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Legendre-Gould Hopper-Based Sheffer Polynomials and Operational Methods

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Abstract: A remarkably large of number of polynomials have been presented and studied. Among several important polynomials, Legendre polynomials, Gould-Hopper polynomials, and Sheffer polynomials have been intensively investigated. In this paper, we aim to incorporate the above-referred three polynomials to introduce the Legendre-Gould Hopper-based Sheffer polynomials by modifying the classical generating function of the Sheffer polynomials. In addition, we investigate diverse properties and formulas for these newly introduced polynomials.

Keywords: Gould-Hopper polynomials; Legendre polynomials; Sheffer sequences with Appell sequences and associated sequences; Legendre-Gould Hopper-based Sheffer polynomials; Quasi-monomials; operational techniques

MSC: 11B83; 33C45; 33C99; 33E20

1. Introduction and Preliminaries

Various special polynomials have found diverse and vital applications in a number of fields such as mathematics, applied mathematics, mathematical physics, and engineering. According to the necessity of solving certain specific problems in diverse fields or pure mathematical interests, recently, a remarkably large number of new polynomials and numbers as well as a variety of generalizations (or extensions) and variants of some known polynomials have been established and investigated. In this regard, the Gould-Hopper polynomials incorporated with the Legendre polynomials were extended to be named as the Legendre-Gould Hopper polynomials by using operational methods. Also, certain Sheffer polynomials based on some known polynomials have been presented and investigated (see, e.g., [1–4]). In this paper, we aim to introduce the Legendre-Gould Hopper-based Sheffer polynomials by modifying the classical generating function (2) of the Sheffer polynomials. Since these newly introduced polynomials are found to be Sheffer type polynomials, we show that several properties and identities of the Sheffer polynomials are employed to give the corresponding results to these new polynomials. Moreover, other properties and formulas for these new polynomials such as quasi-monomials, other operational and certain integral representations are presented. We conclude that this method is pointed out to be easily applicable to other known polynomials, which are quasi-monomials with respect to some multiplicative and derivative operators, to yield certain Sheffer polynomials based on the known polynomials. In addition to this conclusion, the Legendre-Gould Hopper-based Sheffer polynomials generated by other generating functions with

investigating their properties and formulas are poised as problems which are left to the interested researcher and the authors for future investigation.

For our purpose, some notations with modified ones and certain known facts are recalled and introduced. In what follows, let \mathcal{P} denote the algebra of formal power series in the variable t over the field \mathbb{C} of characteristic zero, say, \mathbb{C} the field of complex numbers. If $f(t)$ and $g(t)$ in \mathcal{P} satisfy

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \quad (a_0 = 0, a_1 \neq 0) \quad \text{and} \quad g(t) = \sum_{k=0}^{\infty} b_k t^k \quad (b_0 \neq 0),$$

then $f(t)$ and $g(t)$ are said to be a delta series and an invertible series, respectively. With each pair of a delta series $f(t)$ and an invertible series $g(t)$, there exists a unique sequence $s_n(x)$ of polynomials gratifying the orthogonality conditions (see [5], p. 17, Theorem 2.3.1)

$$\langle g(t) f(t)^k | s_n(x) \rangle = n! \delta_{n,k} \quad (n, k \in \mathbb{N}_0), \tag{1}$$

where $\delta_{n,k}$ is the Kronecker delta function defined by $\delta_{n,k} = 1$ ($n = k$) and $\delta_{n,k} = 0$ ($n \neq k$). The operator $\langle \cdot | \cdot \rangle$ remains the same as in (see [5], Chapter 2). Here and elsewhere, let \mathbb{N} be the set of positive integers and, also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The sequence $s_n(x)$ satisfying (1) is called the Sheffer sequence for $(g(t), f(t))$, or $s_n(x)$ is Sheffer for $(g(t), f(t))$, which is usually denoted as $s_n(x) \sim (g(t), f(t))$. In particular, if $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$, or $s_n(x)$ is associated with $f(t)$; if $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence for $g(t)$, or $s_n(x)$ is Appell for $g(t)$ (see ([5], p. 17); see also [6,7]). Among diverse characterizations of Sheffer sequences, we recall the generating function (see, e.g., ([5], Theorem 2.3.4)): The sequence ${}_{[g,f]}s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x \bar{f}(t)} = \sum_{k=0}^{\infty} \frac{{}_{[g,f]}s_k(x)}{k!} t^k \tag{2}$$

for all x in \mathbb{C} , where $\bar{f}(t) = f^{-1}(t)$ is the compositional inverse of $f(t)$. Here and elsewhere, the ordered-pair notation $[g, f]$ implies that the first element g is an invertible series and the second one f is a delta series.

A sequence $u_n(x)$ ($\deg u_n(x) = n$) of polynomials is called Sheffer A -type zero if it has a generating function of the form (see, e.g., ([8], p. 222, Theorem 72); see also ([5], p. 19)).

$$A(t) \exp(xH(t)) = \sum_{n=0}^{\infty} u_n(x) t^n, \tag{3}$$

where $A(t)$ and $H(t)$ are an invertible series and a delta series, respectively. Thus, $s_n(x)$ is a Sheffer sequence if and only if $s_n(x)/n!$ is a sequence of Sheffer A -type zero. It is noted that sequences of Sheffer A -type zero were suggested to be named as *poweroids* by Steffensen ([9], p. 335), from which the concept of monomiality arose (see also ([5], p. 19)). The quasi-monomial treatment of special polynomials has been proved to be a powerful tool for the investigation of the properties of a wide class of polynomials such as Sheffer polynomials (see, e.g., [9–12]). A sequence of polynomials $p_n(x)$ ($n \in \mathbb{N}_0, x \in \mathbb{C}$) of degree n is said to be quasi-monomials with respect to operators \hat{M} and \hat{P} (called, respectively, multiplicative and derivative operators) acting on polynomials if they do exist and satisfy

$$\hat{M} \{p_n(x)\} = p_{n+1}(x) \quad \text{and} \quad \hat{P} \{p_n(x)\} = n p_{n-1}(x). \tag{4}$$

It is easily found from (4) that

$$\hat{M} \hat{P} \{p_n(x)\} = n p_n(x) \tag{5}$$

and

$$\hat{P} \hat{M} \{p_n(x)\} = (n+1) p_n(x). \quad (6)$$

The algebra of \hat{P} , \hat{M} , the identity operator $\hat{1}$ and the zero operator $\hat{0}$ are seen to satisfy the commutation relations

$$[\hat{P}, \hat{M}] := \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}, \quad [\hat{P}, \hat{1}] = [\hat{M}, \hat{1}] = \hat{0}. \quad (7)$$

The algebra (7) is called the Heisenberg-Weyl algebra. The polynomials $p_n(x)$ are derived via the action of \hat{M}^n on $p_0(x)$:

$$p_n(x) = \hat{M}^n \{p_0(x)\}. \quad (8)$$

For $p_0(x) = 1$, $p_n(x) = \hat{M}^n \{1\}$ and the exponential generating function of $p_n(x)$ is

$$\mathcal{F}(x, t) = e^{t\hat{M}} \{1\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{M}^n \{1\} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}. \quad (9)$$

The simplest ones of (4) are the multiplication $\hat{M} = X$ and derivative $\hat{P} = D = \frac{d}{dx}$ operators acting on the space of polynomials. They act on monomials as follows:

$$Xx^n = x^{n+1} \quad \text{and} \quad Dx^n = nx^{n-1}, \quad (10)$$

which lead to $[D, X] = 1$ and, in view of linearity, are utilized to act on polynomials and formal power series.

For more details and applications of operational methods and quasi-monomials, one may be referred, for example, to [1–3,5,9–21].

We recall the Gould-Hopper polynomials (see ([22], Equation (6.2))) (also called sometimes as higher-order Hermite or Kampé de Fériet polynomials) $H_n^{(s)}(x, y)$ defined by

$$H_n^{(s)}(x, y) = n! \sum_{k=0}^{\lfloor \frac{n}{s} \rfloor} \frac{y^k x^{n-sk}}{k! (n-sk)!}. \quad (11)$$

A generating function of $H_n^{(s)}(x, y)$ is given by (see, e.g., ([22], Equation (6.3)))

$$\exp(xt + yt^s) = \sum_{n=0}^{\infty} H_n^{(s)}(x, y) \frac{t^n}{n!}. \quad (12)$$

They appear as the solution of the generalized heat equation (see, e.g., [13]; see also ([23], p. 96))

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial^s}{\partial x^s} f(x, y), \quad f(x, 0) = x^n. \quad (13)$$

Under the operational formalism (or a formal solution of (13) when it is considered to be a first-order differential equation with respect to the variable y with the initial condition), they are defined as

$$H_n^{(s)}(x, y) = \exp\left(y \frac{\partial^s}{\partial x^s}\right) \{x^n\}. \quad (14)$$

These polynomials are quasi-monomial under the action of the operators (see [13])

$$\begin{aligned} \hat{M}_H &:= x + sy \frac{\partial^{s-1}}{\partial x^{s-1}}, \\ \hat{P}_H &:= \frac{\partial}{\partial x}. \end{aligned} \quad (15)$$

We recall the Legendre polynomials $S_n(x, y)$ and $\frac{R_n(x, y)}{n!}$ which are defined by the generating functions (see, e.g., [24]):

$$\exp(yt) C_0(-xt^2) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!} \quad (16)$$

and

$$C_0(xt) C_0(-yt) = \sum_{n=0}^{\infty} \frac{R_n(x, y)}{n!} \frac{t^n}{n!}, \quad (17)$$

respectively, where $C_0(x)$ denotes the 0th-order Bessel-Tricomi function. The n th order Bessel-Tricomi function $C_n(x)$ is defined by the following series (see, e.g., ([11], p. 150)):

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! (n+k)!} \quad (n \in \mathbb{N}_0), \quad (18)$$

where $J_n(x)$ is the ordinary cylindrical Bessel function of the first kind (see, e.g., [23]). The operational definition of the 0th-order Bessel-Tricomi function $C_0(x)$ is given by (see, e.g., ([25], p. 86))

$$C_0(\alpha x) = \exp\left(-\alpha D_x^{-1}\right) \{1\}, \quad (19)$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and is defined by means of the Riemann-Liouville fractional integral (see, e.g., ([26], p. 69))

$$D_x^{-\alpha} \{f\} := \frac{1}{\Gamma(\alpha)} \int_0^x (x-\eta)^{\alpha-1} f(\eta) d\eta \quad (x > 0; \Re(\alpha) > 0). \quad (20)$$

It is easy to see that

$$\left(D_x^{-1}\right)^n \{1\} := D_x^{-n} \{1\} = \frac{x^n}{n!} \quad (n \in \mathbb{N}). \quad (21)$$

Yasmin [25] introduced and studied the Legendre-Gould Hopper polynomials (LeGHP) ${}_s H_n^{(r)}(x, y, z)$ and $\frac{{}_R H_n^{(r)}(x, y, z)}{n!}$, which are defined, respectively, by the following generating functions:

$$C_0(-xt^2) \exp(yt + zt^r) = \sum_{n=0}^{\infty} {}_s H_n^{(r)}(x, y, z) \frac{t^n}{n!} \quad (22)$$

and

$$C_0(xt) C_0(-yt) \exp(zt^r) = \sum_{n=0}^{\infty} \frac{{}_R H_n^{(r)}(x, y, z)}{n!} \frac{t^n}{n!}. \quad (23)$$

The polynomials ${}_s H_n^{(r)}(x, y, z)$ and $\frac{{}_R H_n^{(r)}(x, y, z)}{n!}$ are quasi-monomials under the action of the multiplicative and derivative operators (see ([25], Equations (2.18a,b) and (2.19a,b)))

$$\begin{cases} \hat{M}_{SH} := y + 2D_x^{-1} \frac{\partial}{\partial y} + rz \frac{\partial^{r-1}}{\partial y^{r-1}}, \\ \hat{P}_{SH} = \frac{\partial}{\partial y} \end{cases} \quad (24)$$

and

$$\begin{cases} \hat{M}_{RH} := -D_x^{-1} + D_y^{-1} + rz \frac{\partial^{r-1}}{\partial y^{r-1}}, \\ \hat{P}_{RH} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}, \end{cases} \quad (25)$$

respectively.

2. Legendre-Gould Hopper Based Sheffer Polynomials

In this section, Legendre-Gould Hopper-based Sheffer polynomials are introduced, and their quasi-monomial properties and differential equations are established.

Definition 1. The Legendre-Gould Hopper-based Sheffer polynomials ${}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]}$ for the functions g and f are defined by the following generating function

$$\begin{aligned} \mathcal{F}_{[g,f]}(x, y, z : t) &:= \frac{1}{g(f^{-1}(t))} C_0 \left(-x(f^{-1}(t))^2 \right) \exp \left(yf^{-1}(t) + z(f^{-1}(t))^r \right) \\ &= \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \frac{t^n}{n!}, \end{aligned} \tag{26}$$

where $f(t)$ and $g(t)$ are a delta series and an invertible series, respectively.

Remark 1. The Sheffer-type polynomials in Definition 1 are well-defined in the following sense: We find from (24) that

$$\mathcal{F}_{[g,f]}(x, y, z : t) = \frac{1}{g(f^{-1}(t))} \exp \left(\hat{M}_{SH}f^{-1}(t) \right) = \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \frac{t^n}{n!}. \tag{27}$$

Furthermore, in view of (2), we can write

$$\mathcal{F}_{[g,f]}(x, y, z : t) = \frac{1}{g(f^{-1}(t))} \exp \left(\hat{M}_{SH}f^{-1}(t) \right) = \sum_{n=0}^{\infty} {}_{[g,f]}s_n(\hat{M}_{SH}) \frac{t^n}{n!}. \tag{28}$$

Equating the last series in (27) and (28) and comparing the coefficients of t^n on both sides of the resulting identity gives

$${}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} = {}_{[g,f]}s_n(\hat{M}_{SH}) = {}_{[g,f]}s_n \left(y + 2D_x^{-1} \frac{\partial}{\partial y} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} \right). \tag{29}$$

Also, in view of (22), we have

$$\sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))} \sum_{n=0}^{\infty} {}_sH_n^{(r)}(x, y, z) \frac{(f^{-1}(t))^n}{n!}. \tag{30}$$

Definition 2. The Legendre-Gould Hopper-based Sheffer polynomials $\frac{{}_RLeGH^{(r)}s_n(x, y, z)_{[g,f]}}{n!}$ for the functions g and f are defined by the following generating function

$$\begin{aligned} \mathcal{H}_{[g,f]}(x, y, z : t) &:= \frac{1}{g(f^{-1}(t))} C_0 \left(xf^{-1}(t) \right) C_0 \left(-yf^{-1}(t) \right) \exp \left(z(f^{-1}(t))^r \right) \\ &= \sum_{n=0}^{\infty} \frac{{}_RLeGH^{(r)}s_n(x, y, z)_{[g,f]}}{(n!)^2} t^n, \end{aligned} \tag{31}$$

where $f(t)$ and $g(t)$ are a delta series and an invertible series, respectively.

Remark 2. As in Remark 1, the Sheffer-type polynomials in Definition 2 are also well-defined in the following sense: Indeed, it follows from (25) that

$$\mathcal{H}_{[g,f]}(x, y, z : t) = \frac{1}{g(f^{-1}(t))} \exp \left(\hat{M}_{RH}f^{-1}(t) \right). \tag{32}$$

Moreover, in terms of (2), we get

$$\mathcal{H}_{[g,f]}(x, y, z : t) = \frac{1}{g(f^{-1}(t))} \exp(\hat{M}_{RH} f^{-1}(t)) = \sum_{n=0}^{\infty} {}_{[g,f]}s_n(\hat{M}_{RH}) \frac{t^n}{n!}. \tag{33}$$

Identifying the last series in (31) and (33) and equating the coefficients of t^n on both sides of the resultant identity yields

$$\frac{{}_RLeGH^{(r)}s_n(x, y, z)_{[g,f]}}{n!} = {}_{[g,f]}s_n(\hat{M}_{RH}) = {}_{[g,f]}s_n\left(-D_x^{-1} + D_y^{-1} + rz \frac{\partial^{r-1}}{\partial y^{r-1}}\right). \tag{34}$$

Also, in view of (23), we find

$$\sum_{n=0}^{\infty} {}_RLeGH^{(r)}s_n(x, y, z)_{[g,f]} \frac{t^n}{(n!)^2} = \frac{1}{g(f^{-1}(t))} \sum_{n=0}^{\infty} \frac{{}_RH_n^{(r)}(x, y, z)}{n!} \frac{(f^{-1}(t))^n}{n!}. \tag{35}$$

The next two theorems show that the two polynomials introduced in Definitions 1 and 2 are quasi monomials for some derived multiplicative and derivative operators.

Theorem 1. The Legendre-Gould Hopper-based Sheffer polynomials ${}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]}$ for the functions g and f are quasi-monomial under the action of the following multiplicative and derivative operators:

$${}_s\hat{M}_{LeGH_{[g,f]}} := \left(y + 2D_x^{-1} \frac{\partial}{\partial y} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right) \frac{1}{f'(\partial_y)} \tag{36}$$

and

$${}_s\hat{P}_{LeGH_{[g,f]}} := f(\partial_y), \tag{37}$$

respectively. Here and elsewhere $\partial_y := \frac{\partial}{\partial y}$.

Proof. We find

$$\partial_y \left\{ \exp\left(yf^{-1}(t) + z(f^{-1}(t))^r\right) \right\} = f^{-1}(t) \exp\left(yf^{-1}(t) + z(f^{-1}(t))^r\right). \tag{38}$$

Since f^{-1} denotes the compositional inverse of the function f and $f(t)$ has a series expansion in t , we have

$$\begin{aligned} f(\partial_y) \left\{ \exp\left(yf^{-1}(t) + z(f^{-1}(t))^r\right) \right\} &= f\left(f^{-1}(t)\right) \exp\left(yf^{-1}(t) + z(f^{-1}(t))^r\right) \\ &= t \exp\left(yf^{-1}(t) + z(f^{-1}(t))^r\right). \end{aligned} \tag{39}$$

Differentiating both sides of the second equality in (27) with respect to t affords

$$\begin{aligned} \left({}_s\hat{M}_{SH} - \frac{g'(f^{-1}(t))}{g(f^{-1}(t))} \right) \frac{1}{f'(f^{-1}(t))} \mathcal{F}_{[g,f]}(x, y, z : t) \\ = \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_{n+1}(x, y, z)_{[g,f]} \frac{t^n}{n!}. \end{aligned} \tag{40}$$

Since $f(t)$ is a delta series in t , $f'(t)$ becomes an invertible series in t . Therefore $\frac{1}{f'(f^{-1}(t))}$ can be a series expansion of $f^{-1}(t)$. Since $g(t)$ is an invertible series in t , $\frac{1}{g(f^{-1}(t))}$ can be a series expansion of $f^{-1}(t)$. It is found from (26) that $\mathcal{F}_{[g,f]}(x, y, z : t)$ is a product of $\frac{1}{g(f^{-1}(t))}$ and another function which

can be expanded in powers of $f^{-1}(t)$. So we observe that $\mathcal{F}_{[g,f]}(x, y, z : t)$ can be a series expansion in powers of $f^{-1}(t)$.

Considering this observation and using (38), it follows from (40) that

$$\begin{aligned} \left({}_s\hat{M}_{SH} - \frac{g'(\partial_y)}{g(\partial_y)} \right) \frac{1}{f'(\partial_y)} \mathcal{F}_{[g,f]}(x, y, z : t) \\ = \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_{n+1}(x, y, z)_{[g,f]} \frac{t^n}{n!}. \end{aligned} \tag{41}$$

Using (26) and (36) in (41), we obtain

$${}_s\hat{M}_{LeGH}_{[g,f]} \left\{ {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \right\} = {}_sLeGH^{(r)}s_{n+1}(x, y, z)_{[g,f]} \quad (n \in \mathbb{N}_0). \tag{42}$$

Thus, Equation (42) satisfies the first identity of (4) for quasi monomials.

Using (39), we get

$$f(\partial_y) \left\{ \mathcal{F}_{[g,f]}(x, y, z : t) \right\} = t \mathcal{F}_{[g,f]}(x, y, z : t). \tag{43}$$

Employing the series definition of (26) in Equation (43) provides

$$\sum_{n=0}^{\infty} f(\partial_y) \left\{ {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \right\} \frac{t^n}{n!} = \sum_{n=1}^{\infty} {}_sLeGH^{(r)}s_{n-1}(x, y, z)_{[g,f]} \frac{t^n}{(n-1)!},$$

both sides of which, upon equating the coefficients of t^n , gives

$$f(\partial_y) \left\{ {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \right\} = n {}_sLeGH^{(r)}s_{n-1}(x, y, z)_{[g,f]} \quad (n \in \mathbb{N}_0). \tag{44}$$

Therefore Equation (44) fulfills the second identity of (4) for quasi monomials. \square

Theorem 2. The Legendre-Gould Hopper-based Sheffer polynomials $\frac{{}_RLeGH^{(r)}s_n(x, y, z)_{[g,f]}}{n!}$ for the functions g and f are quasi-monomials under the action of the following multiplicative and derivative operators

$${}_R\hat{M}_{LeGH} := \left(-D_x^{-1} + D_y^{-1} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)} \right) \frac{1}{f'(\partial_y)} \tag{45}$$

and

$${}_R\hat{P}_{LeGH} := f \left(\frac{-\partial}{\partial x} x \frac{\partial}{\partial y} \right), \tag{46}$$

respectively. Here and elsewhere $\partial_x := \frac{\partial}{\partial x}$.

Proof. The proof would run parallel with that of Theorem 1. The involved details are omitted. \square

3. Results Derivable from the Previous Section

We begin with the following remark which depicts some simple properties about invertible and delta series.

Remark 3. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are

$$f(t) = t + \sum_{k=2}^{\infty} a_k t^k \tag{47}$$

and

$$g(t) = 1 + \sum_{k=1}^{\infty} b_k t^k, \quad (48)$$

where $\{a_k\}_{k \geq 2}$ and $\{b_k\}_{k \in \mathbb{N}}$ are sequences of constants in \mathbb{C} . Then the following properties are satisfied:

- (i) $f(0) = 0$, $f'(0) = 1$ and $g(0) = 1$;
- (ii) The compositional inverse $f^{-1}(t)$ of $f(t)$ exists (for any delta series) and is of the same form as in (47);
- (iii) The function $\frac{1}{g(t)}$ is defined (for any invertible series) and is of the same form as in (48).

The following theorems contain some properties which are easily derivable from those quasi monomials of those series given in Definitions 1 and 2.

Theorem 3. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are the same as in (47) and (48). Also let $n \in \mathbb{N}_0$. Then we find

- (a) ${}_s\text{LeGH}^{(r)}s_0(x, y, z)_{[g,f]} = 1$ and so

$${}_s\hat{M}_{\text{LeGH}_{[g,f]}}^n \{1\} = {}_s\text{LeGH}^{(r)}s_n(x, y, z)_{[g,f]};$$

- (b) A generating function is

$$\exp\left({}_s\hat{M}_{\text{LeGH}_{[g,f]}} t\right) \{1\} = \sum_{k=0}^{\infty} {}_s\text{LeGH}^{(r)}s_k(x, y, z)_{[g,f]} \frac{t^k}{k!};$$

- (c) A differential equation is

$$\begin{aligned} \left(y + 2D_x^{-1} \frac{\partial}{\partial y} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right) \frac{f(\partial_y)}{f'(\partial_y)} {}_s\text{LeGH}^{(r)}s_n(x, y, z)_{[g,f]} \\ = n {}_s\text{LeGH}^{(r)}s_n(x, y, z)_{[g,f]}. \end{aligned}$$

Proof. The assertions here follow immediately from the results in Theorem 1 by using (5), (8), and (9). \square

Theorem 4. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are the same as in (47) and (48). Also let $n \in \mathbb{N}_0$. Then we find

- (a) ${}_R\text{LeGH}^{(r)}s_0(x, y, z)_{[g,f]} = 1$ and so

$${}_R\hat{M}_{\text{LeGH}_{[g,f]}}^n \{1\} = \frac{{}_R\text{LeGH}^{(r)}s_n(x, y, z)_{[g,f]}}{n!};$$

- (b) A generating function is

$$\exp\left({}_R\hat{M}_{\text{LeGH}_{[g,f]}} t\right) \{1\} = \sum_{k=0}^{\infty} \frac{{}_R\text{LeGH}^{(r)}s_k(x, y, z)_{[g,f]}}{k!} \frac{t^k}{k!};$$

- (c) A differential equation is

$$\begin{aligned} \left(-D_x^{-1} + D_y^{-1} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right) \frac{f\left(\frac{-\partial}{\partial x} x \frac{\partial}{\partial y}\right)}{f'(\partial_y)} {}_R\text{LeGH}^{(r)}s_n(x, y, z)_{[g,f]} \\ = n {}_R\text{LeGH}^{(r)}s_n(x, y, z)_{[g,f]}. \end{aligned}$$

Proof. The assertions here follow immediately from the results in Theorem 2 by using (5), (8), and (9). □

For further assertions in the next sections, here is a suitable position to give some observations in the following remark.

Remark 4.

(a) The operator $\langle \cdot | \cdot \rangle$ in (1) can be expressed as follows:

$$\langle g(t) f(t)^k | s_n(x) \rangle = g(\partial_x) f(\partial_x)^k \{s_n(x)\} \quad (n, k \in \mathbb{N}_0).$$

More generally, for $f(t) \in \mathcal{P}$ and $p(x)$ being a series in the variable x , we find

$$\langle f(t) | p(x) \rangle = f(\partial_x) \{p(x)\}.$$

(b) In view of ([5] p. 20, Theorem 2.3.7), the identity (44) implies that the sequence ${}_sLeGH^{(r)}s_{n-1}(x, y, z)_{[g,f]}$ is Sheffer for $(g(t), f(t))$, when the sequence is looked upon as a function of the variable y .

4. Sheffer Sequences

In this section and in the sequel, unless otherwise stated, we consider the sequence ${}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]}$ as a function of the variable y and its associated sequence is

$${}_sLeGH^{(r)}s_n(x, y, z)_{[1,f]} \quad (n \in \mathbb{N}_0). \tag{49}$$

Then, in view of Remark 4, the assertions of the theorems in this section can be verified by the corresponding theorems in ([5], pp. 17–24). In this regard, proofs are omitted.

Theorem 5. The following identity holds.

$$g(\partial_y) f(\partial_y)^k \left\{ {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \right\} = n! \delta_{n,k} \quad (n, k \in \mathbb{N}_0). \tag{50}$$

Theorem 6. For any $h(t) \in \mathcal{P}$,

$$h(t) = \sum_{k=0}^{\infty} h(\partial_y) \left\{ {}_sLeGH^{(r)}s_k(x, y, z)_{[g,f]} \right\} \frac{g(t) f(t)^k}{k!}. \tag{51}$$

Theorem 7. For any polynomial $p(y)$,

$$p(y) = \sum_{k \geq 0} g(\partial_y) f(\partial_y)^k \{p(y)\} \frac{{}_sLeGH^{(r)}s_k(x, y, z)_{[g,f]}}{k!}. \tag{52}$$

Theorem 8. The following expansion holds.

$${}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} = \sum_{k=0}^n \left(g \left(f^{-1}(\partial_y) \right)^{-1} \left(f^{-1}(\partial_y) \right)^k \right) \{y^n\} \frac{y^k}{k!}. \tag{53}$$

The Equation (53) follows from the modified Equation (2.3.4) in ([5], p. 19) which should be corrected with replacing $f(t)^k$ by $(f^{-1}(t))^k$.

Theorem 9. For any $h(t) \in \mathcal{P}$,

$$\begin{aligned} & h(\partial_y) \left\{ {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \right\} \\ &= \sum_{k=0}^n \binom{n}{k} h(\partial_y) \left\{ {}_sLeGH^{(r)}s_k(x, y, z)_{[g,f]} \right\} {}_sLeGH^{(r)}s_{n-k}(x, y, z)_{[1,f]}. \end{aligned} \tag{54}$$

Theorem 10. The Sheffer identity is

$$\begin{aligned} & {}_sLeGH^{(r)}s_n(x, y + u, z)_{[g,f]} \\ &= \sum_{k=0}^n \binom{n}{k} {}_sLeGH^{(r)}s_k(x, u, z)_{[1,f]} {}_sLeGH^{(r)}s_{n-k}(x, y, z)_{[g,f]} \end{aligned} \tag{55}$$

for all $u \in \mathbb{C}$.

Theorem 11. For any $h(t)$ and $l(t)$ in \mathcal{P} ,

$$\begin{aligned} & (h(\partial_y) l(\partial_y)) \left\{ {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \right\} \\ &= \sum_{k=0}^n \binom{n}{k} \left(h(\partial_y) \left\{ {}_sLeGH^{(r)}s_k(x, y, z)_{[g,f]} \right\} \right) \left(l(\partial_y) \left\{ {}_sLeGH^{(r)}s_{n-k}(x, y, z)_{[1,f]} \right\} \right). \end{aligned} \tag{56}$$

5. Associated

For the sequence ${}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]}$ as a function of the variable y and its associated sequence is

$${}_sLeGH^{(r)}s_n(x, y, z)_{[1,f]} \quad (n \in \mathbb{N}_0). \tag{57}$$

Then, in view of Remark 4, the assertions of the theorems in this section can be validated by the corresponding theorems in ([5], pp. 25–26). In this regard, proofs are omitted.

Theorem 12. The generating function is

$$C_0 \left(-x(f^{-1}(t))^2 \right) \exp \left(yf^{-1}(t) + z(f^{-1}(t))^r \right) = \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_n(x, y, z)_{[1,f]} \frac{t^n}{n!}. \tag{58}$$

Theorem 13. For any $h(t) \in \mathcal{P}$,

$$h(t) = \sum_{k=0}^{\infty} h(\partial_y) \left\{ {}_sLeGH^{(r)}s_k(x, y, z)_{[1,f]} \right\} \frac{f(t)^k}{k!}. \tag{59}$$

Theorem 14. For any polynomial $p(y)$,

$$p(y) = \sum_{k \geq 0} f(\partial_y)^k \{p(y)\} \frac{{}_sLeGH^{(r)}s_k(x, y, z)_{[1,f]}}{k!}. \tag{60}$$

Theorem 15. The conjugate representation is

$${}_sLeGH^{(r)}s_n(x, y, z)_{[1,f]} = \sum_{k=1}^n f^{-1}(\partial_y)^k \{y^n\} \frac{y^k}{k!}. \tag{61}$$

Theorem 16. For any $h(t) \in \mathcal{P}$,

$$\begin{aligned}
 h(t) {}_sLeGH^{(r)}S_n(x, y, z)_{[1,f]} &= \sum_{k=0}^n \binom{n}{k} h(\partial_y) \left\{ {}_sLeGH^{(r)}S_k(x, y, z)_{[1,f]} \right\} {}_sLeGH^{(r)}S_{n-k}(x, y, z)_{[1,f]}.
 \end{aligned}
 \tag{62}$$

Theorem 17. The binomial identity is

$$\begin{aligned}
 {}_sLeGH^{(r)}S_n(x, y + u, z)_{[1,f]} &= \sum_{k=0}^n \binom{n}{k} {}_sLeGH^{(r)}S_k(x, u, z)_{[1,f]} {}_sLeGH^{(r)}S_{n-k}(x, y, z)_{[1,f]}
 \end{aligned}
 \tag{63}$$

for all u in \mathbb{C} .

6. Appell Sequences

For the sequence ${}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]}$ as a function of the variable y , its Appell sequence is

$${}_sLeGH^{(r)}S_n(x, y, z)_{[g,t]} := {}_sLeGH^{(r)}S_n(x, y, z)_{[g,f(t)=t]} \quad (n \in \mathbb{N}_0).
 \tag{64}$$

Then, in view of Remark 4, the assertions of the theorems in this section can be justified by the corresponding theorems in ([5], pp. 26–28). In this respect, proofs are omitted.

Theorem 18. The generating function is

$$\frac{1}{g(t)} C_0(-xt^2) \exp(yt + zt^r) = \sum_{n=0}^{\infty} {}_sLeGH^{(r)}S_n(x, y, z)_{[g,t]} \frac{t^n}{n!}.
 \tag{65}$$

Theorem 19. For any $h(t) \in \mathcal{P}$,

$$h(t) = \sum_{k=0}^{\infty} h(\partial_y) \left\{ {}_sLeGH^{(r)}S_k(x, y, z)_{[g,t]} \right\} g(t) \frac{t^k}{k!}.
 \tag{66}$$

Theorem 20. For any polynomial $p(y)$,

$$p(y) = \sum_{k \geq 0} g(\partial_y) \left\{ p^{(k)}(y) \right\} \frac{{}_sLeGH^{(r)}S_k(x, y, z)_{[g,t]}}{k!},
 \tag{67}$$

where $p^{(k)}(y)$ is the k th derivative of $p(y)$.

Theorem 21. The conjugate representation is

$${}_sLeGH^{(r)}S_n(x, y, z)_{[g,t]} = \sum_{k=0}^n \binom{n}{k} g(\partial_y)^{-1} \left\{ y^{n-k} \right\} y^k.
 \tag{68}$$

Theorem 22. The Appell sequence satisfies

$${}_sLeGH^{(r)}S_n(x, y, z)_{[g,t]} = g(\partial_y)^{-1} \left\{ y^n \right\}.
 \tag{69}$$

Theorem 23. For any $h(t) \in \mathcal{P}$,

$$h(\partial_y) \left\{ {}_sLeGH^{(r)}S_n(x, y, z)_{[g,t]} \right\} = \sum_{k=0}^n \binom{n}{k} h(\partial_y) \left\{ {}_sLeGH^{(r)}S_k(x, y, z)_{[g,t]} \right\} y^{n-k}.
 \tag{70}$$

Theorem 24. *The Appell identity is*

$${}_sLeGH^{(r)}S_n(x, y + u, z)_{[g,t]} = \sum_{k=0}^n \binom{n}{k} {}_sLeGH^{(r)}S_k(x, u, z)_{[g,t]} y^{n-k}. \tag{71}$$

7. Particular Cases

In view of choices of f and g and the variables $x, y,$ and z in Definitions 1 and 2, in fact, there may be a myriad of particular cases. Here we choose to illustrate several ones which are listed in Examples.

Example 1. *Comparing (22) and (23) with (26) and (31), respectively, we find*

$${}_sLeGH^{(r)}S_n(x, y, z)_{[1,t]} = {}_sH_n^{(r)}(x, y, z) \tag{72}$$

and

$$\frac{{}_RLeGH^{(r)}S_n(x, y, z)_{[1,t]}}{n!} = \frac{{}_RH_n^{(r)}(x, y, z)}{n!}, \tag{73}$$

which are the Legendre-Gould Hopper polynomials (LeGHP) and were investigated intensively by Yasmin [25]. Yasmin illustrated several special cases of the LeGHP connected with known polynomials (see ([25], p. 91, Table 2.1)).

Example 2. *Continuing to Example 1, here we take one case given in ([25], p. 91, II, Table 2.1). Setting $z = 0$ in (22) gives the 2-variable Legendre polynomials denoted by ${}_2L_n(x, y) := {}_sH_n^{(r)}(x, y, 0)$ and generated by (see also [15])*

$$C_0(-xt^2) \exp(yt) = \sum_{n=0}^{\infty} {}_2L_n(x, y) \frac{t^n}{n!}. \tag{74}$$

Furthermore, for the choice of $z = 0$ in Definition 1, the 2-variable Legendre-based Sheffer polynomials ${}_2LegS_n(x, y)_{[g,f]}$ for the functions g and f are defined by the following generating function

$$\frac{1}{g(f^{-1}(t))} C_0(-x(f^{-1}(t))^2) \exp(yf^{-1}(t)) = \sum_{n=0}^{\infty} {}_2LegS_n(x, y)_{[g,f]} \frac{t^n}{n!}, \tag{75}$$

where $f(t)$ and $g(t)$ are a delta series and an invertible series, respectively. As in Theorem 1, the 2-variable Legendre-based Sheffer polynomials ${}_2LegS_n(x, y)_{[g,f]}$ for the functions g and f are quasi-monomial under the action of the following multiplicative and derivative operators:

$$\hat{M}_{2Leg[g,f]} := \left(y + 2D_x^{-1} \frac{\partial}{\partial y} - \frac{g'(\partial_y)}{g(\partial_y)} \right) \frac{1}{f'(\partial_y)} \tag{76}$$

and

$$\hat{P}_{2Leg[g,f]} := f(\partial_y), \tag{77}$$

respectively.

Also as in Theorem 3, if $f(t)$ and $g(t)$ have the same forms as in (47) and (48) and $n \in \mathbb{N}_0$, then

(a) ${}_2LegS_0(x, y)_{[g,f]} = 1$ and so

$$\hat{M}_{2Leg[g,f]}^n \{1\} = {}_2LegS_n(x, y)_{[g,f]};$$

(b) A generating function is

$$\exp\left(\hat{M}_{2Leg[g,f]} t\right) \{1\} = \sum_{k=0}^{\infty} {}_2LegS_k(x, y)_{[g,f]} \frac{t^k}{k!};$$

(c) A differential equation is

$$\left(y + 2D_x^{-1} \frac{\partial}{\partial y} - \frac{g'(\partial_y)}{g(\partial_y)} \right) \frac{f(\partial_y)}{f'(\partial_y)} {}_2\text{Le}gS_n(x, y)_{[g,f]} = n {}_2\text{Le}gS_n(x, y)_{[g,f]}.$$

Example 3. From (26) and (31), we obtain

$$e^{-t} C_0(-xt^2) \exp(yt + zt^r) = \sum_{n=0}^{\infty} {}_s\text{Le}GH^{(r)}S_n(x, y, z)_{[e^t, t]} \frac{t^n}{n!} \tag{78}$$

and

$$e^{-t} C_0(xt) C_0(-yt) \exp(zt^r) = \sum_{n=0}^{\infty} {}_R\text{Le}GH^{(r)}S_n(x, y, z)_{[e^t, t]} \frac{t^n}{(n!)^2}. \tag{79}$$

Using (22) and (23) in (78) and (79), respectively, gives

$${}_s\text{Le}GH^{(r)}S_n(x, y, z)_{[e^t, t]} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} {}_sH_k^{(r)}(x, y, z) \tag{80}$$

and

$$\frac{{}_R\text{Le}GH^{(r)}S_n(x, y, z)_{[e^t, t]}}{n!} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{{}_RH_k^{(r)}(x, y, z)}{k!}. \tag{81}$$

In addition, the sequence ${}_s\text{Le}GH^{(r)}S_n(x, y, z)_{[e^t, t]}$ is found to belong to Appell sequence. Therefore the results in Section 6 may apply to this sequence to yield more involved identities. For example,

$${}_s\text{Le}GH^{(r)}S_n(x, y + u, z)_{[e^t, t]} = \sum_{k=0}^n \binom{n}{k} {}_s\text{Le}GH^{(r)}S_k(x, u, z)_{[e^t, t]} y^{n-k}. \tag{82}$$

Example 4. Let $f(t) = t$ and define a function $g : \mathbb{C} \rightarrow \mathbb{C}$ by

$$g(t) := \begin{cases} \frac{e^t - 1}{t} & (t \neq 0), \\ 1 & (t = 0). \end{cases} \tag{83}$$

Then it is easy to see that g is an entire function and an invertible series in \mathcal{P} . Now use $f(t) = t$ and $g(t) = g(t)$ in (26) and set $x = z = 0$, and replace y by x . We get

$$\sum_{n=0}^{\infty} {}_s\text{Le}GH^{(r)}S_n(0, x, 0)_{[g, t]} \frac{t^n}{n!} = \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi). \tag{84}$$

Hence

$${}_s\text{Le}GH^{(r)}S_n(0, x, 0)_{[g, t]} = B_n(x) \quad (n \in \mathbb{N}_0), \tag{85}$$

which are the classical Bernoulli polynomials (see, e.g., ([27] Section 1.7).

Furthermore, using the Appell identity (71), we obtain

$${}_s\text{Le}GH^{(r)}S_n(0, x + y, 0)_{[g, t]} = \sum_{k=0}^n \binom{n}{k} {}_s\text{Le}GH^{(r)}S_k(0, x, 0)_{[g, t]} y^{n-k}. \tag{86}$$

Or, equivalently,

$$B_n(x + y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k} \quad (n \in \mathbb{N}_0), \tag{87}$$

which is a well known identity (see, e.g., ([27] p. 82, Equation (13))).

8. Other Operational and Certain Integral Representations

We begin by recalling the Crofton-type identity (see, e.g., ([28], Equation (3.18) and Theorem 3.5); see also ([25], Equation (2.6))

$$f\left(y + m\lambda \frac{d^{m-1}}{dy^{m-1}}\right)\{1\} = \exp\left(\lambda \frac{d^m}{dy^m}\right)\{f(y)\}. \quad (88)$$

Theorem 25. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are the same as in (47) and (48). Also let $n \in \mathbb{N}_0$. Then we obtain

$${}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]} = \exp\left(D_x^{-1} \frac{\partial^2}{\partial y^2} + z \frac{\partial^r}{\partial y^r}\right)\{[g,f]S_n(y)\} \quad (89)$$

and

$$\frac{{}_rLeGH^{(r)}S_n(x, y, z)_{[g,f]}}{n!} = \exp\left(-D_x^{-1} \frac{\partial}{\partial y} + D_y^{-1} \frac{\partial}{\partial y} + z \frac{\partial^r}{\partial y^r}\right)\{[g,f]S_n(0)\}. \quad (90)$$

Proof. The use of (88) in (29) and (34) yields (89) and (90), respectively. \square

Theorem 26. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are the same as in (47) and (48). Also let $n \in \mathbb{N}_0$. Then we get

$${}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]} = \exp\left(z \frac{\partial^r}{\partial y^r}\right) {}_2LeGS_n(x, y)_{[g,f]}. \quad (91)$$

Proof. From (26) we obtain

$$\frac{\partial^r}{\partial y^r} {}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]} = \frac{\partial}{\partial z} {}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]}. \quad (92)$$

Comparing the generating functions (26) and (75), we get

$${}_sLeGH^{(r)}S_n(x, y, 0)_{[g,f]} = {}_2LeGS_n(x, y)_{[g,f]}. \quad (93)$$

Now we can consider (92) as a first-order differential equation of the variable z with the initial condition (93). Then solving this first-order differential equation gives the solution (91). \square

Theorem 27. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are the same as in (47) and (48). Also let $n \in \mathbb{N}_0$. Then the following integral representation holds:

$${}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]} = \int_0^\infty e^{-u} {}_sLeGH^{(r)}S_n(x, y, uD_z^{-1})_{[g,f]} du. \quad (94)$$

Proof. Combining (26) and (22) provides

$$\sum_{n=0}^{\infty} {}_sLeGH^{(r)}S_n(x, y, z)_{[g,f]} \frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))} \sum_{n=0}^{\infty} {}_sH_n^{(r)}(x, y, z) \frac{(f^{-1}(t))^n}{n!}. \quad (95)$$

Using the following integral formula (see ([25], Theorem 3.1))

$${}_sH_n^{(r)}(x, y, z) = \int_0^\infty e^{-u} {}_sH_n^{(r)}(x, y, uD_z^{-1}) du \quad (96)$$

in the right-hand side of (95) gives

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} e^{-u} \left\{ \frac{1}{g(f^{-1}(t))} {}_sH_n^{(r)}(x, y, uD_z^{-1}) \frac{(f^{-1}(t))^n}{n!} \right\} du. \end{aligned} \tag{97}$$

Employing (30) in the braces of the right-hand side of (97), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_sLeGH^{(r)}s_n(x, y, z)_{[g,f]} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ \int_0^{\infty} e^{-u} {}_sLeGH^{(r)}s_n(x, y, uD_z^{-1})_{[g,f]} du \right\} \frac{t^n}{n!}, \end{aligned}$$

both sides of which, upon equating the coefficients of t^n , yields (94). □

Theorem 28. Let $f(t)$ and $g(t)$ be delta and invertible series, respectively, whose power series forms are the same as in (47) and (48). Also let $n \in \mathbb{N}_0$. Then the following integral representation holds:

$${}_RLeGH^{(r)}s_n(x, y, z)_{[g,f]} = \int_0^{\infty} e^{-u} {}_RLeGH^{(r)}s_n(x, y, uD_z^{-1})_{[g,f]} du. \tag{98}$$

Proof. Recalling the following integral formula (see ([25], Theorem 3.2))

$${}_RH_n^{(r)}(x, y, z) = \int_0^{\infty} e^{-u} {}_RH_n^{(r)}(x, y, uD_z^{-1}) du \tag{99}$$

and using (35), similarly as in the proof of Theorem 27, we can obtain (98). The details are omitted. □

9. Concluding Remarks

Certain *symmetries* are ubiquitous in a wide range of mathematical properties and formulas (e.g., (86) or (87) is symmetric with respect to the variables x and y).

In this paper, we introduced the so-called Legendre-Gould Hopper-based Sheffer polynomials whose generating functions are constructed by substituting the generating functions of the well-developed Legendre-Gould Hopper polynomials in [25] after t is replaced by $f^{-1}(t)$ for the exponential part of that of Sheffer polynomials in (2). Then the newly introduced polynomials were found to be still Sheffer type polynomials and, therefore, many important properties and identities for these new polynomials were easily established by employing those in the well-developed theory of Sheffer polynomials (see, e.g., [5]; see also [29]).

The essential part of introducing these new Sheffer polynomials is found that the based polynomials, here Legendre-Gould Hopper polynomials, are quasi-monomials with respect to a pair of certain multiplicative and derivative operators. In this point and in a little detail, if some known polynomials of, say, function of variable x , are quasi-monomials with respect to a pair (\hat{M}, \hat{P}) of certain multiplicative and derivative operators which are given by $\hat{P} = \frac{\partial}{\partial x}$, certain new Sheffer polynomials based on these known polynomials may be devised in the same method as in this paper.

Posing a Problem

As in (3), instead of the generating functions in Definitions 1 and 2, if we consider the following type generating functions:

$$A(t)C_0 \left(-x(H(t))^2 \right) \exp \left(yH(t) + z(H(t))^r \right)$$

and

$$A(t)C_0 \left(x f^{-1}(t) \right) C_0(-y f^{-1}(t)) \exp \left(z(H(t))^r \right),$$

where $A(t)$ is an invertible series and $H(t)$ is a delta series, we are believed to be able to establish corresponding results as those in this paper. This posed problem is left to the interested researcher and the authors for future investigation.

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