Rational Type Contractions in Extended $b$-Metric Spaces

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Abstract: In this paper, we establish the existence of fixed points of rational type contractions in the setting of extended $b$-metric spaces. Our results extend considerably several well-known results in the existing literature. We present some nontrivial examples to show the validity of our results. Furthermore, as applications, we obtain the existence of solution to a class of Fredholm integral equations.

Keywords: comparison function; a-admissible; rational type contraction; extended $b$-metric space; Fredholm integral equation

1. Introduction and Preliminaries

The concept of distance between two abstract objects has received importance not only for mathematical analysis but also for its related fields. Bakhtin [1] introduced $b$-metric spaces as a generalization of metric spaces (see also Czerwik [2]). Recently, Kamran et al. [3] gave the notion of extended $b$-metric space and presented a counterpart of Banach contraction mapping principle. On the other hand, fixed point results dealing with general contractive conditions involving rational type expression are also interesting. Some well-known results in this direction are involved (see [4–10]).

First, of all, we recall some fixed point theorems for rational type contractions in metric spaces.

Theorem 1 ([5]). Let $T$ be a continuous self mapping on a complete metric space $(X, d)$. If $T$ is a rational type contraction, there exist $\alpha, \beta \in [0, 1)$, where $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(x, Tx)d(y, Ty)}{d(x, y)},$$

for all $x, y \in X$, $x \neq y$, then $T$ has a unique fixed point in $X$.

Theorem 2 ([4]). Let $T$ be a continuous self mapping on a complete metric space $(X, d)$. If $T$ is a rational type contraction, there exist $\alpha, \beta \in [0, 1)$, where $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)},$$

for all $x, y \in X$, then $T$ has a unique fixed point in $X$.

Theorem 3 ([11]). Let $T$ be a self mapping on a complete metric space $(X,d)$. If $T$ is a rational type contraction, $T$ satisfies the inequality
\[ d(Tx, Ty) \leq k \left\{ \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty) + d(y,Tx)} \right\}, \]
for all $x, y \in X$, where $0 \leq k < 1$. Then, $T$ has a unique fixed point in $X$.


Theorem 4 ([14]). Let $T$ be a self mapping on a complete generalized metric space $(X,d_g)$. If $T$ is a rational type contraction, $T$ satisfies the inequality
\[ d_g(Tx, Ty) \leq k \left\{ \max \left\{ d_g(x,y), \frac{d_g(x,Tx)d_g(x,Ty) + d_g(y,Ty)d_g(y,Tx)}{A_0(x,y)} \right\} \right\}, \]
for all $x, y \in X, x \neq y$, where $0 \leq k < 1$ and $A_0(x,y) = \max\{d_g(x,Ty),d_g(y,Tx)\}$. Then, $T$ has a unique fixed point in $X$.

Let us recall some basic concepts in $b$-metric spaces as follows.

Definition 1 ([1,2]). Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to [0, +\infty)$ is called a $b$-metric on $X$, if, for all $x, y, z \in X$, the following conditions hold:
\begin{enumerate}
\item[(d_b1)] $d_b(x, y) = 0$ if and only if $x = y$;
\item[(d_b2)] $d_b(x, y) = d_b(y, x)$;
\item[(d_b3)] $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$.
\end{enumerate}

In this case, the pair $(X,d_b)$ is called a $b$-metric space.

It is well-known that any $b$-metric space will become a metric space if $s = 1$. However, any metric space does not necessarily be a $b$-metric space if $s > 1$. In other words, $b$-metric spaces are more general than metric spaces (see [15]).

The following example gives us evidence that $b$-metric space is indeed different from metric space.

Example 1 ([16]). Let $(X,d)$ be a metric space and $d_b(x, y) = (d(x, y))^p$ for all $x, y \in X$, where $p > 1$ is a real number. Then, $(X,d_b)$ is a $b$-metric space with $s = 2^{p-1}$. However, $(X,d_b)$ is not a metric space.

Definition 2 ([17]). Let $\{x_n\}$ be a sequence in a $b$-metric space $(X,d_b)$. Then,
\begin{enumerate}
\item[(i)] $\{x_n\}$ is called a convergent sequence, if, for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_b(x_n, x) < \epsilon$, for all $n \geq n_0$, and we write $\lim_{n \to n_0} x_n = x$;
\item[(ii)] $\{x_n\}$ is called a Cauchy sequence, if, for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_b(x_n, x_m) < \epsilon$, for all $n, m \geq n_0$;
\item[(iii)] $(X,d_b)$ is said to be complete if every Cauchy sequence is convergent in $X$.
\end{enumerate}

The following theorem is a basic theorem for Banach type contraction in $b$-metric space.

Theorem 5 ([18]). Let $T$ be a self mapping on a complete $b$-metric space $(X,d_b)$. Then, $T$ has a unique fixed point in $X$ if
\[ d_b(Tx, Ty) \leq kd_b(x, y) \]
Definition 3 is satisfied. (see [15,19]). For fixed point results and more examples in b-space, and extended b-space reduces to a refer to [15–18].

Let \( X = [0, +\infty) \). Define two mappings \( \theta : X \times X \to [1, +\infty) \) and \( d_\theta : X \times X \to [0, +\infty) \) as follows: \( \theta(x, y) = 1 + x + y \), for all \( x, y \in X \), and

\[
d_\theta(x, y) = \begin{cases} x + y, & x, y \in X, x \neq y, \\ 0, & x = y. \end{cases}
\]

Then, \((X, d_\theta)\) is an extended b-metric space.

Indeed, \((d_\theta1)\) and \((d_\theta2)\) in Definition 3 are clear. Let \( x, y, z \in X \). We prove that \((d_\theta3)\) in Definition 3 is satisfied.

(i) If \( x = y \), then \((d_\theta3)\) is clear.

(ii) If \( x \neq y, x = z \), then

\[
\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + x + y)[0 + (z + y)]
\]

\[
= (1 + x + y)(x + y)
\]

\[
\geq x + y = d_\theta(x, y).
\]

(iii) If \( x \neq y, y = z \), then

\[
\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + x + y)[(x + z) + 0]
\]

\[
= (1 + x + y)(x + y)
\]

\[
\geq x + y = d_\theta(x, y).
\]

(iv) If \( x \neq y, y \neq z, x \neq z \), then

\[
\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + x + y)[(x + z) + (z + y)]
\]

\[
\geq x + 2z + y
\]

\[
\geq x + y = d_\theta(x, y).
\]

Consider the above cases, it follows that \((d_\theta3)\) holds. Hence, the claim holds.
Example 3. Let $X = \mathbb{R}$. Define two mappings $\theta : X \times X \to [1, +\infty)$ and $d_\theta : X \times X \to [0, +\infty)$ as follows: $\theta(x, y) = 1 + |x| + |y|$, for all $x, y \in X$ and

$$d_\theta(x, y) = \begin{cases} x^2 + y^2, & x, y \in X, x \neq y, \\ 0, & x = y. \end{cases}$$

Then, $(X, d_\theta)$ is an extended b-metric space.

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ in Definition 3 are obvious. Let $x, y, z \in X$. We prove that $(d_\theta 3)$ in Definition 3 is satisfied.

(i) If $x = y$, then $(d_\theta 3)$ is obvious.

(ii) If $x \neq y, x = z$, then

$$\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + |x| + |y|)[0 + (z^2 + y^2)]$$

$$= (1 + |x| + |y|)(x^2 + y^2)$$

$$\geq x^2 + y^2 = d_\theta(x, y).$$

(iii) If $x \neq y, y = z$, then

$$\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + |x| + |y|)[(x^2 + z^2) + 0]$$

$$= (1 + |x| + |y|)(x^2 + y^2)$$

$$\geq x^2 + y^2 = d_\theta(x, y).$$

(iv) If $x \neq y, y \neq z, x \neq z$, then

$$\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + |x| + |y|)[(x^2 + z^2) + (z^2 + y^2)]$$

$$\geq (1 + |x| + |y|)(x^2 + y^2)$$

$$\geq x^2 + y^2 = d_\theta(x, y).$$

Consider the above cases, it follows that $(d_\theta 3)$ holds. Hence, the claim holds.

Example 4. Let $X = \mathbb{R}$. Define two mappings $d_\theta : X \times X \to [0, +\infty)$ and $\theta : X \times X \to [1, +\infty)$ as follows:

$$d_\theta(x, y) = \begin{cases} \frac{|x| + |y|}{1 + |x| + |y|}, & x, y \in X, x \neq y, \\ 0, & x = y. \end{cases}$$

and $\theta(x, y) = 1 + |x| + |y|$, for all $x, y \in X$. Then, $(X, d_\theta)$ is an extended b-metric space.

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ in Definition 3 are valid. Let $x, y, z \in X$. We prove that $(d_\theta 3)$ in Definition 3 is satisfied.

(i) If $x = y$, then $(d_\theta 3)$ holds.

(ii) If $x \neq y, x = z$, then

$$\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + |x| + |y|)[0 + \frac{|z| + |y|}{1 + |z| + |y|}]$$

$$= (1 + |x| + |y|) \cdot \frac{|x| + |y|}{1 + |x| + |y|}$$

$$\geq \frac{|x| + |y|}{1 + |x| + |y|}$$

$$= d_\theta(x, y).$$
(iii) If \( x \neq y, y = z \), then
\[
\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + |x| + |y|)
\left(\frac{|x| + |z| + |y|}{1 + |x| + |z| + |y|} \theta \right)
\geq \frac{|x| + |z| + |y|}{1 + |x| + |z| + |y|} = d_\theta(x, y).
\]

(iv) If \( x \neq y, y \neq z, x \neq z \), then, by the fact that \( f(t) = \frac{t}{1 + t} \) is nondecreasing on \([0, +\infty)\) and \(|x| + |y| \leq |x| + |z| + |y|\), it follows that
\[
\theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] = (1 + |x| + |y|)
\left(\frac{|x| + |z| + |y|}{1 + |x| + |z| + |y|} \theta \right)
\geq \frac{|x| + |z| + |y|}{1 + |x| + |z| + |y|} = d_\theta(x, y).
\]

Consider the above cases, it follows that \((d_\theta3)\) holds. Hence, the claim holds.

**Example 5.** Let \( X = [0, +\infty) \) and \( \theta(x, y) = \frac{3 + x + y}{2} \) be a function on \( X \times X \). Define a mapping \( d_\theta : X \times X \to [0, +\infty) \) as follows:

\[
d_\theta(x, y) = 0, \text{ for all } x, y \in X, x = y,
\]

\[
d_\theta(x, y) = d_\theta(y, x) = 5, \text{ for all } x, y \in X \setminus \{0\}, x \neq y,
\]

\[
d_\theta(x, 0) = d_\theta(0, x) = 2, \text{ for all } x \in X \setminus \{0\}.
\]

Then, \((X, d_\theta)\) is an extended-b metric space.

As a matter of fact, obviously, \((d_\theta1)\) and \((d_\theta2)\) hold. For \((d_\theta3)\), we have the following cases:

(i) Let \( x, y, z \in X \setminus \{0\} \) such that \( x, y \) and \( z \) are distinct each other, then
\[
d_\theta(x, y) = 5 \leq 5(3 + x + y) = \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)].
\]

(ii) Let \( x, y \in X \setminus \{0\}, x \neq y \) and \( z = 0 \), then
\[
d_\theta(x, y) = 5 \leq 2(3 + x + y) = \theta(x, y)[d_\theta(x, 0) + d_\theta(0, y)].
\]

(iii) Let \( x, z \in X \setminus \{0\}, x \neq z \) and \( y = 0 \), then
\[
d_\theta(x, 0) = 2 \leq \frac{7}{2}(3 + x) = \theta(x, 0)[d_\theta(x, z) + d_\theta(z, y)].
\]

Therefore, \((d_\theta3)\) in Definition 3 holds. Thus, the claims hold.

**Remark 1.** Examples 2–5 are extended b-metric spaces but not b-metric spaces.

Similar to Definition 2, we recall some concepts in extended b-metric spaces as follows.

**Definition 4 ([3]).** Let \( \{x_n\} \) be a sequence in an extended b-metric space \((X, d_\theta)\). Then,
(i) \( \{x_n\} \) is called a convergent sequence, if, for each \( \epsilon > 0 \), there exists \( n_0 = n_0(\epsilon) \in \mathbb{N} \) such that \( d_0(x_n, x) < \epsilon \), for all \( n \geq n_0 \), and we write \( \lim_{n \to \infty} x_n = x \);
(ii) \( \{x_n\} \) is called a Cauchy sequence, if, for each \( \epsilon > 0 \), there exists \( n_0 = n_0(\epsilon) \in \mathbb{N} \) such that \( d_0(x_n, x_m) < \epsilon \), for all \( n, m \geq n_0 \);
(iii) \((X, d_0)\) is said to be complete if every Cauchy sequence is convergent in \( X \).

As we know, the limit of convergent sequence in extended \( b \)-metric space \((X, d_0)\) is unique provided that \( d_0 \) is a continuous mapping (see [3]).

Definition 5 ([20,21]). Let \( T \) be a self mapping on an extended \( b \)-metric space \((X, d_0)\). For \( x_0 \in X \), the set
\[
O(x_0, T) = \{x_0, Tx_0, T^2x_0, T^3x_0, \ldots \}
\]
is said to be an orbit of \( T \) at \( x_0 \). \( T \) is said to be orbitally continuous at \( \xi \in X \) if \( \lim_{k \to \infty} T^kx_0 = \xi \) implies \( \lim_{k \to \infty} TT^kx_0 = T\xi \). Moreover, if every Cauchy sequence of the form \( \{T^kx_0\}_{k=1}^{\infty} \) is convergent to some point in \( X \), then \((X, d_0)\) is said to be a \( T \)-orbitally complete space.

Note that, if \((X, d_0)\) is complete extended \( b \)-metric space, then \( X \) is \( T \)-orbitally complete for any self-mapping \( T \) on \( X \). Moreover, if \( T \) is continuous, then it is obviously orbitally continuous in \( X \). However, the converse may not be true.

In the sequel, unless otherwise specified, we always denote \( \text{Fix}(T) = \{x \in X | Tx = x\} \).

Definition 6 ([23]). Let \( X \) be a nonempty set and \( \alpha : X \times X \to \mathbb{R} \) be a mapping. A mapping \( T : X \to X \) is called \( \alpha \)-admissible, if for all \( x, y \in X \), \( \alpha(x, y) \geq 1 \) implies \( \alpha(Tx, Ty) \geq 1 \).

Definition 7 ([23]). Let \( X \) be a nonempty set and \( \alpha : X \times X \to \mathbb{R} \) be a mapping. Then, \( T : X \to X \) is called \( \alpha^* \)-admissible if it is a \( \alpha \)-admissible mapping and \( \alpha(x, y) \geq 1 \) holds for all \( x, y \in \text{Fix}(T) \neq \emptyset \).

Example 6. Let \( X = [0, +\infty) \) and \( T : X \to X \) be a mapping defined by \( Tx = \frac{x(1+x)}{2} \). Let \( \alpha : X \times X \to \mathbb{R} \) be a function defined by
\[
\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}
\]
Then, \( T \) is \( \alpha \)-admissible and \( \text{Fix}(T) = \{0, 1\} \). Moreover, \( \alpha(x, y) \geq 1 \) is satisfied for all \( x, y \in \text{Fix}(T) \). Consequently, \( T \) is \( \alpha^* \)-admissible.

Example 7 ([23]). Let \( X = [0, +\infty) \) and \( T : X \to X \) be a mapping defined by \( Tx = \sqrt{\frac{x(x^2+2)}{3}} \). Let \( \alpha : X \times X \to [0, +\infty) \) be a function defined by
\[
\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}
\]
Then, \( T \) is a \( \alpha \)-admissible mapping and \( \text{Fix}(T) = \{0, 1, 2\} \). However, \( \alpha(x, 2) = \alpha(2, x) = 0 \) is satisfied for \( x \in \{0, 1\} \). Thus, \( T \) is not \( \alpha^* \)-admissible.

Definition 8 ([24]). Let \( T \) be a self mapping on a nonempty set \( X \). Then, \( T \) is called \( \alpha \)-orbitally admissible if, for all \( x \in X \), \( \alpha(x, Tx) \geq 1 \) leads to \( \alpha(Tx, T^2x) \geq 1 \).

It is mentioned that each \( \alpha \)-admissible mapping must be an \( \alpha \)-orbitally admissible mapping (for more details, see [24]). For the uniqueness of fixed point, we will use the following definition frequently.
Definition 9. An $\alpha$-orbitally admissible mapping $T$ is called $\alpha$-orbitally admissible if $x, x^* \in \text{Fix}(T) \neq \emptyset$ implies $\alpha(x, x^*) \geq 1$.

Definition 10 ([17,25]). A function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be a comparison function, if it is nondecreasing and $\lim_{n \to \infty} \psi^n(t) = 0$ for all $t > 0$, where $\psi^n$ denotes the $n^{th}$ iteration of $\psi$.

In what follows, the set of all comparison functions is denoted by $\Psi$. Some examples for comparison functions, the reader may refer to [26].

Lemma 1 ([27]). Let $\psi \in \Psi$. Then, $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

The following lemmas will be used in the sequel.

Lemma 2 ([28]). Let $(X, d_\theta)$ be an extended $b$-metric space, $x_0 \in X$ and $\{x_n\}$ be a sequence in $X$. If $\psi \in \Psi$ satisfies

$$\lim_{n,m \to \infty} \frac{\theta(x_n, x_m)\psi^n(d_\theta(x_0, x_1))}{\psi^{n-1}(d_\theta(x_0, x_1))} < 1$$

(1)

and

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n))$$

for all $m > n \geq 2$, $n, m \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in $X$.

Proof. From the given conditions, we get

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)) \leq \cdots \leq \psi^n(d_\theta(x_0, x_1)).$$

On taking limit as $n \to \infty$, we have

$$\lim_{n \to \infty} d_\theta(x_n, x_{n+1}) = 0.$$

Setting $\theta_i = \theta(x_i, x_{i+p})$ for each $i \in \mathbb{N}$, $p \geq 1$ and $d_\theta(x_0, x_1) = t$, we obtain

$$d_\theta(x_n, x_{n+p}) \leq \theta(x_n, x_{n+p})[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+p})]$$

$$\leq \theta(x_n, x_{n+p})d_\theta(x_{n-1}, x_{n+1}) + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})d_\theta(x_{n+1}, x_{n+2})$$

$$\leq \theta(x_n, x_{n+p})d_\theta(x_{n-2}, x_{n+1}) \cdot \theta(x_{n+1}, x_{n+2}) + \cdots + \theta(x_n, x_{n+p})d_\theta(x_{n+p-2}, x_{n+p-1})$$

$$\leq \theta(x_n, x_{n+p})d_\theta(x_{n-3}, x_{n+1}) \cdot \theta(x_{n+1}, x_{n+2}) + \cdots + \theta(x_n, x_{n+p})d_\theta(x_{n+p-3}, x_{n+p-1})$$

$$\leq \theta(x_n, x_{n+p})d_\theta(x_{n-4}, x_{n+1}) \cdot \theta(x_{n+1}, x_{n+2}) + \cdots + \theta(x_n, x_{n+p})d_\theta(x_{n+p-4}, x_{n+p-1})$$

$$\leq \theta(x_n, x_{n+p})d_\theta(x_{n-5}, x_{n+1}) \cdot \theta(x_{n+1}, x_{n+2}) + \cdots + \theta(x_n, x_{n+p})d_\theta(x_{n+p-5}, x_{n+p-1})$$

$$\leq \theta_n \psi^n(d_\theta(x_0, x_1)) + \theta_n \theta_{n+1} \psi^{n-1}(d_\theta(x_0, x_1))$$

$$+ \cdots + \theta_n \theta_{n+p-1} \psi^{n-p}(d_\theta(x_0, x_1))$$

$$= \sum_{i=1}^{n+p-1} \psi(t) \prod_{j=1}^{i} \theta_j \leq \sum_{i=n}^{n+p-1} \psi(t) \prod_{j=1}^{i} \theta_j$$

$$= \sum_{i=n}^{n+p-1} \psi(t) \prod_{j=1}^{i} \theta_j \leq \sum_{i=n}^{n+p-1} \psi(t) \prod_{j=1}^{i} \theta_j.$$
Theorem 6. Let T be a self mapping on a T-orbitally complete extended b-metric space

\[ d \sum_{i=1}^{n+p-1} \psi^i(t) \prod_{j=1}^{i} \theta_j - \sum_{i=1}^{n-1} \psi^i(t) \prod_{j=1}^{i} \theta_i. \]

Notice that

\[ \lim_{n \to \infty} \frac{\theta(x_n, x_{n+p}) \psi^n(d_\theta(x_0, x_1))}{\psi^{n-1}(d_\theta(x_0, x_1))} = \lim_{n \to \infty} \frac{\theta_n \psi^n(t)}{\psi^{n-1}(t)} < 1, \]

then, by the Ratio test the series, \( \sum_{i=1}^{\infty} \psi^i(t) \prod_{j=1}^{i} \theta_j \) converges.

Let \( S = \sum_{i=1}^{\infty} \psi^i(t) \prod_{j=1}^{i} \theta_j \) and \( S_n = \sum_{i=1}^{n} \psi^i(t) \prod_{j=1}^{i} \theta_j \) be the sequence of partial sum. Consequently, for any \( n \geq 1 \) and \( p \geq 1 \), we obtain

\[ d_\theta(x_n, x_{n+p}) \leq S_{n+p-1} - S_{n-1}. \]

Taking the limit as \( n \to \infty \) from both side of the above inequality, we make a conclusion that \( \{x_n\} \) is a Cauchy sequence in \( X \).

Lemma 3 ([29]). Let \( \{x_n\} \) be a sequence in an extended b-metric space \((X, d_\theta)\) such that

\[ \lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{k} \]

and

\[ 0 < d_\theta(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n) \]

for any \( m > n \geq 2, n, m \in \mathbb{N} \), where \( k \in [0, 1) \), then \( \{x_n\} \) is a Cauchy sequence in \( X \).

Proof. Choose \( \psi(t) = kt \), where \( k \in [0, 1) \) in Lemma 2. Then, the proof is completed.

2. Fixed Points of Rational Type Contractions

In this section, we assume that \((X, d_\theta)\) is an extended b-metric space with the continuous functional \( d_\theta \). Let \( T : X \to X \) be a mapping. For \( x, y \) in \( X \), we always denote

\[ \mathcal{N}(x, y) = \max \left\{ \frac{d_\theta(x, y)}{d_\theta(x, y)} \right\}, \frac{d_\theta(x, Ty) d_\theta(x, Tx)}{1 + d_\theta(x, Ty)}, \frac{d_\theta(y, Ty) [1 + d_\theta(x, Ty)]}{1 + d_\theta(x, y)} \right\}, \]

\[ \mathcal{K}(x, y) = \max \left\{ \frac{d_\theta(x, y)}{\max \{d_\theta(x, Ty), d_\theta(y, Ty)\}} \right\}, \frac{d_\theta(x, Tx) d_\theta(y, Ty) + d_\theta(x, Ty) d_\theta(y, Tx)}{\max \{d_\theta(x, Ty), d_\theta(y, Tx)\}} \right\}. \]

Theorem 6. Let \( T \) be a self mapping on a T-orbitally complete extended b-metric space \((X, d_\theta)\).

Assume that there exist two functions \( \alpha : X \times X \to [0, +\infty) \), \( \psi \in \Psi \) such that

\[ \alpha(x, y) d_\theta(Tx, Ty) \leq \psi(\mathcal{N}(x, y)) \]

for all \( x, y \in X, x \neq y \). That is, \( T \) is a rational type contraction. If

(i) \( T \) is \( \alpha \)-orbitally admissible;

(ii) there exists \( x_0 \in X \) satisfying \( \alpha(x_0, Tx_0) \geq 1 \);

(iii) (i) is satisfied for \( x_n = T^n x_0 (n = 0, 1, 2, \ldots) \);
(iv) $T$ is either continuous or, orbitally continuous on $X$.

Then, $T$ possesses a fixed point $z \in X$. Moreover, the sequence \( \{T^n x_0\}_{n \in \mathbb{N}} \) converges to $z \in X$.

**Proof.** By (ii), define a sequence \( \{x_n\} \) in $X$ such that $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N} \cup \{0\}$.

If $x_n = x_{n+1}$, for some $n \in \mathbb{N} \cup \{0\}$, then $x_n$ is a fixed point of $T$. This completes the proof. Without loss of generality, we therefore assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$.

Based on (i), $\alpha(x_0, x_1) = \alpha(Tx_0, Tx_1) \geq 1$ implies that $\alpha(x_1, x_2) = \alpha(Tx_1, Tx_2) \geq 1$.

Then, $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$. Continuing this process, one has $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$.

Taking $x = x_{n-1}$ and $y = x_n$, for all $n \in \mathbb{N}$ in (2), we have

\[
d_\theta(x_n, x_{n+1}) = d_\theta(Tx_{n-1}, Tx_n) \\
\leq \alpha(x_{n-1}, x_n)d_\theta(Tx_{n-1}, Tx_n) \\
\leq \psi(N(x_{n-1}, x_n)),
\]

where

\[
N(x_{n-1}, x_n) = \max \left\{ d_\theta(x_{n-1}, x_n), \frac{d_\theta(x_n, Tx_n)d_\theta(x_{n-1}, Tx_{n-1})}{d_\theta(x_{n-1}, x_n)}, \frac{d_\theta(x_{n-1}, Tx_{n-1})[1 + d_\theta(x_n, Tx_n)]}{1 + d_\theta(x_{n-1}, x_n)}, \frac{d_\theta(x_n, Tx_n)[1 + d_\theta(x_{n-1}, Tx_{n-1})]}{1 + d_\theta(x_{n-1}, x_n)} \right\}.
\]

Similar to ([10], Theorem 2.1), we can prove

\[
0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)), \quad \text{for all } n \in \mathbb{N}.
\]

In fact, we finish the proof via three cases.

(i) If $N(x_{n-1}, x_n) = d_\theta(x_{n-1}, x_n)$, then by (3), it follows that

\[
0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)).
\]

This is (5).

(ii) If $N(x_{n-1}, x_n) = d_\theta(x_n, x_{n+1})$, then by (3), we have

\[
0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_n, x_{n+1})) < d_\theta(x_n, x_{n+1}),
\]

which is a contradiction.

(iii) If $N(x_{n-1}, x_n) = \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}$, then by (4), it is easy to say that

\[
\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \leq \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}.
\]

In this case, we discuss it with two subcases.

(i) If $\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_{n-1}, x_n)$, then

\[
d_\theta(x_{n-1}, x_n) > d_\theta(x_n, x_{n+1}).
\]
By (6), we get
\[ d_\theta(x_{n-1}, x_n) \leq \frac{d_\theta(x_{n-1}, x_n)}{1 + d_\theta(x_{n-1}, x_n)} \]
which means that
\[ d_\theta(x_{n-1}, x_n) \leq d_\theta(x_n, x_{n+1}). \]
This is in contradiction with (7).

(ii) If \( \max \{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_n, x_{n+1}) \), then
\[ d_\theta(x_n, x_{n+1}) > d_\theta(x_{n-1}, x_n). \]

By (6), we get
\[ d_\theta(x_n, x_{n+1}) \leq \frac{d_\theta(x_{n-1}, x_n)}{1 + d_\theta(x_{n-1}, x_n)}, \]
which establishes that
\[ d_\theta(x_n, x_{n+1}) \leq d_\theta(x_{n-1}, x_n). \]
This is in contradiction with (8).

This is to say, (iii) does not occur. Thus, (5) is satisfied. Accordingly, we speculate that
\[ d_\theta(x_n, x_{n+1}) \leq d_\theta(x_{n-1}, x_n) \leq \ldots \leq d_\theta(x_0, x_1). \]

Letting \( n \to \infty \), we obtain that \( \lim_{n \to \infty} d_\theta(x_n, x_{n+1}) = 0 \).

It follows from Lemma 2 that \( \{T^n x_0\} \) is a Cauchy sequence in \( X \). Since \( (X, d_\theta) \) is \( T \)-orbitally complete, then there is \( z \in X \) such that \( \lim_{n \to \infty} T^n x_0 = z \).

Assume that \( T \) is continuous, then
\[ d_\theta(z, Tz) = \lim_{n \to \infty} d_\theta(x_n, T x_n) = \lim_{n \to \infty} d_\theta(x_n, x_{n+1}) = 0. \]
Therefore, \( T \) possesses a fixed point \( z \) in \( X \).

Assume that \( T \) is orbitally continuous on \( X \), thus, \( x_{n+1} = T x_n = T(T^n x_0) \to Tz \) as \( n \to \infty \). Since the limit of sequence in extended \( b \)-metric space is unique, then \( z = Tz \). Thus, \( T \) possesses a fixed point \( z \) in \( X \), i.e., \( \text{Fix}(T) \neq \emptyset \). \( \square \)

**Example 8.** Under all the conditions of Example 3, let \( T : X \to X \) be a continuous mapping defined by
\[ T x = \begin{cases} \frac{2x}{3}, & 0 \leq x \leq 1, \\ 2x - \frac{4}{3}, & \text{otherwise}. \end{cases} \]

In addition, we define a mapping \( \alpha : X \times X \to [0, +\infty) \) as
\[ \alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases} \]

Let \( x_0 \in X \) be a point with \( \alpha(x_0, T x_0) \geq 1 \), then \( x_0 \in [0, 1] \subset X \) and \( \alpha(T x_0, T^2 x_0) = \alpha \left( \frac{2x_0}{3}, \frac{4}{3} x_0 \right) \geq 1 \). Therefore, \( T \) is \( \alpha \)-orbitally admissible.

Set \( \psi(t) = kt \), for all \( t > 0 \), where \( k = \frac{4}{3} \), then \( \psi^a(t) = k^a t \).
For all distinct \(x, y\) in \(X\), ones have
\[
a(x,y)d_\varnothing(Tx,Ty) \leq \frac{4}{9}(x^2 + y^2) = kd_\varnothing(x,y) \leq kN(x,y).
\]

Moreover, there is \(x_0 \in X\) with \(a(x_0,Tx_0) \geq 1\), then \(a(Tx_0,T^2x_0) \geq 1\). Now, we deduce inductively that \(a(x_n,x_{n+1}) \geq 1\), where \(x_n = T^n x_0 = (\frac{4}{9})^n x_0\) for all \(n \in \mathbb{N} \cup \{0\}\). Obviously, \(x_n \to 0\) as \(n \to \infty\). Thus, \((X,d_\varnothing)\) is \(T\)-orbitally complete.

Note that \(\lim_{n,m \to \infty} \theta(x_n,x_m) = 1 < \frac{4}{9} = \frac{1}{k}\), where \(k = \frac{4}{9}\), that is to say,
\[
\lim_{n,m \to \infty} k \theta(x_n,x_m) = \lim_{n,m \to \infty} \frac{\theta(x_n,x_m)\psi^n(d_\varnothing(x_0,x_1))}{\psi^{n-1}(d_\varnothing(x_0,x_1))} < 1.
\]

Thus, all the conditions of Theorem 6 hold and hence \(T\) possesses a fixed point in \(X\) and \(\text{Fix}(T) = \{0, \frac{4}{9}\}\).

**Theorem 7.** In addition to all the conditions of Theorem 6, suppose that the \(T\) is \(a^*\)-orbitally admissible. Then, \(T\) possesses a unique fixed point \(z \in X\).

**Proof.** Following Theorem 6, \(T\) possesses a fixed point in \(X\). Thus, \(\text{Fix}(T) \neq \emptyset\). Assume that \(T\) is \(a^*\)-orbitally admissible. If possible, there exist \(z, z^* \in \text{Fix}(T)\), \(z \neq z^*\) such that \(Tz = z\) and \(Tz^* = z^*\), then \(a(z,z^*) = a(Tz, Tz^*) \geq 1\).

Taking \(x = z\), \(y = z^*\) in (2), we obtain
\[
d_\varnothing(z,z^*) = d_\varnothing(Tz,Tz^*) \leq a(z,z^*)d_\varnothing(Tz,Tz^*) \leq \psi(N(z,z^*))
\]
\[
= \psi(\max \left\{ d_\varnothing(z,z^*), \frac{d_\varnothing(z,z^*)}{d_\varnothing(z,z)^*}, \frac{d_\varnothing(z,z^*)}{d_\varnothing(z,z)^*}, \frac{1}{1 + d_\varnothing(z,z^*)}, \frac{1}{1 + d_\varnothing(z,z^*)} \right\})
\]
\[
= \psi(d_\varnothing(z,z^*))
\]
\[
< d_\varnothing(z,z^*),
\]
which is a contradiction. Therefore, \(T\) possesses a unique fixed point in \(X\). \(\Box\)

**Corollary 1.** ([10], Theorem 2.1) Let \(T\) be a continuous self mapping on a complete extended \(b\)-metric space \((X,d_\varnothing)\) such that
\[
d_\varnothing(Tx,Ty) \leq kN(x,y)
\]
for all \(x, y \in X, x \neq y\), where \(k \in [0,1)\). That is, \(T\) is a rational type contraction. In addition, suppose that for all \(x_0 \in X\),
\[
\lim_{n,m \to \infty} \theta(x_n,x_m) < \frac{1}{k},
\]
(9)
where \(x_n = T^n x_0, m > n \geq 1\). Then, \(T\) has a unique fixed point \(z \in X\). Moreover, the sequence \(\{T^n x_0\}_{n \in \mathbb{N}}\) converges to \(z \in X\).

**Proof.** Setting \(a(x,y) = 1\), for all \(x, y \in X\), then \(a(x,Tx) \geq 1\) implies that \(a(Tx, T^2x) \geq 1\). Therefore, \(T\) is \(a\)-orbitally admissible.

Let \(\psi(t) = kt\), for all \(t > 0\), where \(0 \leq k < 1\), then \(\psi^n(t) = k^n t\). Using (iii) of Theorem 6. In view of (9), then (iii) of Theorem 6 is satisfied. Thus, all the conditions of Theorem 6 hold. Therefore, \(T\) possesses a fixed point in \(X\), i.e., \(\text{Fix}(T) \neq \emptyset\). Because of \(\text{Fix}(T) \subseteq X\), then \(T\) is \(a^*\)-orbitally admissible and hence, by Theorem 7, \(T\) has a unique fixed point in \(X\). \(\Box\)
Remark 2. (i) The uniqueness of fixed point is not guaranteed if $T$ is not a* -orbitally admissible. In Example 8, $T$ is a -orbitally admissible and $\text{Fix}(T) = \{0, \frac{4}{3}\}$. However, $\alpha(\frac{4}{3}, T\frac{4}{3}) = 0$ so $T$ is not a* -orbitally admissible. Therefore, Theorem 7 is not applicable in this case.

(ii) In Example 8, for $x = 1$ and $y = 2$, we obtain

$$d_\theta(T1, T2) = \frac{68}{9} > d_\theta(1, 2) = 5.$$ 

Therefore, ([3], Theorem 2) and ([10], Theorem 2.1) are not applicable in this case.

Motivated by Piri et al. [14], we extend a fixed point theorem for Khan type from metric spaces to extended b-metric spaces.

**Theorem 8.** Let $T$ be a self mapping on a $T$-orbitally complete extended b-metric space $(X, d_\theta)$. Suppose that $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ are two functions satisfying

$$\alpha(x, y)d_\theta(Tx, Ty) \leq \begin{cases} \psi(K(x, y)), & \text{whenever } A(x, y) \neq 0 \text{ and } B(x, y) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \quad (10)$$

for all $x, y \in X$, $x \neq y$, where

$$A(x, y) = \max\{d_\theta(x, Ty), d_\theta(y, Tx)\}, \quad B(x, y) = \max\{d_\theta(y, Ty), d_\theta(y, Tx)\}.$$ 

If

(i) $T$ is $\alpha$ -orbitally admissible;

(ii) there exists $x_0 \in X$ and $\alpha(x_0, Tx_0) \geq 1$;

(iii) (1) is satisfied for $x_n = T^n x_0$ ($n = 0, 1, 2, \ldots$).

Then, $T$ possesses a fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to $z \in X$.

**Proof.** By (ii), define a sequence $\{x_n\}$ in $X$ such that $x_{n+1} = Tx_n = T^{n+1} x_0$, for all $n \in \mathbb{N} \cup \{0\}$. Since $T$ is $\alpha$-orbitally admissible, then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ implies $\alpha(x_1, x_2) = \alpha(Tx_0, T^2 x_0) \geq 1$. Thus, inductively, we obtain that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. In order to show that $T$ possesses a fixed point $z \in X$, we assume that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$. We divide the proof into the following two cases:

**Case 1**

Suppose that

$$\max\{d_\theta(x_{n-1}, Tx_n), d_\theta(x_n, Tx_{n-1})\} \neq 0$$

and

$$\max\{d_\theta(x_n, Tx_n), d_\theta(x_n, Tx_{n-1})\} \neq 0,$$

for all $n \in \mathbb{N}$. From (10), we obtain that

$$d_\theta(x_n, x_{n+1}) = d_\theta(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)d_\theta(Tx_{n-1}, Tx_n) \leq \psi(K(x_{n-1}, x_n)),$$

where

$$K(x_{n-1}, x_n) = \max\left\{d_\theta(x_{n-1}, x_n), \right.\]}

$$d_\theta(x_{n-1}, Tx_{n-1})d_\theta(x_{n-1}, Tx_n) + d_\theta(x_n, Tx_n)d_\theta(x_n, Tx_{n-1})\right\},$$

$$d_\theta(x_{n-1}, Tx_{n-1})d_\theta(x_n, Tx_n) + d_\theta(x_{n-1}, Tx_n)d_\theta(x_n, Tx_{n-1})\right\},$$

$$\max\{d_\theta(x_n, Tx_n), d_\theta(x_n, Tx_{n-1})\} = \max\left\{d_\theta(x_{n-1}, x_n), \right.\]}

$$\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, x_{n+1}) + d_\theta(x_n, x_{n+1})d_\theta(x_n, x_n)\},$$

$$\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_n)\}.$$
\[
\frac{d_\theta(x_{n-1}, x_n)d_\theta(x_n, x_{n+1}) + d_\theta(x_{n-1}, x_{n+1})d_\theta(x_n, x_n)}{\max\{d_\theta(x_n, x_{n+1}), d_\theta(x_n, x_n)\}}
\]

\[
= d_\theta(x_{n-1}, x_n).
\]

Therefore,

\[
0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)).
\]

Furthermore,

\[
0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)) \leq \cdots \leq \psi^n(d_\theta(x_0, x_1)).
\]

Letting \(n \to \infty\), we have

\[
\lim_{n \to \infty} d_\theta(x_n, x_{n+1}) = 0.
\]

It follows from Condition (iii) and Lemma 2 that \(\{T^n x_0\}\) is a Cauchy sequence in \(X\). Notice that \(X\) is \(T\)-orbitally complete, thus, there is \(z \in X\) with \(x_n = T^n x_0 \to z\) as \(n \to \infty\).

Assume, if possible, \(Tz \neq z\). From (10) and the triangular inequality, we obtain

\[
d_\theta(z, Tz) \leq \theta(z, Tz)[d_\theta(Tz, Tx_n) + d_\theta(Tx_n, z)]
= \theta(z, Tz)d_\theta(Tz, Tx_n) + \theta(z, Tz)d_\theta(Tx_n, z)
\leq \theta(z, Tz)\alpha(z, x_n)d_\theta(Tz, Tx_n) + \theta(z, Tz)d_\theta(x_n+1, z)
\leq \theta(z, Tz)\psi(K(z, x_n)) + \theta(z, Tz)d_\theta(x_n+1, z)
< \theta(z, Tz)K(z, x_n) + \theta(z, Tz)d_\theta(x_n+1, z),
\]

(11)

where

\[
K(z, x_n) = \max \left\{d_\theta(z, x_n), \frac{d_\theta(z, Tz)d_\theta(z, Tx_n) + d_\theta(x_n, Tx_n)d_\theta(x_n, Tz)}{\max\{d_\theta(x_n, Tx_n), d_\theta(x_n, Tz)\}}, \frac{d_\theta(z, Tz)d_\theta(x_n, Tx_n) + d_\theta(z, Tz)d_\theta(x_n, Tz)}{\max\{d_\theta(x_n, Tx_n), d_\theta(x_n, Tz)\}} \right\}
= \max \left\{d_\theta(z, x_n), \frac{d_\theta(z, Tz)d_\theta(z, x_{n+1}) + d_\theta(x_n, x_{n+1})d_\theta(x_n, Tz)}{\max\{d_\theta(z, x_{n+1}), d_\theta(x_n, Tz)\}}, \frac{d_\theta(z, Tz)d_\theta(x_n, x_{n+1}) + d_\theta(z, x_{n+1})d_\theta(x_n, Tz)}{\max\{d_\theta(x_n, x_{n+1}), d_\theta(x_n, Tz)\}} \right\}.
\]

Taking \(n \to \infty\) from both sides of (11), we have \(d_\theta(z, Tz) \leq 0\), which is in contradiction with \(Tz \neq z\).

Case 2

Assume that

\[
\max\{d_\theta(x_{n-1}, Tx_n), d_\theta(x_n, Tx_{n-1})\} = 0
\]

or

\[
\max\{d_\theta(x_n, Tx_n), d_\theta(x_n, Tx_{n-1})\} = 0,
\]

for all \(n \in \mathbb{N}\). Consider (10), it follows that

\[x_n = x_{n+1} = Tx_n.\]
Thus, $T$ possesses a fixed point in $X$, i.e., $\text{Fix}(T) \neq \emptyset$. □

**Example 9.** Under all the conditions of Example 5, let $T : X \to X$ be a mapping defined by

$$
T_0 = \begin{cases} 
0, & 0 \leq x < \frac{3}{2}, \\
1, & \frac{3}{2} \leq x < 500, \\
100, & x \geq 500.
\end{cases}
$$

We also define a mapping $\alpha : X \times X \to [0, +\infty)$ as

$$
\alpha(x, y) = \begin{cases} 
1, & x, y \in [0, \frac{3}{2}), \\
0, & \text{otherwise}.
\end{cases}
$$

Let $x \in X$ be a point such that $\alpha(x, Tx) \geq 1$, then $x \in [0, \frac{3}{2}) \subset X$ and $\alpha(Tx, T^2x) \geq 1$. Therefore, $T$ is $\alpha$-orbitally admissible.

Set $\psi(t) = kt$, for all $t > 0$, where $k = \frac{1}{2}$. For all $x, y \in X$, we obtain

$$
\alpha(x, y)d_\theta(Tx, Ty) \leq k\max\{\alpha(x, y)\}
$$

Clearly, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(Tx_0, T^2x_0) \geq 1$. Therefore, by the mathematical induction, we have $\alpha(x_n, x_{n+1}) = \alpha(T^n x_0, T^{n+1}x_0) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Consequently, $T^n x_0 \to 0$ as $n \to \infty$. This shows that $(X, d_\theta)$ is a $T$-orbitally complete extended $b$-metric space.

Moreover, it is easy to see that

$$
\lim_{m,n \to \infty} \theta(T^m x_0, T^n x_0) = \frac{3}{2} < 2 = \frac{1}{k}.
$$

Accordingly, all the conditions of Theorem 8 hold and, therefore, $T$ possesses a fixed point and $\text{Fix}(T) = \{0, 2\}$.

**Theorem 9.** In addition to Theorem 8, suppose that $T$ is $\alpha^*$-orbitally admissible. Then, $T$ possesses a unique fixed point $z \in X$.

**Proof.** By Theorem 8, $T$ possesses a fixed point in $X$, i.e., $\text{Fix}(T) \neq \emptyset$. For the uniqueness, let $z, z^* \in \text{Fix}(T)$ such that $z \neq z^*$. Then, by the $\alpha^*$-orbital admissibility of $T$, we have $\alpha^*(z, z^*) \geq 1$.

As in Theorem 8, we also divide the proof into two cases as follows:

**Case 1**

Suppose that

$$
\max\{d_\theta(z, Tx^*), d_\theta(z^*, Tz)\} \neq 0
$$

and

$$
\max\{d_\theta(z^*, Tz^*), d_\theta(z^*, Tz)\} \neq 0.
$$

From (10), we obtain

$$
d_\theta(z, z^*) = d_\theta(Tz, Tz^*) \leq \alpha(z, z^*)d_\theta(Tz, Tz^*) \leq \psi(\mathcal{K}(z, z^*)),
$$

where

$$
\mathcal{K}(z, z^*) = \max \left\{\frac{d_\theta(z, z^*)}{\max\{d_\theta(z, z^*), d_\theta(z^*, Tz)\}}, \frac{d_\theta(z, Tz)d_\theta(z, Tz^*) + d_\theta(z^*, Tz^*)d_\theta(z^*, Tz)}{\max\{d_\theta(z, Tz^*), d_\theta(z^*, Tz)\}}\right\}
$$

$$
= d_\theta(z, z^*).
$$
Therefore,
\[ d_\theta(z,z^*) \leq \psi\left(d_\theta(z,z^*)\right) < d_\theta(z,z^*). \]

This is a contradiction.

Case 2
Assume that
\[ \max\{d_\theta(z,Tz^*),d_\theta(z^*,Tz)\} = 0 \]
or
\[ \max\{d_\theta(z^*,Tz^*),d_\theta(z^*,Tz)\} = 0. \]

Consequently, \( z = Tz^* = Tz = z^* \).
Thus, \( T \) possesses a unique fixed point in \( X \). This completes the proof. \( \square \)

**Corollary 2.** Let \( T \) be a self mapping on a complete extended b-metric space \((X,d_\theta)\) such that
\[
 d_\theta(Tx,Ty) \leq k \begin{cases} 
 K(x,y), & \text{whenever } \mathcal{A}(x,y) \neq 0 \text{ and } \mathcal{B}(x,y) \neq 0, \\
 0, & \text{otherwise,}
\end{cases}
\]
for all \( x,y \in X, x \neq y \), where \( 0 \leq k < 1 \), \( \mathcal{A}(x,y) \) and \( \mathcal{B}(x,y) \) are defined in Theorem 8. Furthermore, suppose, for all \( x_0 \in X \), that (9) is satisfied. Then, \( T \) has a unique fixed point \( z \in X \). Moreover, the sequence \( \{T^n x_0\}_{n \in \mathbb{N}} \) converges to \( z \).

**Corollary 3.** Let \( T \) be a self mapping on a complete extended b-metric space \((X,d_\theta)\) such that
\[
 d_\theta(Tx,Ty) \leq k \max\left\{ \frac{d_\theta(x,y) + d_\theta(x,Ty) + d_\theta(y,Tx)}{\mathcal{A}(x,y)}, \frac{d_\theta(x,Ty) + d_\theta(y,Tx)}{\mathcal{A}(x,y)} \right\}, \]
if \( \mathcal{A}(x,y) \neq 0 \),
\[
 0, \quad \text{if } \mathcal{A}(x,y) = 0,
\]
for all \( x,y \in X, x \neq y \), where \( 0 \leq k < 1 \) and \( \mathcal{A}(x,y) = \max\{d_\theta(x,Ty),d_\theta(y,Tx)\} \). Further suppose, for all \( x_0 \in X \), that (9) is satisfied. Then, \( T \) has a unique fixed point \( z \in X \). Moreover, the sequence \( \{T^n x_0\}_{n \in \mathbb{N}} \) converges to \( z \).

**Corollary 4.** ([10], Theorem 2.2) Let \( T \) be a self mapping on a complete extended b-metric space \((X,d_\theta)\) such that
\[
 d_\theta(Tx,Ty) \leq k \max\left\{ \frac{d_\theta(x,y) + d_\theta(x,Ty) + d_\theta(y,Tx)}{\mathcal{A}(x,y)}, \frac{d_\theta(x,Ty) + d_\theta(y,Tx)}{\mathcal{A}(x,y)} \right\}, \]
if \( \mathcal{B}(x,y) \neq 0 \),
\[
 0, \quad \text{if } \mathcal{B}(x,y) = 0,
\]
for all \( x,y \in X, x \neq y \), where \( 0 \leq k < 1 \) and \( \mathcal{B}(x,y) = d_\theta(x,Ty) + d_\theta(y,Tx) \). Further assume, for all \( x_0 \in X \), that (9) is satisfied. Then, \( T \) has a unique fixed point \( z \in X \). Moreover, the sequence \( \{T^n x_0\}_{n \in \mathbb{N}} \) converges to \( z \).

**Remark 3.** (i) In Example 9, \( T \) is \( \alpha \)-orbitally admissible. Since \( \text{Fix}(T) = \{0,2\} \), but \( \alpha_0(2,2) = \alpha_0(2,2) = 0 \), \( T \) is not \( \alpha^* \)-orbitally admissible. In this case, Theorem 9 is not applicable in Example 9.
(ii) In Example 9, if \( x = 2 \) and \( y = 500 \), then
\[
 d_\theta(Tx,Ty) = d_\theta(T2,T500) = d_\theta(2,100) = 5 > \frac{1}{2} \max\left\{5, \frac{5}{2}\right\}.
\]
This shows that Corollaries 2–4 are not applicable in Example 9.

**3. Applications**

In this section, by using fixed point theorems mentioned above, we cope with some problems for the unique solution to a class of Fredholm integral equations.
Let $X = C[a, b]$ be a set of all real valued continuous functions on $[a, b]$. Define two mappings $d_\theta : X \times X \to [0, +\infty)$ by

$$d_\theta(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^p,$$

and $\theta : X \times X \to [1, +\infty)$ by

$$\theta(x, y) = 2^{p-1} + |x(t)| + |y(t)|,$$

where $p > 1$ is a constant. Then, $(X, d_\theta)$ is a complete extended $b$-metric space.

Define a Fredholm integral equation by

Let $T : X \to X$ be an integral operator defined by

$$Tx(t) = \eta(t) + \lambda \int_a^b I(t,s,x(s))ds,$$

where $t \in [a,b]$, $|\lambda| > 0$ and $I : [a,b] \times [a,b] \times X \to \mathbb{R}$ and $\eta : [a,b] \to \mathbb{R}$ are continuous functions. Let $T : X \to X$ be an integral operator defined by

$$Tx(t) = \eta(t) + \lambda \int_a^b I(t,s,x(s))ds.$$  \hspace{1cm} \text{(12)}$$

**Theorem 10.** Let $T : X \to X$ be an integral operator defined in (12). Suppose that the following assumptions hold:

(i) for any $x_0 \in X$, \( \lim_{n \to +\infty} \theta(T^n x_0, T^m x_0) < \frac{1}{k}, \) where $k = \frac{1}{2^p}$,

(ii) for any $x, y \in X$, $x \neq y$, it satisfies

$$|I(t,s,x(s)) - I(t,s,y(s))| \leq \xi(t,s)|x(s) - y(s)|,$$

where $(s,t) \in [a,b] \times [a,b]$ and $\xi : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function satisfying

$$\sup_{t \in [a,b]} \int_a^b \xi^p(t,s)ds < \frac{1}{2^p|\lambda|^p(b-a)^{p-1}}.$$  \hspace{1cm} \text{(14)}$$

Then, the integral operator $T$ has a unique solution in $X$.

**Proof.** Let $x_0 \in X$ and define a sequence $\{x_n\}$ in $X$ by $x_n = T^n x_0$, $n \geq 1$. From (12), we obtain

$$x_{n+1} = Tx_n(t) = \eta(t) + \lambda \int_a^b I(t,s,x_n(s))ds.$$  

Let $q > 1$ be a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Making full use of (13) and the Hölder’s inequality, we speculate that

$$|Tx(t) - Ty(t)|^p = \left| \lambda \int_a^b I(t,s,x(s))ds - \lambda \int_a^b I(t,s,y(s))ds \right|^p$$

$$\leq \left( \int_a^b |\lambda||I(t,s,x(s)) - I(t,s,y(s))||ds \right)^p$$

$$\leq \left( \int_a^b |\lambda|^q ds \right)^\frac{p}{q} \left( \int_a^b |I(t,s,x(s)) - I(t,s,y(s))|^p ds \right)^\frac{1}{p}$$

$$= |\lambda|^p(b-a)^{p-1} \int_a^b \xi^p(t,s)|x(s) - y(s)|^p ds.$$  \hspace{1cm} \text{(15)}$$
Making the most of (15) and (14), we deduce that
\[
d_\theta(Tx, Ty) = \sup_{t \in [a, b]} \left| Tx(t) - Ty(t) \right|^p
\leq |\lambda|^p(b - a)^{p-1} \sup_{t \in [a, b]} \left| \int_a^b \xi^p(t, s) |x(s) - y(s)|^p ds \right|
\leq |\lambda|^p(b - a)^{p-1} \sup_{s \in [a, b]} |x(s) - y(s)|^p \left( \sup_{t \in [a, b]} \int_a^b \xi^p(t, s) ds \right)
\leq \frac{1}{2^p} N(x, y).
\]

Setting \( k = \frac{1}{2^p} \), we obtain that
\[
d_\theta(Tx, Ty) \leq k N(x, y).
\]
Thus, all the conditions of Corollary 1 are satisfied and hence \( T \) possesses a unique fixed point in \( X \). \( \Box \)

**Theorem 11.** Let \( T : X \to X \) be an integral operator defined by (12). Assume that the following assumptions hold:

(i) \( \lim_{n,m \to \infty} \theta(T^n x_0, T^m x_0) < \frac{1}{k} \), where \( k = \frac{1}{2^p} \) for any \( x_0 \in X \);

(ii) for all distinct \( x, y \) in \( X \), one has
\[
\left| I(t, s, x(s)) - I(t, s, y(s)) \right| \leq \begin{cases} 
\xi(t, s) K(x(s), y(s)), & \text{where } A \neq 0 \text{ and } B \neq 0, \\
0, & \text{otherwise},
\end{cases}
\]

where
\[
A = A(x(s), y(s)) = \sup \{|x(s) - Ty(s)|^p, |y(s) - Tx(s)|^p\}, \quad B = B(x(s), y(s)) = \sup \{|y(s) - Ty(s)|^p, |y(s) - Tx(s)|^p\},
\]
\((s, t) \in [a, b] \times [a, b] \text{ and } \xi : [a, b] \times [a, b] \to \mathbb{R} \text{ is a continuous function such that}
\[
\sup_{t \in [a, b]} \int_a^b \xi^p(t, s) ds < \frac{1}{2^p |\lambda|^p(b - a)^{p-1}}.
\]

Then, the integral operator \( T \) has a unique solution in \( X \).

**Example 10.** Let \( X = C[0, 1] \) be a set of all real valued continuous functions defined on \([0, 1] \). Then, \((X, d_\theta)\) is a complete extended \( b \)-metric space equipped with \( d_\theta(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|^2 \), where \( \theta(x, y) = 2 + |x(t)| + |y(t)| \), for all \( x, y \in X \). Let \( T : X \to X \) be an operator defined by
\[
Tx(t) = \eta(t) + \int_0^1 I(t, s, x(s)) ds,
\]
where \( \eta(t) = \frac{1}{4} \) and \( I(t, s, x(s)) = \frac{t(1+x^2(s))}{3} \), for all \((t, s) \in [0, 1] \times [0, 1] \).

We have
\[
|Tx(t) - Ty(t)|^2 = \left| \int_0^1 I(t, s, x(s)) ds - \int_0^1 I(t, s, y(s)) ds \right|^2
\leq \int_0^1 \left| \frac{1}{3} (x^2(s) - y^2(s)) \right|^2 ds.
\]

(16)
Taking the supremum on both sides of (16), for all \( t \in [0, 1] \), we obtain

\[
d_\theta(Tx, Ty) = \sup_{t \in [0, 1]} |Tx(t) - Ty(t)|^2 \leq \frac{1}{9}d_\theta(x, y) < \frac{1}{6}N(x, y).
\]

In addition, \( \lim_{m,n \to \infty} \theta(T^m x_0, T^n x_0) = 2 < \frac{1}{k} \), where \( k = \frac{1}{6} \) and \( x_0(t) = \frac{1}{4} \). Thus, all the conditions of Theorem 10 are satisfied and hence the integral operator \( T \) has a unique solution.

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**References**


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20. Alsulami, H.H.; Karapınar, E.; Rakočević, V. Ćirić type non-unique fixed point theorems on $b$-metric spaces. *Filomat* 2017, 3, 3147–3156. [CrossRef]


22. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $\alpha$-$\psi$-contractive type mappings. *Nonlinear Anal.* 2012, 75, 2154–2165. [CrossRef]

23. Mahendra Singh, Y.; Khan, M.S.; Kang, S.M. $F$-convex contraction via admissible mapping and related fixed point theorems with an application. *Mathematics* 2018, 6, 105. [CrossRef]

24. Popescu, O. Some new fixed point theorems for $\alpha$-Geraghty contraction type maps in metric spaces. *Fixed Point Theory Appl.* 2014, 2014, 190. [CrossRef]


29. Alqahtani, B.; Fulga, A.; Karapınar, E. Non-unique fixed point results in extended $b$-metric space. *Mathematics* 2018, 6, 68. [CrossRef]