symmetry

MDPI

Article

Quasi-Arithmetic Type Mean Generated by the Generalized Choquet Integral

Sebastian Wójcik

Institute of Mathematics, University of Rzeszów, Pigonia 1, 35-959 Rzeszów, Poland; swojcik@ur.edu.pl

Received: 24 November 2020; Accepted: 14 December 2020; Published: 17 December 2020

Abstract: It is known that the quasi-arithmetic means can be characterized in various ways, with an essential role of a symmetry property. In the expected utility theory, the quasi-arithmetic mean is called the certainty equivalent and it is applied, e.g., in a utility-based insurance contracts pricing. In this paper, we introduce and study the quasi-arithmetic type mean in a more general setting, namely with the expected value being replaced by the generalized Choquet integral. We show that a functional that is defined in this way is a mean. Furthermore, we characterize the equality, positive homogeneity, and translativity in this class of means.

Keywords: quasi-arithmetic mean; Choquet integral; positive homogeneity; translativity; certainty equivalent

MSC: 39B22; 91B30

1. Introduction

The notion of quasi-arithmetic mean, playing an important role in several branches of mathematics and its applications, was introduced in the book by Hardy, Littlewood, and Pólya [1]. Recall that, if $I \subseteq \mathbb{R}$ is an interval and $u : I \to \mathbb{R}$ is a strictly monotone continuous function, then the quasi-arithmetic mean $M_u : \bigcup_{n\in\mathbb{N}} I^n \to I$, which is generated by u, is given by

$$M_u(\mathbf{x}) = u^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} u(x_i)\right) \quad \text{for } \mathbf{x} = (x_1, ..., x_n) \in I^n,\ n \in \mathbb{N}.$$

Various axiomatic characterizations of the quasi-arithmetic mean have been independently established by de Finetti [2], Kolmogorov [3], and Nagumo [4]. For these characterizations, the symmetry property of the mean is essential. Quasi-arithmetic means have been generalized in several directions. In particular, Bajraktarević [5] introduced quasi-arithmetic means that were weighted by a weight function. Bajraktarević means were characterized by Páles [6]. A further generalization of the quasi-arithmetic means has been developed by Daróczy [7,8], who introduced deviation means.

The notion of the quasi-arithmetic mean can be considered in more general settings. In particular, if $X : S \to \mathbb{R}$ is an $\mathcal{F}-$measurable essentially bounded function on a given probability space $(S, \mathcal{F}, P)$, then the quasi-arithmetic mean of X, generated by a strictly monotone continuous function $u : \mathbb{R} \to \mathbb{R}$, is defined as follows

$$M_{(u,P)}(X) = u^{-1}\left(\mathbb{E}_P[u(X)]\right), \tag{1}$$

where $\mathbb{E}_P$ denotes the expected value with respect to P.

In the expected utility theory developed by von Neumann and Morgenstern [9], the quasi-arithmetic mean, as given by (1), is known as the certainty equivalent. It establishes a symmetry in preferences between a risk that is represented by X and a deterministic payoff $M_{(u,P)}(X)$, in a

sense that a decision maker is indifferent between taking the risk X and receiving $M_{(u,P)}(X)$ for sure. Quasi-arithmetic means are also applied in a utility-based insurance contracts pricing. In this setting, the quasi-arithmetic mean that is defined by (1) is called the mean-value premium. It expresses a premium principle from a client's point of view for a risk represented by X.

Replacing, in (1), the probability measure P by a capacity μ, that is a monotone normed set function defined on measurable space $(S, \mathcal{F})$, we obtain

$$M_{(u,\mu)}(X) = u^{-1}\left(\mathbb{E}_\mu[u(X)]\right), \tag{2}$$

where $\mathbb{E}_\mu$ is the Choquet integral with respect to μ (see Section 2). It turns out that, for any $\mathcal{F}-$measurable function $X : S \to \mathbb{R}$, $\bar{\mu}-$essentially bounded from below, and $\mu-$essentially bounded from above $M_{(u,\mu)}(X)$ is a well-defined mean (see Sections 2 and 3 for details). The functional $M_{u,\mu,\nu}$, as given by (2), refers to the certainty equivalent under the model of Schmeidler [10]. Some particular cases of $M_{(u,\mu)}$ have been investigated in [11]. Inspired by the results in [11], in this paper, we study the properties of the functional

$$M_{(u,\mu,\nu)}(X) = u^{-1}\left(\mathbb{E}_{\mu\nu}[u(X)]\right), \tag{3}$$

defined on a family of all $\mathcal{F}-$measurable functions $X : S \to \mathbb{R}$, which are $\nu-$essentially bounded from below and $\mu-$essentially bounded from above. Here $\mathbb{E}_{\mu\nu}$ stands for the generalized Choquet integral with respect to capacities μ and ν on $(S, \mathcal{F})$. We show that $M_{(u,\mu,\nu)}$ is well-defined and it is a mean, which is

$$\inf X \leq M_{(u,\mu,\nu)}(X) \leq \sup X. \tag{4}$$

Furthermore, we establish the characterizations of the equality, positive homogeneity, and translativity in the class of means defined by (3). In the whole paper $(S, \mathcal{F})$ stands for a measurable space.

2. Generalized Choquet Integral

A set function $\mu : \mathcal{F} \to [0, 1]$ is called *the capacity on* $(S, \mathcal{F})$ provided that it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$,
(2) $\mu(S) = 1$, and
(3) if $A, B \in \mathcal{F}$ and $A \subset B$. then $\mu(A) \leq \mu(B)$.

Obviously, every probability measure on $(S, \mathcal{F})$ is a capacity on $(S, \mathcal{F})$. Furthermore, simple calculations show that, if μ is a capacity on $(S, \mathcal{F})$, then so is the set function $\bar{\mu}$ of the form

$$\bar{\mu}(A) = 1 - \mu(S \backslash A) \ \text{ for } \ A \in \mathcal{F}. \tag{5}$$

It is called the *conjugation of* μ.

If μ is a capacity on $(S, \mathcal{F})$, then an $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ is called *μ-essentially bounded from above* provided that there exists $t \in \mathbb{R}$, such that $\mu(\{s \in S : \ X(s) > t\}) = 0$. In such a case the *$\mu$-essential supremum* of X is defined in the following way

$$\mu - \operatorname{ess\,sup} X := \inf\{t \in \mathbb{R} : \ \mu(\{s \in S : \ X(s) > t\}) = 0\}.$$

An $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ is said to be a *μ-essentially bounded from below* if there exists $t \in \mathbb{R}$, such that $\mu(\{s \in S : \ X(s) < t\}) = 0$. In that case,

$$\mu - \operatorname{ess\,inf} X := \sup\{t \in \mathbb{R} : \ \mu(\{s \in S : \ X(s) < t\}) = 0\}$$

is called the *μ-essential infimum* of X.

Remark 1. *Assume that μ is a capacity on $(S, \mathcal{F})$ and $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function. Note that:*

- *if X is μ-essentially bounded from above, then $-\infty \leq \mu - ess\,sup\,X < \infty$;*
- *if X is μ-essentially bounded from below then $-\infty < \mu - ess\,inf\,X \leq \infty$.*

The following example shows that, in general, $\mu - ess\,sup\,X$ and $\mu - ess\,inf\,X$ need not be finite.

Example 1. *Let $(S, \mathcal{F}) = (\mathbb{R}, 2^{\mathbb{R}})$ and $\mu : \mathcal{F} \to [0, 1]$ be given by*

$$\mu(A) = \begin{cases} 0 \ \textit{for} \quad A \neq S, \\ 1 \ \textit{for} \quad A = S. \end{cases} \tag{6}$$

Obviously, μ is a capacity on $(S, \mathcal{F})$. Furthermore, $X = id_{\mathbb{R}}$ is $\mathcal{F}$-measurable and, for every $t \in \mathbb{R}$, we have $\mu(\{s \in \mathbb{R} : \ X(s) > t\}) = \mu((t, \infty)) = 0$ and $\mu(\{s \in \mathbb{R} : \ X(s) < t\}) = \mu((-\infty, t)) = 0$. Hence, X is μ-essentially bounded, but $\mu - ess \ \ sup\,X = -\infty$ and $\mu - ess \ \ inf\,X = \infty$.

Lemma 1. *Assume that μ is a capacity on $(S, \mathcal{F})$ and $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below. Then $\bar{\mu} - ess\,inf\,X$ and $\mu - ess\,sup\,X$ are finite and*

$$\bar{\mu} - ess\,inf\,X \leq \mu - ess\,sup\,X. \tag{7}$$

Proof. Suppose that

$$\mu - \operatorname{ess\,sup} X < \bar{\mu} - \operatorname{ess\,inf} X$$

and fix $t_1, t_2 \in \mathbb{R}$ such that

$$\mu - \operatorname{ess\,sup} X < t_1 < t_2 < \bar{\mu} - \operatorname{ess\,inf} X.$$

Then

$$1 = \mu(\{X > t_1\}) + \bar{\mu}(\{X \leq t_1\}) \leq \mu(\{X > t_1\}) + \bar{\mu}(\{X < t_2\}) = 0$$

which yields a contradiction. □

Remark 2. *Assume that $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function. Subsequently, for every capacity μ on $(S, \mathcal{F})$, we have $\mu(\{X < \inf X\}) = \mu(\{X > \sup X\}) = \mu(\emptyset) = 0$ and $\mu(\{X \leq \sup X\}) = \mu(\{X \geq \inf X\}) = \mu(S) = 1$. Therefore:*

(i) *if X is bounded from below (above) then X is μ-essentially bounded from below (above) for every capacity μ on $(S, \mathcal{F})$; and,*

(ii) *if, for some capacity μ on $(S, \mathcal{F})$, X is μ-essentially bounded from below then*

$$inf\,X \leq \mu - ess\ inf\,X \leq sup\,X; \tag{8}$$

(iii) *if, for some capacity μ on $(S, \mathcal{F})$, X is μ-essentially bounded from above, then*

$$inf\,X \leq \mu - ess\,sup\,X \leq sup\,X. \tag{9}$$

In what follows, for a given strictly increasing function $u : \mathbb{R} \to \mathbb{R}$, we adopt the standard convention $u(-\infty) := \lim_{x \to -\infty} u(x)$ and $u(\infty) := \lim_{x \to \infty} u(x)$.

Lemma 2. *Assume that μ is a capacity on $(S, \mathcal{F})$ and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function.*

(i) *If $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from above, then so is $u \circ X$. Furthermore,*

$$\mu - ess\,sup\ u \circ X = u(\mu - ess\,sup\ X).$$

(ii) *If $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from below, then so is $u \circ X$. Moreover,*

$$\mu - ess\,inf\; u \circ X = u(\mu - ess\,inf\; X).$$

Proof. Let $X : S \to \mathbb{R}$ be an $\mathcal{F}$-measurable function μ-essentially bounded from above. Because u is continuous, $u \circ X$ is $\mathcal{F}$-measurable. Moreover, as u is strictly increasing, we have

$$\mu(\{u \circ X > u(t)\}) = \mu(\{X > t\}) \text{ for } t \in \mathbb{R}.$$

Therefore, because X is μ-essentially bounded from above, so is $u \circ X$. We show that

$$\mu - \operatorname{ess\,sup}\; u \circ X \leq u(\mu - \operatorname{ess\,sup}\; X). \tag{10}$$

Suppose that (10) does not hold and fix $t \in \mathbb{R}$, such that

$$u(\mu - \operatorname{ess\,sup}\; X) < t < \mu - \operatorname{ess\,sup}\; u \circ X.$$

Thus, $\mu(\{u \circ X > t\}) > 0$, hence $t < u(\infty)$. Consequently $t = u(s)$ for some $s \in \mathbb{R}$ and so, we obtain

$$\mu(\{X > s\}) = \mu(\{u \circ X > u(s)\}) = \mu(\{u \circ X > t\}) > 0.$$

Hence, $s \leq \mu - \operatorname{ess\,sup}\; X$, that is $t = u(s) \leq u(\mu - \operatorname{ess\,sup}\; X)$, which yields a contradiction and proves (10). Now, we prove that

$$u(\mu - \operatorname{ess\,sup}\; X) \leq \mu - \operatorname{ess\,sup}\; u \circ X. \tag{11}$$

Suppose that (11) is not true and fix $t \in \mathbb{R}$, such that

$$\mu - \operatorname{ess\,sup}\; u \circ X < t < u(\mu - \operatorname{ess\,sup}\; X).$$

Subsequently, $\mu(\{u \circ X > t\}) = 0$, which implies that $t > u(-\infty)$. Hence, $t = u(s)$ for some $s \in \mathbb{R}$ and we obtain

$$\mu(\{X > s\}) = \mu(\{u \circ X > u(s)\}) = \mu(\{u \circ X > t\}) = 0,$$

that is $s \geq \mu - \operatorname{ess\,sup}\; X$. Thus $t = u(s) \geq u(\mu - \operatorname{ess\,sup}\; X)$, which gives a contradiction and proves (11). In this way, we have proved (i). A proof of (ii) is similar. □

If μ is a capacity on $(S, \mathcal{F})$ and $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below, then the Choquet integral of X is defined in following way (cf. [12])

$$E_\mu[X] := \int_0^\infty \mu(\{X > x\})dx - \int_{-\infty}^0 \bar{\mu}(\{X < x\})dx, \tag{12}$$

where the integrals on the right-hand side of (12) are the Riemann integrals. Note that both of the integrals on the right-hand side of (12) are finite. Several details concerning various properties of Choquet integral can be found in [13]. In particular, in [13] (Proposition 5.1), it is proved that the Choquet integral is positively homogeneous, translative, and monotonic. Moreover, it is asymmetric, which is, for every $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below, it holds

$$E_\mu[-X] = -E_{\bar{\mu}}[X]. \tag{13}$$

Remark 3. *Note that, for all capacities μ and ν on $(S, \mathcal{F})$ and any $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and ν-essentially bounded from below, we have*

$$E_\mu[X^+] = \int_0^\infty \mu(\{X^+ > x\})dx = \int_0^\infty \mu(\{X > x\})dx \tag{14}$$

and

$$E_\nu[X^-] = \int_0^\infty \nu(\{X^- > x\})dx = \int_0^\infty \nu(\{-X > x\})dx = \int_{-\infty}^0 \nu(\{X < x\})dx. \tag{15}$$

In particular, taking $\nu = \bar{\mu}$, in view of (12), we obtain

$$E_\mu[X] = E_\mu[X^+] - E_{\bar{\mu}}[X^-]. \tag{16}$$

Proposition 1. *Assume that μ is a capacity on $(S, \mathcal{F})$. Subsequently, for every $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below, it holds*

$$\bar{\mu} - ess\ inf\ X \leq E_\mu[X] \leq \mu - ess\ sup\ X. \tag{17}$$

Proof. Assume that $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below and put $m = \bar{\mu} - \text{ess inf } X$ and $M = \mu - \text{ess sup } X$. Subsequently, $\bar{\mu}(\{X < x\}) = 0$ for $x \in (-\infty, m)$ and $\mu(\{X > x\}) = 0$ for $x \in (M, \infty)$. Furthermore, from Corollary 1, we derive that m and M are finite and $m \leq M$. Therefore, the following three cases are possible:

(i) $0 \leq m \leq M$,
(ii) $m \leq M \leq 0$, and
(iii) $m < 0 < M$.

In the first case, using the monotonicity of μ, for every $x \in (-\infty, m)$, we get

$$\mu(\{X > x\}) \geq \mu\left(\left\{X \geq \frac{x+m}{2}\right\}\right) = 1 - \bar{\mu}\left(\left\{X < \frac{x+m}{2}\right\}\right) = 1.$$

Hence

$$m = \int_0^m \mu(\{X > x\})dx \leq \int_0^M \mu(\{X > x\})dx \leq M.$$

On the other hand, applying Remark 3 with $\nu = \bar{\mu}$, we obtain

$$E_\mu[X] = \int_0^\infty \mu(\{X > x\})dx - \int_{-\infty}^0 \bar{\mu}(\{X < x\})dx = \int_0^M \mu(\{X > x\})dx.$$

Therefore, (17) holds.

In the second case, for every $x \in (M, \infty)$, we obtain

$$\bar{\mu}(\{X < x\}) \geq \bar{\mu}\left(\left\{X \leq \frac{x+M}{2}\right\}\right) = 1 - \mu\left(\left\{X > \frac{x+M}{2}\right\}\right) = 1,$$

that is

$$m \leq -\int_m^0 \bar{\mu}(\{X < x\})dx \leq -\int_M^0 \bar{\mu}(\{X < x\})dx = M.$$

Furthermore, while taking Remark 3 with $\nu = \bar{\mu}$ into account, we conclude that

$$E_\mu[X] = \int_0^\infty \mu(\{X > x\})dx - \int_{-\infty}^0 \bar{\mu}(\{X < x\})dx = -\int_m^0 \bar{\mu}(\{X < x\})dx.$$

Hence, (17) is valid.

In the third case, while using the monotonicity of Choquet integral, in view of Remark 3, we get

$$m \leq -\int_m^0 \bar{\mu}(\{X < x\})dx = -\int_{-\infty}^0 \bar{\mu}(\{X < x\})dx \leq E_\mu[X]$$

$$\leq \int_0^\infty \mu(\{X > x\})dx = \int_0^M \mu(\{X > x\})dx \leq M,$$

which yields the assertion. □

Now, assume that μ and ν are capacities on $(S, \mathcal{F})$ and $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable μ-essentially bounded from above and ν-essentially bounded from below function. The generalized Choquet integral of X is given by

$$E_{\mu\nu}[X] := E_\mu[X^+] - E_\nu[X^-], \tag{18}$$

where X^+ and X^- denote the positive and negative part of X, respectively, which is $X^+ := \max\{X, 0\}$ and $X^- := \max\{-X, 0\}$. Note that X^+ and X^- are $\mathcal{F}$-measurable, X^+ is μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below (cf. Remark 2 (i)), while X^- is ν-essentially bounded from below $\bar{\nu}$-essentially bounded from above. Thus, the generalized Choquet integral is well-defined by (18). Generalized Choquet integral was introduced in [14] for discrete random variables.

Remark 4. *It follows from* (16)–(18) *that*

$$E_{\mu\bar{\mu}}[X] = E_\mu[X] \tag{19}$$

for $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and $\bar{\mu}$-essentially bounded from below. Hence, the Choquet integral is a particular case of the generalized Choquet integral. Another important particular case of the generalized Choquet integral is the Šipoš integral, which refers to case $\mu = \nu$.

Remark 5. *Because the Choquet integral is positively homogeneous and monotonic, so is the generalized Choquet integral. Moreover, in view of* (13)*, for every capacities μ and ν on $(S, \mathcal{F})$ $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and ν-essentially bounded from below, we obtain*

$$E_{\mu\nu}[-X] = -E_{\nu\mu}[X]. \tag{20}$$

Remark 6. *Assume that μ and ν are capacities on $(S, \mathcal{F})$ and $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from above and ν-essentially bounded from below. Let $m = \nu - ess\ inf\ X$ and $M = \mu - ess\ sup\ X$. Subsequently, $\mu(\{X > x\}) = 0$ for $x \in (M, \infty)$ and $\mu(\{X < x\}) = 0$ for $x \in (-\infty, m)$. Therefore, making use of* (14), (15) *and* (18)*, we obtain that:*

(i) *if $m > 0$ and $M \geq 0$ then*

$$E_{\mu\nu}[X] = \int_0^M \mu(\{X > x\})dx;$$

(ii) *if $m \leq 0$ and $M < 0$ then*

$$E_{\mu\nu}[X] = -\int_m^0 \nu(\{X < x\})dx;$$

(iii) *if $m \leq 0 \leq M$ then*

$$E_{\mu\nu}[X] = \int_0^M \mu(\{X > x\})dx - \int_m^0 \nu(\{X < x\})dx;$$

(iv) *if $M < 0 < m$ then $E_{\mu\nu}[X] = 0$.*

Remark 7. *Assume that μ and ν are capacities on $(S, \mathcal{F})$. Subsequently, in view of Remark 6, we have*

$$\min\{\nu - ess\ inf\ X, 0\} \leq E_{\mu\nu}[X] \leq \max\{\mu - ess\ sup\ X, 0\} \tag{21}$$

for every $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and ν-essentially bounded from below.

We complete with this section with the following result.

Proposition 2. *Assume that μ and ν are the capacities on $(S, \mathcal{F})$ and $X : S \to \mathbb{R}$ is of the form*

$$X = x_1 \mathbb{1}_A + x_2 \mathbb{1}_{S \setminus A}, \tag{22}$$

with some $A \in \mathcal{F}$ and $x_1, x_2 \in \mathbb{R}$, such that $x_1 < x_2$. Subsequently, X is $\mathcal{F}$-measurable μ-essentially bounded from above and ν-essentially bounded from below. Furthermore,

$$E_{\mu\nu}[X] = (1 - \mu(S \setminus A))x_1 + \mu(S \setminus A)x_2 \quad \textit{whenever} \quad 0 \leq x_1, \tag{23}$$

$$E_{\mu\nu}[X] = \nu(A)x_1 + \mu(S \setminus A)x_2 \quad \textit{whenever} \quad x_1 < 0 < x_2, \tag{24}$$

$$E_{\mu\nu}[X] = \nu(A)x_1 + (1 - \nu(A))x_2 \quad \textit{whenever} \quad x_2 \leq 0. \tag{25}$$

Proof. An $\mathcal{F}$-measurability of X is obvious. Note that $\nu(\{X < x_1\}) = \nu(\emptyset) = 0$ and $= \mu(\{X > x_2\}) = \mu(\emptyset) = 0$, hence X is μ-essentially bounded from above and ν-essentially bounded from below. Furthermore, we have

$$\mu(\{X > x\}) = \begin{cases} 1 & \text{for} \quad x \in (-\infty, x_1), \\ \mu(S \setminus A) & \text{for} \quad x \in [x_1, x_2), \\ 0 & \text{for} \quad x \in [x_2, \infty) \end{cases}$$

and

$$\nu(\{X < x\}) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, x_1], \\ \nu(A) & \text{for} \quad x \in (x_1, x_2], \\ 1 & \text{for} \quad x \in (x_2, \infty). \end{cases}$$

Therefore, while taking into account (14), (15) and (18), after a standard computation, we obtain the assertion. □

3. Quasi-Arithmetic Type Mean Generated by the Generalized Choquet Integral

In this section, we prove that $M_{(u,\mu,\nu)}$ defined on a family of all $\mathcal{F}-$measurable function $X : S \to \mathbb{R}$ $\nu-$essentially bounded from below and $\mu-$essentially bounded from above, as given by (3), is well-defined and it is a mean.

Theorem 1. *Assume that μ and ν are capacities on $(S, \mathcal{F})$ and $u : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Subsequently, for every $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and ν-essentially bounded from below, there exists a unique $M_{(u,\mu,\nu)}(X) \in \mathbb{R}$, such that*

$$u(M_{(u,\mu,\nu)}(X)) = E_{\mu\nu}[u(X)]. \tag{26}$$

Furthermore, $M_{(u,\mu,\nu)}$ is a mean, which is (4) holds for every $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and ν-essentially bounded from below.

Proof. Assume that $X : S \to \mathbb{R}$ is an $\mathcal{F}$-measurable function μ-essentially bounded from above and ν-essentially bounded from below. Subsequently, according to Lemma 2, so is $u \circ X$. Let $M =$

$\mu - \operatorname{ess\,sup} X$ and $m = \nu - \operatorname{ess\,inf} X$. If there exist $s_1, s_2 \in S$, such that $(u \circ X)(s_1) < 0 < (u \circ X)(s_2)$ then, applying Lemma 2 and Corollary 7, we obtain

$$u(\inf X) \leq \min\{u(m), u(X(s_1))\} = \min\{\nu - \operatorname{ess\,inf} u \circ X, u(X(s_1))\}$$

$$\leq \min\{\nu - \operatorname{ess\,inf} u(X), 0\} \leq E_{\mu\nu}[u(X)] \leq \max\{\mu - \operatorname{ess\,sup} u \circ X, 0\}$$

$$\leq \max\{\mu - \operatorname{ess\,sup} u \circ X, u(X(s_2))\} = \max\{u(M), u(X(s_2))\} \leq u(\sup X).$$

Because u is continuous, this means that (26) holds with some $M_{(u,\mu,\nu)}(X) \in \mathbb{R}$. Furthermore, as u is strictly increasing, such an $M_{(u,\mu,\nu)}(X)$ is unique and it satisfies (4).

Now, assume that $u \circ X$ takes only non-positive values. Afterwards, according to Remark 2 (ii), (iii), we have $\nu - \operatorname{ess\,inf} u \circ X \leq 0$ and $\mu - \operatorname{ess\,sup} u \circ X \leq 0$. Hence, applying Remark 6, in view of (13), we have

$$E_{\mu\nu}[u(X)] = -E_{\nu}[-u(X)] = E_{\bar{\nu}}[u(X)]. \tag{27}$$

Hence, applying Proposition 1, Lemma 2, and Remark 2, we obtain

$$u(\inf X) \leq u(\nu - \operatorname{ess\,inf} X) = \nu - \operatorname{ess\,inf} u \circ X \leq E_{\mu\nu}[u(X)]$$

$$\leq \bar{\nu} - \operatorname{ess\,sup} u \circ X = u(\bar{\nu} - \operatorname{ess\,sup} X) \leq u(\sup X).$$

Thus, arguing as previously, we conclude that there exists a unique $M_{(u,\mu,\nu)}(X) \in \mathbb{R}$, such that (26) holds and, moreover, (4) is valid.

If $u \circ X$ takes only non-negative values, applying Remark 2 (ii), (iii), we have $\nu - \operatorname{ess\,inf} u \circ X \geq 0$ and $\mu - \operatorname{ess\,sup} u \circ X \geq 0$. Thus, making use of Remark 6, we conclude that

$$E_{\mu\nu}[u(X)] = E_{\mu}[u(X)], \tag{28}$$

hence, applying Proposition 1, Lemma 2, and Remark 2, we get

$$u(\inf X) \leq u(\bar{\mu} - \operatorname{ess\,inf} X) = \bar{\mu} - \operatorname{ess\,inf} u \circ X \leq E_{\mu\nu}[u(X)]$$

$$\leq \mu - \operatorname{ess\,sup} u \circ X = u(\mu - \operatorname{ess\,sup} X) \leq u(\sup X).$$

Therefore, as u is continuous and strictly increasing, we have the assertion. □

Remark 8. *It follows from* (26) *that, for arbitrary capacities μ and ν on $(S, \mathcal{F})$ and for every continuous strictly increasing function $u : \mathbb{R} \to \mathbb{R}$, the functional that is given by* (3) *is a well-defined mean on the family of all $\mathcal{F}$-measurable functions $X : S \to \mathbb{R}$ $\mu-$essentially bounded from above and $\nu-$essentially bounded from below. In particular, while taking* (19) *into account, we conclude that* (2) *defines a mean on a family of all $\mathcal{F}$-measurable functions $X : S \to \mathbb{R}$ $\mu-$essentially bounded from above and $\bar{\mu}-$essentially bounded from below.*

Remark 9. *Assume that $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function. It follows from* (3) *and* (23)–(25) *that, if $X : S \to \mathbb{R}$ is of the form* (22) *with some $A \in \mathcal{F}$ and $x, y \in \mathbb{R}$, such that $x < y$, then*

$$M_{(u,\mu,\nu)}(X) = \begin{cases} u^{-1}((1-\mu(S\backslash A))u(x) + \mu(S\backslash A)u(y)) & \text{if } 0 \leq u(x), \\ u^{-1}(\nu(A))u(x) + \mu(S\backslash A)u(y)) & \text{if } u(x) < 0 < u(y), \\ u^{-1}(\nu(A))u(x) + (1-\nu(A))u(y)) & \text{if } u(y) \leq 0. \end{cases} \tag{29}$$

4. Main Properties of the Mean

In this section, we are going to investigate some of the properties of means defined by (3). From now on, we assume that μ and ν are capacities on $(S, \mathcal{F})$. By $\mathcal{X}$, we denote a family of all $\mathcal{F}$-measurable function $X : S \to \mathbb{R}$ μ-essentially bounded from above and ν-essentially bounded from

below. A subfamily of $\mathcal{X}$ of functions $X : S \to \mathbb{R}$ of the form (22), where $A \in \mathcal{F}$ and $x, y \in \mathbb{R}$, such that $x < y$, will be denoted by $\mathcal{X}^{(2)}$.

We begin with a characterization of the equality in this class of means. The following result will play an essential role in our considerations.

Lemma 3. *[11] (Lemma 5.1) Let $I \subset \mathbb{R}$ be a non-degenerate interval, $f : I \to \mathbb{R}$ be a non-constant continuous function, and $\alpha, \beta \in \mathbb{R}$. If f satisfies equation*

$$f((1-\alpha)x + \alpha y) = (1-\beta)f(x) + \beta f(y) \quad \textit{for} \quad x, y \in I, x < y, \tag{30}$$

then $\alpha = \beta$. If, moreover, α, $\beta \notin \{0, 1\}$, then there exist $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, such that

$$f(x) = ax + b \quad \textit{for} \quad x \in I. \tag{31}$$

Remark 10. *In [11], the above result is formulated for functions $f : \mathbb{R} \to \mathbb{R}$. However, it remains true with the same proof for $f : I \to \mathbb{R}$.*

Theorem 2. *Assume that $u, v : \mathbb{R} \to \mathbb{R}$ are strictly increasing continuous functions and there exists a set $A \in \mathcal{F}$, such that*

$$\mu(A),\ \nu(S \backslash A) \in (0, 1). \tag{32}$$

Subsequently, the following conditions are equivalent:

(i)

$$M_{(v,\mu,\nu)}(X) = M_{(u,\mu,\nu)}(X) \ \textit{for} \quad X \in \mathcal{X}^{(2)}, \tag{33}$$

(ii)

$$M_{(v,\mu,\nu)}(X) = M_{(u,\mu,\nu)}(X) \ \textit{for} \quad X \in \mathcal{X}, \tag{34}$$

(iii) *there exist $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$ such that*

$$v(x) = \alpha u(x) + \beta \quad \textit{for} \quad x \in \mathbb{R} \tag{35}$$

and

$$\nu = \bar{\mu} \quad \textit{whenever} \quad u(x)v(x) < 0 \ \textit{for some} \ x \in \mathbb{R}. \tag{36}$$

Proof. In order to prove the implication $(i) \Rightarrow (iii)$ assume that (33) holds. The proof is based on some ideas in [11]. Suppose that

$$u(x)v(x) \geq 0 \ \text{ for } \ x \in \mathbb{R}. \tag{37}$$

Let $I^- := \{x \in \mathbb{R} : u(x) < 0 \text{ and } v(x) < 0\}$ and $I^+ := \{x \in \mathbb{R} : u(x) > 0 \text{ and } v(x) > 0\}$. Making use of (29), for every $x, y \in I^+$ with $x < y$, we obtain

$$v^{-1}((1-\mu(S\backslash A))v(x) + \mu(S\backslash A)v(y)) = u^{-1}((1-\mu(S\backslash A))u(x) + \mu(S\backslash A)u(y)), \tag{38}$$

and, for every $x, y \in I^-$ with $x < y$, we obtain

$$v^{-1}(\nu(A)v(x) + (1-\nu(A))v(y)) = u^{-1}(\nu(A)u(x) + (1-\nu(A))u(y)). \tag{39}$$

Therefore, defining $f : u(\mathbb{R}) \to \mathbb{R}$ in the following way

$$f(x) = (v \circ u^{-1})(x) \ \text{ for } \ x \in u(\mathbb{R}) \tag{40}$$

and replacing in (38) and (39) x by $u^{-1}(x)$ and y by $u^{-1}(y)$, we obtain

$$f((1-\mu(S\backslash A))x+\mu(S\backslash A)y) = (1-\mu(S\backslash A))f(x)+\mu(S\backslash A)f(y)) \text{ for } x,y \in u(I^+),\ x<y \tag{41}$$

and

$$f(\nu(A)x+(1-\nu(A))y) = \nu(A)f(x)+(1-\nu(A))f(y)) \text{ for } x,y \in u(I^-),\ x<y, \tag{42}$$

respectively. Furthermore, because u and v are strictly increasing and continuous, so is f. Moreover, in view of (37), we have either $I^- = \mathbb{R}$ or $I^+ = \mathbb{R}$ or $I^- = (-\infty, z)$ and $I^+ = (z, \infty)$ for some $z \in \mathbb{R}$. If $I^+ = \mathbb{R}$, while taking into account (32) and applying Lemma 3, we conclude that there exist $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$, such that

$$f(x) = \alpha x + \beta \quad \text{for} \quad x \in u(\mathbb{R}). \tag{43}$$

Hence, in view of (40), we obtain (35). If $I^- = \mathbb{R}$, the same conclusion follows from (39). In the case where $I^- = (-\infty, z)$ and $I^+ = (z, \infty)$ for some $z \in \mathbb{R}$, making use of (32) and applying Lemma 3, we obtain that there exist $\alpha^+, \alpha^- \in (0,\infty)$ and $\beta^+, \beta^- \in \mathbb{R}$, such that

$$v(x) = \begin{cases} \alpha^- u(x) + \beta^- & \text{for} \quad x \in (z,\infty), \\ \alpha^+ u(x) + \beta^+ & \text{for} \quad x \in (-\infty, z) \end{cases} \tag{44}$$

Moreover, as u and v are continuous, we get $u(z) = v(z) = 0$. Hence, we have $\beta^- = \beta^+ = 0$, hence

$$v(x) = \begin{cases} \alpha^- u(x) & \text{for} \quad x \in (z,\infty), \\ \alpha^+ u(x) & \text{for} \quad x \in (-\infty, z). \end{cases} \tag{45}$$

Additionally, note that making use of (29) and (33), for every $x \in (-\infty, z)$ and $y \in (z, \infty)$, we get

$$v^{-1}(\nu(A)v(x)+\mu(S\backslash A)v(y)) = u^{-1}(\nu(A)u(x)+\mu(S\backslash A)u(y)).$$

Hence, in view of (45), for every $x \in (-\infty, z)$ and $y \in (z, \infty)$, we obtain

$$v^{-1}(\nu(A)\alpha^- u(x)+\mu(S\backslash A)\alpha^+ u(y)) = u^{-1}(\nu(A)u(x)+\mu(S\backslash A)u(y)). \tag{46}$$

Furthermore, for every $x \in (-\infty, z)$ and $y \in (z, \infty)$ sufficiently close to z, we have

$$\nu(A)\alpha^- u(x)+\mu(S\backslash A)\alpha^+ u(y) < 0,$$

that is

$$u^{-1}(\nu(A)\alpha^- u(x)+\mu(S\backslash A)\alpha^+ u(y)) \in (-\infty, z).$$

Therefore, applying (45), from (46), we derive that

$$\alpha^-\nu(A)u(x)+\alpha^+\mu(S\backslash A)u(y) = \alpha^-\nu(A)u(x)+\alpha^-\mu(S\backslash A)u(y)$$

for every $x \in (-\infty, z)$ and $y \in (z, \infty)$ sufficiently close to z. Hence, in view of (32), we get $\alpha^+ = \alpha^-$ and, so, while taking into account (45), we obtain (35) with $\alpha := \alpha^+ = \alpha^-$ and $\beta = 0$.

Now, suppose that (37) does not hold. Subsequently, as u and v are continuous, there exists a non-degenerate interval $J \subseteq \mathbb{R}$, such that $u(x)v(x) < 0$ for $x \in J$. Thus, either

$$u(J) \subset (0,\infty) \text{ and } v(J) \subset (-\infty, 0)$$

or

$$u(J) \subset (-\infty, 0) \text{ and } v(J) \subset (0,\infty).$$

In the first case, in view of (29), we obtain

$$v^{-1}(\nu(B)v(x) + (1-\nu(B))v(y)) = u^{-1}((1-\mu(S\backslash B))u(x) + \mu(S\backslash B)u(y))$$

for every $B \in \mathcal{F}$ and $x, y \in J$, such that $x < y$. Hence, replacing x by $u^{-1}(x)$ and y by $u^{-1}(y)$, for every $B \in \mathcal{F}$ and $x, y \in u(J)$ with $x < y$, we get

$$f((1-\mu(S\backslash B))x + \mu(S\backslash B)y) = \nu(B)f(x) + (1-\nu(B))f(y)$$

where $f : u(J) \to \mathbb{R}$ is given by (40). Accordingly, as f is continuous, applying Lemma 3, we conclude that $\nu(B) = 1 - \mu(S\backslash B)$ for $B \in \mathcal{F}$, which is $\nu = \bar{\mu}$. Using the similar arguments, in the second case, for every $B \in \mathcal{F}$ and $x, y \in J$, such that $x < y$, we obtain

$$f(\mu(B)x + (1-\mu(B))y) = (1-\nu(S\backslash B))f(x) + \nu(S\backslash B)f(y).$$

Hence, according to Lemma 3, we obtain $\mu(B) = 1 - \nu(S\backslash B)$ for $B \in \mathcal{F}$, which again gives $\nu = \bar{\mu}$.

Because $\nu = \bar{\mu}$, from (29) and (33), we derive that

$$v^{-1}((1-\mu(S\backslash A))v(x) + \mu(S\backslash A)v(y)) = u^{-1}((1-\mu(S\backslash A))u(x) + \mu(S\backslash A)u(y))$$

for every $x, y \in \mathbb{R}$, such that $x < y$. Thus,

$$f((1-\mu(S\backslash B))x + \mu(S\backslash B)y) = \nu(B)f(x) + (1-\nu(B))f(y)$$

for every $x, y \in \mathbb{R}$ with $x < y$, where f is given by (40). Hence, taking (32) into account and applying Lemma 3, we obtain that there exist $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$, such that (43) holds. Hence , in view of (40), we get (35). In this way, we proved the implication $(i) \Rightarrow (iii)$.

Now, we prove that $(iii) \Rightarrow (ii)$. Assume that (iii) holds. Subsequently, by (35), we have

$$v^{-1}(x) = u^{-1}\left(\frac{1}{\alpha}(x-\beta)\right) \quad \text{for} \quad x \in v(\mathbb{R}). \tag{47}$$

If (37) is valid, then, as we have already noted, either $I^- = \mathbb{R}$ or $I^+ = \mathbb{R}$ or $I^- = (-\infty, z)$ and $I^+ = (z, \infty)$ for some $z \in \mathbb{R}$. In the first case, in view of (13) and (18), for every $X \in \mathcal{X}$, we have (27) and $E_{\mu\nu}[v(X)] = E_{\bar{\nu}}[v(X)]$. Thus, using the fact that Choquet integral is positively homogeneous and translative, taking into account (3), (35), and (47), for every $X \in \mathcal{X}$, we obtain

$$M_{(v,\mu,\nu)}(X) = v^{-1}\left(E_{\mu\nu}[v(X)]\right) = v^{-1}\left(E_{\bar{\nu}}[v(X)]\right) = u^{-1}\left(\frac{1}{\alpha}\left(E_{\bar{\nu}}[\alpha u(X) + \beta] - \beta\right)\right)$$

$$= u^{-1}\left(E_{\bar{\nu}}[u(X)]\right) = u^{-1}\left(E_{\mu\nu}[u(X)]\right) = M_{(u,\mu,\nu)}(X).$$

In the second case, in view of (18), for every $X \in \mathcal{X}$, we have (28) and $E_{\mu\nu}[v(X)] = E_{\mu}[v(X)]$. Hence, arguing, as previously, we obtain (34). In the third case from the continuity of u and v, we derive that $u(z) = v(z) = 0$. Hence, in view of (35), we get $\beta = 0$ and, so, as the generalized Choquet integral is positively homogeneous, making use of (47), for every $X \in \mathcal{X}$, we obtain

$$M_{(v,\mu,\nu)}(X) = v^{-1}\left(E_{\mu\nu}[v(X)]\right) = u^{-1}\left(\frac{1}{\alpha}E_{\mu\nu}\alpha[u(X)]\right) = u^{-1}\left(E_{\mu\nu}[u(X)]\right) = M_{(u,\mu,\nu)}(X).$$

If (37) does not hold then, according to (36), we have $\nu = \bar{\mu}$. Therefore, taking (19), (35), and (47) into account, for every $X \in \mathcal{X}$, we have obtain

$$M_{(v,\mu,\nu)}(X) = M_{(v,\mu,\bar{\mu})}(X) = v^{-1}\left(E_{\mu\bar{\mu}}[v(X)]\right) = v^{-1}\left(E_{\mu}[v(X)]\right) = u^{-1}\left(\frac{1}{\alpha}\left(E_{\mu}[\alpha u(X) + \beta] - \beta\right)\right)$$

$$= u^{-1}\left(E_{\mu}[u(X)]\right) = u^{-1}\left(E_{\mu\bar{\mu}}[u(X)]\right) = M_{(u,\mu,\bar{\mu})}(X) = M_{(u,\mu,\nu)}(X).$$

This proves that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

Applying Theorem 2, we are going to characterize positive homogeneity and translativity in the class of means given by (3).

Theorem 3. *Let $u : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Assume that there exists a set $A \in \mathcal{F}$, such that (32) holds. Subsequently, the following conditions are equivalent:*

(i)

$$M_{(u,\mu,\nu)}(tX) = tM_{(u,\mu,\nu)}(X) \ \ for \ \ X \in \mathcal{X}^{(2)},\ t \in (0,\infty), \tag{48}$$

(ii)

$$M_{(u,\mu,\nu)}(tX) = tM_{(u,\mu,\nu)}(X) \ \ for \ \ X \in \mathcal{X},\ t \in (0,\infty), \tag{49}$$

(iii) *there exist $\alpha, \beta, r \in (0,\infty)$ and $\delta \in \mathbb{R}$, such that*

$$u(x) = \begin{cases} -\beta(-x)^r + \delta \ \ for \ \ x \in (-\infty, 0), \\ \alpha x^r + \delta \ \ for \ \ x \in [0,\infty) \end{cases} \tag{50}$$

and

$$\nu = \bar{\mu} \quad whenever \quad \delta \neq 0. \tag{51}$$

Proof. Assume that (i) holds. Let, for every $t \in (0,\infty)$, $u_t : \mathbb{R} \to \mathbb{R}$, be given by

$$u_t(x) = u(tx) \ \ \text{for} \ \ x \in \mathbb{R}. \tag{52}$$

Subsequently,

$$u_t^{-1}(x) = \frac{1}{t}u^{-1}(x) \ \ \text{for} \ \ x \in u(\mathbb{R}),\ t \in (0,\infty).$$

Moreover, while using the positive homogeneity of the generalized Choquet integral, in view of (48), for every $X \in \mathcal{X}^{(2)}$ and $t \in (0,\infty)$, we obtain

$$M_{(u_t,\mu,\nu)}(X) = u_t^{-1}\left(E_{\mu\nu}[u_t(X)]\right) = \frac{1}{t}u^{-1}\left(E_{\mu\nu}[u(tX)]\right) = \frac{1}{t}M_{(u,\mu,\nu)}(tX) = M_{(u,\mu,\nu)}(X).$$

Therefore

$$M_{(u_t,\mu,\nu)}(X) = M_{(u,\mu,\nu)}(X) \ \ \text{for} \ \ X \in \mathcal{X}^{(2)},\ t \in (0,\infty). \tag{53}$$

Hence, applying Theorem 2, we conclude that, for every $t \in (0,\infty)$, there exist $a(t) \in (0,\infty)$ and $b(t) \in \mathbb{R}$, such that

$$u_t(x) = a(t)u(x) + b(t) \ \ \text{for} \ \ x \in \mathbb{R}.$$

Thus, in view of (52), we obtain

$$u(tx) = a(t)u(x) + b(t) \ \ \text{for} \ \ x \in \mathbb{R},\ t \in (0,\infty). \tag{54}$$

In particular, we have

$$u(tx) = a(t)u(x) + b(t) \ \ \text{for} \ \ (x,t) \in (0,\infty)^2.$$

Therefore, applying [15] (Corollary 3), we obtain

$$u(x) = \alpha m(x) + \delta \ \ for \ \ x \in (0,\infty) \tag{55}$$

where $\alpha \in (0,\infty)$, $\delta \in \mathbb{R}$ and $m : (0,\infty) \to \mathbb{R}\backslash\{0\}$ is a multiplicative function, that is

$$m(x+y) = m(x)m(y) \quad \text{for} \quad x, y \in (0,\infty).$$

Moreover, as u is continuous and strictly increasing, in view of (55), so is m. Thus, according to [16] (Theorem 13.1.6), there exists $r \in (0,\infty)$, such that $m(x) = x^r$ for $x \in (0,\infty)$. Hence, making use of (55) and using the continuity of u again, we obtain

$$u(x) = \alpha x^r + \delta \quad for \quad x \in [0,\infty). \tag{56}$$

Therefore, setting in (54) $x = 0$ and then $x = 1$, after a standard computation, we obtain that $a(t) = t^r$ for $t \in (0,\infty)$ and $b(t) = \delta(1-t^r)$ for $t \in (0,\infty)$. Thus, putting in (54) $x = -1$, we obtain

$$u(-t) = (u(-1) - \delta)t^r + \delta \quad \text{for} \quad t \in (0,\infty).$$

Hence, when taking $\beta := \delta - u(-1)$, we obtain

$$u(x) = -\beta(-x)^r + \delta \quad for \quad x \in (-\infty, 0)$$

which, together with (56), gives (50). Because u is strictly increasing, we also have $\beta \in (0,\infty)$.

Assume that $\delta \neq 0$. Because u a bijection on $\mathbb{R}$, it has a unique zero, say x_0. Moreover, as $\delta \neq 0$, in view of (50), $x_0 \neq 0$. Furthermore, taking an arbitrary $t \in (0,\infty)\backslash\{1\}$ and making use of (52), we conclude that $\frac{x_0}{t}$ is a unique zero of u_t. Thus, u and u_t have different zeroes and, so, as they are strictly increasing, we have $u_t(x)u(x) < 0$ for every x between these zeroes. Therefore, while taking (53) into account and applying Theorem 2, we obtain $\nu = \bar{\mu}$. Consequently, (51) holds and, so, the implication $(i) \Rightarrow (iii)$ is proven.

In order to prove the implication $(iii) \Rightarrow (ii)$, assume that u is of the form (50) with some $\alpha, \beta, r \in (0,\infty)$, $\delta \in \mathbb{R}$ and (51) holds. It follows from (50) that

$$u(tx) = t^r u(x) + \delta(1-t^r) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0,\infty) \tag{57}$$

and

$$u^{-1}(t^r x + \delta(1-t^r)) = tu^{-1}(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in u(0,\infty). \tag{58}$$

Therefore, if $\delta = 0$, then, while using the positive homogeneity of the Choquet integral, in view of (3), for every $X \in \mathcal{X}$ and $t \in (0,\infty)$, we obtain

$$M_{(u,\mu,\nu)}(tX) = u^{-1}\left(E_{\mu\nu}[u(tX)]\right) = u^{-1}\left(E_{\mu\nu}[t^r u(X)]\right)$$

$$= u^{-1}\left(t^r E_{\mu\nu}[u(X)]\right) = tu^{-1}\left(E_{\mu\nu}[u(X)]\right) = tM_{(u,\mu,\nu)}(X)$$

which shows that $M_{(u,\mu,\nu)}$ is positively homogeneous.

If $\delta \neq 0$, then, according to (51), we have $\nu = \bar{\mu}$. Hence,as the Choquet integral is positively homogeneous and translative, in view of (19) and (57)–(58), for every $X \in \mathcal{X}$ and $t \in (0,\infty)$, we get

$$M_{(u,\mu,\nu)}(tX) = M_{(u,\mu,\bar{\mu})}(tX) = u^{-1}\left(E_{\mu\bar{\mu}}[u(tX)]\right) = u^{-1}\left(E_{\mu}[u(tX)]\right)$$

$$= u^{-1}\left(E_{\mu}[t^r u(X) + \delta(1-t^r)]\right) = u^{-1}\left(t^r E_{\mu}[u(X)] + \delta(1-t^r)\right)$$

$$= tu^{-1}\left(E_{\mu}[u(X)]\right) = tu^{-1}\left(E_{\mu\bar{\mu}}[u(X)]\right) = tM_{(u,\mu,\bar{\mu})}(X) = tM_{(u,\mu,\nu)}(X).$$

In this way, we have proved that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

Theorem 4. *Let $I \subseteq \mathbb{R}$ be a non-degenerate interval and let $u : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Assume that there exists a set $A \in \mathcal{F}$, such that (32) holds. Subsequently, the following conditions are equivalent:*

(i)
$$M_u(X+t) = M_u(X) + t \ \ for \ \ X \in \mathcal{X}^{(2)},\ t \in I, \tag{59}$$

(ii)
$$M_u(X+t) = M_u(X) + t \ \ for \ \ X \in \mathcal{X},\ t \in I, \tag{60}$$

(iii) one of the subsequent possibilities holds:

(a) $\nu = \bar{\mu}$ and there exist $\alpha \in (0,\infty)$, $\beta \in \mathbb{R}$ such that
$$u(x) = \alpha x + \beta \ for \ x \in \mathbb{R}, \tag{61}$$

(b) there exist $\alpha, \beta, r \in \mathbb{R}$ with $\alpha r > 0$, such that
$$u(x) = \alpha e^{rx} + \beta \ for \ x \in \mathbb{R} \tag{62}$$

and
$$\nu = \bar{\mu} \quad whenever \quad \alpha\beta < 0. \tag{63}$$

Proof. Assume that (i) holds. For every $t \in I$ define a function $u_t : \mathbb{R} \to \mathbb{R}$ in the following way
$$u_t(x) = u(x+t) \ \ \text{for} \ \ x \in \mathbb{R}. \tag{64}$$
Subsequently, for every $t \in I$ u_t is strictly increasing and continuous. Moreover, we have
$$u_t^{-1}(x) = u^{-1}(x) - t \ \ \text{for} \ \ x \in u(\mathbb{R}).$$
Thus, taking (59) into account, for every $X \in \mathcal{X}^{(2)}$ and $t \in I$, we obtain
$$M_{(u_t,\mu,\nu)}(X) = u_t^{-1}\left(E_{\mu\nu}[u_t(X)]\right) = u^{-1}\left(E_{\mu\nu}[u(X+t)]\right) - t = M_{(u,\mu,\nu)}(X+t) - t = M_{(u,\mu,\nu)}(X).$$
Hence,
$$M_{(u_t,\mu,\nu)}(X) = M_{(u,\mu,\nu)}(X) \ \ \text{for} \ \ X \in \mathcal{X}^{(2)},\ t \in I. \tag{65}$$
Therefore, according to Theorem 2, for every $t \in I$, there exist $\alpha(t) \in (0,\infty)$ and $\beta(t) \in \mathbb{R}$, such that
$$u_t(x) = \alpha(t)u(x) + \beta(t) \ \ \text{for} \ \ x \in \mathbb{R}$$
which, in view of (64), gives
$$u(x+t) = \alpha(t)u(x) + \beta(t) \ \ \text{for} \ \ x \in \mathbb{R},\ t \in I.$$
Thus, applying [17] (Theorem 2), we conclude that either (61) holds with some $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$, or (62) is valid with some $\alpha, \beta, r \in \mathbb{R}$, such that $\alpha r > 0$.

If u is of the form (61) or it is of the form (62) with $\alpha\beta < 0$ then it has a unique zero, say x_0. Hence, taking a $t \in I\backslash\{0\}$, we obtain that $x_0 - t$ is a unique zero of u_t. Therefore, as u and u_t are strictly increasing, we have $u_t(x)u(x) < 0$ for every x belonging to the interval with endpoints x_0 and $x_0 - t$. Thus, taking (65) into account and applying Theorem 2, we obtain $\nu = \bar{\mu}$, which proves the implication $(i) \Rightarrow (iii)$.

Now, we prove that $(iii) \Rightarrow (ii)$. Assume that (iii) holds. In the case of (a), we have

$$u^{-1}(x) = \frac{1}{\alpha}(x - \beta) \quad \text{for} \quad x \in \mathbb{R}.$$

Hence, as the Choquet integral is positively homogeneous and translative, taking into account (3) and (19), for every $X \in \mathcal{X}$, we obtain

$$M_{(u,\mu,\nu)}(X) = M_{(u,\mu,\bar{\mu})}(X) = u^{-1}\left(E_{\mu\bar{\mu}}[u(X)]\right) = \frac{1}{\alpha}(E_\mu[\alpha X + \beta] - \beta) = E_\mu[X].$$

Therefore, again using the translativity of the Choquet integral, we obtain (60).

In the case of (b), we get

$$u^{-1}(x) = \frac{1}{r}\ln\left(\frac{x-\beta}{\alpha}\right) \quad \text{for} \quad x \in u(\mathbb{R}). \tag{66}$$

If $\alpha\beta < 0$, then $\nu = \bar{\mu}$ and, so, as the Choquet integral is positively homogeneous an translative, making use of (3), (19) and (66), for every $X \in \mathcal{X}$, we obtain

$$M_{(u,\mu,\nu)}(X) = M_{(u,\mu,\bar{\mu})}(X) = u^{-1}\left(E_{\mu\bar{\mu}}[u(X)]\right) = \frac{1}{r}\ln\left(\frac{E_\mu[\alpha e^{rX} + \beta] - \beta)}{\alpha}\right) = \frac{1}{r}\ln\left(E_\mu[e^{rX}]\right).$$

Thus, while using the positive homogeneity of the Choquet integral, for every $X \in \mathcal{X}$ and $t \in I$, we obtain

$$\begin{aligned} M_{(u,\mu,\nu)}(X+t) &= \frac{1}{r}\ln\left(E_\mu[e^{r(X+t)}]\right) = \frac{1}{r}\ln\left(E_\mu[e^{rt}e^{rX}]\right) = \frac{1}{r}\ln\left(e^{rt}E_\mu[e^{rX}]\right) \\ &= \frac{1}{r}\left(rt + \ln\left(E_\mu[e^{rX}]\right)\right) = \frac{1}{r}\ln\left(E_\mu[e^{rX}]\right) + t = M_{(u,\mu,\nu)}(X) + t, \end{aligned}$$

that is (60) holds. If $\alpha\beta \geq 0$ then either $u(\mathbb{R}) \subset (0,\infty)$ or $u(\mathbb{R}) \subset (-\infty, 0)$. In the first case, by (18), we have (28). Hence, for every $X \in \mathcal{X}$, we obtain

$$M_{(u,\mu,\nu)}(X) = u^{-1}\left(E_{\mu\nu}[u(X)]\right) = \frac{1}{r}\ln\left(\frac{E_\mu[\alpha e^{rX} + \beta] - \beta)}{\alpha}\right) = \frac{1}{r}\ln\left(E_\mu[e^{rX}]\right).$$

Thus, as we have already noted, (60) holds. In the latter case, according to (18), we have (27). Therefore, in a similar way, we get

$$M_{(u,\mu,\nu)}(X) = \frac{1}{r}\ln\left(E_{\bar{\nu}}[e^{rX}]\right) \quad \text{for} \quad X \in \mathcal{X}$$

which implies (60). In this way we have proved that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

Remark 11. *Applying Remark 8, from Theorems 2–4, one can easily deduce the characterizations of the equality, positive homogeneity, and translativity for the functional that is defined by* (2)*. Furthermore, Theorems 2–4 generalize the results in [11] (Theorems 3.1–3.3).*

5. Conclusions

We have introduced a new class of quasi-arithmetic type means generated by the generalized Choquet Integral. Our approach is motivated by recent applications of this type of means in the theory of decision making under risk and in the theory of insurance premium principles. Using the methods of functional equations, we established characterizations of some important properties in the

considered class of means. It is remarkable that the aforementioned characterizations are expressed not only in terms of the relations between functions generating means, but they involve also the properties of capacities under consideration.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Hardy, G.H.; Littlewood, J.E.; Pólya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1952.
2. De Finetti, B. Sul concetto di media. *G. Dell Inst. Ital. Degli Attuarii* **1931**, *2*, 369–396.
3. Kolmogorov, A.N. Sur la notion de la moyenne. *Rend. Accad. dei Lincei* **1930**, *12*, 388–391.
4. Nagumo, M. Über eine Klasse der Mittelwerte. *Jpn. J. Math.* **1930**, *7*, 71–79. [CrossRef]
5. Bajraktarević, M. Sur une generalisation des moyennes quasilineaires. *Publ. Inst. Math. (Beogr.) (N.S.)* **1963**, *3*, 69–76.
6. Páles, Z. On the characterization of quasi-arithmetic means with weight function. *Aequationes Math.* **1987**, *32*, 171–194. [CrossRef]
7. Daróczy, Z. A general inequality for means. *Aequationes Math.* **1971**, *7*, 16–21. [CrossRef]
8. Daróczy, Z. Uber eine Klasse von Mittelwerten. *Publ. Math. Debr.* **1973**, *19*, 211–217.
9. Von Neumann, J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University Press: Princeton, NJ, USA, 2007.
10. Schmeidler, D. Subjective probability and expected utility without additivity. *Econometrica* **1989**, *57*, 571–587. [CrossRef]
11. Chudziak, J. Certainty Equivalent Under Cumulative Prospect Theory. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2018**, *22* 131–295. [CrossRef]
12. Choquet, G. Theory of capacities. *Ann. Inst. Fourier* **1953**, *5*, 131–295. [CrossRef]
13. Denneberg, D. *Lectures on Non-Additive Measure and Integral*; Kluwer Academic Publishers: Boston, MA, USA, 1994.
14. Tversky, A.; Kahneman, D. Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertain.* **1992**, *5*, 297–323. [CrossRef]
15. Sobek, B. Pexider equation on restricted domain. *Demonstr. Math.* **2010**, *1*, 87–88.
16. Kuczma, M. *An Introduction to the Theory of Functional Equations and Inequalities*; Birkhäuser Basel: Berlin, Germany, 2009.
17. Aczél, J. Extension of a generalized Pexider equation. *Proc. Am. Math. Soc.* **2005**, *133*, 3227–3233. [CrossRef]