sigma(t))}{y(t)} \geq \exp \left( - \left( \frac{M}{D^{\gamma}} \right) \frac{1}{\gamma} \int_{\sigma(t)}^{t} \frac{1}{\Psi(s)r^\gamma(s)} ds \right) =: E(t). \hspace{1cm} (23)$$

Next, define the generalized Riccati-type function

$$w(t) = \delta(t)^2 r(t)\Psi(x(t)) \frac{(y'(t))^\gamma}{y^\gamma(t)}. \hspace{1cm} (24)$$

Obviously, $w(t) > 0$. Differentiating $w(t)$, we obtain

$$w'(t) = \delta'(t)w(t) + \delta(t) \frac{r(t)\Psi(x(t)) (y'(t))^\gamma}{y^\gamma(t)}' - \gamma \delta(t)^2 \frac{r(t)\Psi(x(t)) (y'(t))^\gamma+1}{y^{\gamma+1}(t)}.$$

Using (9) and (23), we have

$$w'(t) = \frac{\delta'(t)}{\delta(t)}w(t) + \delta(t) \frac{r(t)\Psi(x(t)) (y'(t))^\gamma}{y^\gamma(t)}' - \gamma \frac{y'(t)}{y(t)}w(t) \leq \frac{\delta'(t)}{\delta(t)}w(t) - k\delta(t)Q(t) \frac{y^\gamma(x(t))}{y^\gamma(t)} - \gamma \frac{\delta(t)^2}{(\delta(t)r(t)\Psi(x(t)))^\gamma} w^{\gamma+1}(t) \leq \frac{\delta'(t)}{\delta(t)}w(t) - k\delta(t)Q(t)E^\gamma(t) - \gamma \frac{\gamma}{(M\delta(t)r(t))^\gamma} w^{\gamma+1}(t). \hspace{1cm} (25)$$
Set
\[ b_1(t) = \frac{\delta'(t)}{\delta(t)}, \quad b_2(t) = \frac{\gamma}{(M\delta(t)r(t))^\gamma}, \quad \nu = w(t). \]

By Lemma 1 and (25), we have
\[ w'(t) \leq -k\rho(t)Q(t)E^{\gamma}(t) + \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(t))^{\gamma+1} r(t)}{(\delta(t))^\gamma}. \tag{26} \]

Integrating (26) from \( t_1 \) to \( t \), we have
\[ w(t) \leq w(t_1) - \int_{t_1}^{t} \left[ k\rho(s)Q(s)E^{\gamma}(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} \right] ds. \]

From (21), we get \( w(t) < 0 \), which contradicts \( w(t) > 0 \). The proof is complete. \( \square \)

The following new Kamenev-type oscillation criteria are obtained by using a similar method in Theorem 2.

**Theorem 4.** Suppose that \((A_1) - (A_5)\) hold and \( \psi(x) > D > 0 \). If there exists a function \( \delta(t) \in C^1([t_0, \infty), (0, \infty)) \), and a constant \( \alpha \in (1, \infty) \), such that
\[ \limsup_{t \to \infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^{\alpha} \left[ k\rho(s)Q(s)E^{\gamma}(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} \right] ds = \infty, \tag{27} \]

where \( E(s) = \exp \left( -\left( \frac{M}{\nu} \right)^{\frac{1}{\gamma}} \int_{\nu(s)}^{s} \frac{1}{\psi(u)} du \right) \), then Equation (1) is oscillatory.

**Proof.** From Theorem 3, we also have (26) hold. Multiplying both sides of the inequality (26) with \((t-s)^\alpha\) for \( s \leq t \), then integrating this inequality from \( t_1 \) to \( t \), we get
\[ -\int_{t_1}^{t} (t-s)^\alpha w'(s)ds \geq \int_{t_1}^{t} (t-s)^\alpha \left[ k\rho(s)Q(s)E^{\gamma}(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} \right] ds. \]

Thus, we have
\[ \left( \frac{t-t_1}{t} \right)^\alpha w(t_1) - \frac{\alpha}{t^\alpha} \int_{t_1}^{t} (t-s)^{\alpha-1} w(s)ds \geq \int_{t_1}^{t} (t-s)^\alpha \left[ k\rho(s)Q(s)E^{\gamma}(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} \right] ds. \]

That is means
\[ \frac{1}{t^\alpha} \int_{t_1}^{t} (t-s)^\alpha \left[ k\rho(s)Q(s)E^{\gamma}(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} \right] ds \leq \left( \frac{t-t_1}{t} \right)^\alpha w(t_1). \]

That is to say
\[ \limsup_{t \to \infty} \frac{1}{t^\alpha} \int_{t_1}^{t} (t-s)^\alpha \left[ k\rho(s)Q(s)E^{\gamma}(s) - \frac{M}{(\gamma + 1)^{\gamma+1}} \frac{(\delta'(s))^{\gamma+1} r(s)}{(\delta(s))^\gamma} \right] ds < w(t_1), \]

which contradicts (27). The proof is complete. \( \square \)
4. Example

Consider the following differential equation for \( t \in (0, \infty) \),

\[
\left( \frac{1}{r} \left( \frac{\arctan(x(t))}{\pi} + \frac{1}{2} \right) \left( x(t) + \sum_{j=1}^{\infty} \frac{1}{2 + j} x(t + j) \right) \right)' + \sum_{i=1}^{2} (21 + 6i) t x(t + 1) = 0. \tag{28}
\]

Let \( \gamma = 1 \), \( r(t) = \frac{1}{t} \), \( q(t) = \frac{\arctan x}{\pi} + \frac{1}{2} \), \( f(x) = x \), \( m = 3 \), \( p_i(t) = \frac{1}{2 + i} \), \( \sigma_j(t) = \frac{1}{2 + j} \), \( n = 2 \), \( q_i(t) = (21 + 6i) t \) and \( \sigma_i(t) = \frac{i}{i + 1} \) in Equation (1).

Then, we have \( k = 1 \), \( 0 < \psi(x) < 1 \), \( \sigma(t) = \frac{1}{3} \leq \min\{\sigma_i(t)\} \), \( P(t) = \frac{13}{50} \), \( Q(t) = 13t \), and \( \Psi(\sigma(t)) = \Psi \left( \frac{1}{3} \right) = \frac{13}{157464} t^6 - \frac{6512}{5832} t^4 + 1 + \frac{1}{18} t^2 - \left( \frac{169}{216} t^6 + \frac{1}{2} t^2 \right) \).

\[
\liminf_{t \to \infty} \int_{\sigma(t)}^{t} Q(s) \Psi'(\sigma(s)) \, ds
= \liminf_{t \to \infty} \int_{\frac{1}{3}}^{t} 13 s \left[ \frac{13}{157464} s^6 - \frac{6512}{5832} s^4 + 1 + \frac{1}{18} s^2 - \left( \frac{169}{216} t^6 + \frac{1}{2} t^2 \right) \right] \, ds
= \liminf_{t \to \infty} \int_{\frac{1}{3}}^{t} \frac{168}{157464} s^7 - \frac{84512}{5832} s^5 + \frac{13 + 13t^4}{18} s^3 - \left( \frac{2179}{216} t^6 + \frac{13}{2} t^2 \right) s \, ds
= +\infty > \frac{1}{e},
\]

then applying the Corollary 1, obviously, Equation (28) is oscillatory.

5. Conclusions

In this paper, by using a well-known first-order delay differential equation and generalized Riccati transformations, we get some new results. By the chain rule, Theorem 1 neglected the condition \( \sigma'(t) > 0 \) which the most articles required. In addition, one of its advantages is that the existing oscillation criterion of the first-order delay equation can be rewritten into the oscillation results of Equation (1) by Theorem 1. While Theorem 2 is more universal, that is to say, function \( \sigma'(t) > 0 \) is needed to establish new Kamenev-type oscillation criteria. Finally, we will try to get some new oscillation criteria for \( \gamma \), which is no longer just a quotient of odd positive integers of Equation (1), in the future work.

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