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Applications of Banach Limit in Ulam Stability

Roman Badora [1], Janusz Brzdęk [2,*] and Krzysztof Ciepliński [2]

[1] Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland; roman.badora@us.edu.pl

[2] Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland; cieplin@agh.edu.pl

* Correspondence: brzdek@agh.edu.pl

Abstract: We show how to get new results on Ulam stability of some functional equations using the Banach limit. We do this with the examples of the linear functional equation in single variable and the Cauchy equation.

Keywords: Banach limit; Ulam stability; functional equation

MSC: 39B82; 39B52

Citation: Badora, R.; Brzdęk, J.; Ciepliński, K. Applications of Banach Limit in Ulam Stability. *Symmetry* **2021**, *13*, 841. https://doi.org/10.3390/sym13050841

Academic Editor: Mihai Postolache

Received: 7 April 2021
Accepted: 3 May 2021
Published: 10 May 2021

1. Introduction

Ulam stability (also known as Hyers–Ulam or Ulam–Hyers stability) concerns the following problem: how much does an approximate solution of an equation differ from a solution to the equation? More information on this is provided in the next two sections. Here, let us only mention that Ulam stability has become a very popular field of research, and we refer the reader to [1–3] for information on the background and the methods commonly used in it. One such method is that of the Banach limit, and in this paper, we show how to increase its utility.

Let us recall that the notion of the Banach limit was motivated by the observation that every convergent sequence of real numbers is bounded, but, unfortunately, the converse statement is not true. Over the years, mathematicians have tried to extend the notion of the limit to larger families and, preferably, to apply it to the space of all bounded sequences. In the early twentieth century, the Lwów school of mathematics provided a tool (now called the Banach limit) that made this possible. Namely, in the memorial book of the first Congress of Polish Mathematicians (7–10 September 1927), Mazur [4] (p. 103) stated the following:

Theorem 1 (S. Mazur)**.** *There exists a linear functional f on the space of all bounded sequences such that:*

(a) $f((a_n)_{n\in\mathbb{N}}) = f((a_{n+1})_{n\in\mathbb{N}})$*;*
(b) $f((a_n)_{n\in\mathbb{N}}) \geq 0$*, if* $a_n \geq 0$ *for all* $n \in \mathbb{N}$*;*
(c) $f((1)_{n\in\mathbb{N}}) = 1$*.*

Moreover, he noticed that this functional f is continuous and

$$f((a_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} a_n$$

for every convergent sequence $(a_n)_{n\in\mathbb{N}}$. The proof of this result, based on the Hahn–Banach theorem, was published in Banach's monograph [5].

Thus, there exists a functional called the Banach limit and usually denoted by LIM, which can be applied to the bounded real sequences as a substitute for the limit operation. The functional is linear, shift invariant and positive, but not unique (because of the application of the Hahn–Banach theorem in the proof). The lack of uniqueness means that, generally, the Banach limit of a sequence is not defined unequivocally. However, there are non-convergent sequences for which the Banach limit is uniquely determined (such sequences are called almost convergent). For example, if $a_n = (-1)^n$ for $n \in \mathbb{N}$, then $(a_n)_{n\in\mathbb{N}} + (a_{n+1})_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$ and, therefore, we obtain

$$\begin{aligned}2\mathrm{LIM}((a_n)_{n\in\mathbb{N}}) &= \mathrm{LIM}((a_n)_{n\in\mathbb{N}}) + \mathrm{LIM}((a_{n+1})_{n\in\mathbb{N}})\\ &= \mathrm{LIM}((a_n)_{n\in\mathbb{N}} + (a_{n+1})_{n\in\mathbb{N}}) = \mathrm{LIM}((0)_{n\in\mathbb{N}}) = 0.\end{aligned}$$

Thus, for any Banach limit, we have $\mathrm{LIM}((-1)^n{}_{n\in\mathbb{N}}) = 0$. Generally we can show that the Banach limit of a bounded periodic sequence is the average value of the period of this sequence.

Further, the Banach limit is not a multiplicative functional. In fact, if as before $a_n = (-1)^n$ for $n \in \mathbb{N}$, then $\mathrm{LIM}((a_n)_{n\in\mathbb{N}}) = 0$, and, consequently, we have

$$\begin{aligned}\mathrm{LIM}((a_n)_{n\in\mathbb{N}} \cdot (a_{n+1})_{n\in\mathbb{N}}) &= \mathrm{LIM}((-1)_{n\in\mathbb{N}})\\ &= -1 \neq 0 = 0 \cdot 0 = \mathrm{LIM}((a_n)_{n\in\mathbb{N}}) \cdot \mathrm{LIM}((a_{n+1})_{n\in\mathbb{N}}).\end{aligned}$$

In this article, we denote traditionally by ℓ^∞ the space of all bounded real sequences (with the supremum norm); c means the space of all convergent real sequences, and LIM stands for the Banach limit, that is for a linear functional defined on the space ℓ^∞ and satisfying the following two conditions:

$$\inf\{a_n : n \in \mathbb{N}\} \le \mathrm{LIM}\big((a_n)_{n\in\mathbb{N}}\big) \le \sup\{a_n : n \in \mathbb{N}\} \tag{1}$$

and

$$\mathrm{LIM}\big((a_{n+k})_{n\in\mathbb{N}}\big) = \mathrm{LIM}\big((a_n)_{n\in\mathbb{N}}\big), \tag{2}$$

for all $(a_n)_{n\in\mathbb{N}} \in \ell^\infty$ and $k \in \mathbb{N}$.

It is not difficult to show that this definition is equivalent to that given in Theorem 1 (with $f = \mathrm{LIM}$). Namely, first note that (a) follows from (2) (with $k = 1$) and (b) and (c) result from (1). Conversely, by an easy induction, we can derive (2) from (a). Further, assume that (b) and (c) are valid and take $(c_n)_{n\in\mathbb{N}} \in \ell^\infty$. Write $\iota := \inf\{c_n : n \in \mathbb{N}\}$ and $\sigma := \sup\{c_n : n \in \mathbb{N}\}$. Then $c_n - \iota \ge 0$ and $\sigma - c_n \ge 0$ for each $n \in \mathbb{N}$, whence by the linearity of LIM, (b), and (c)

$$\mathrm{LIM}\big((c_n)_{n\in\mathbb{N}}\big) - \iota = \mathrm{LIM}\big((c_n - \iota)_{n\in\mathbb{N}}\big) \ge 0,$$

$$\sigma - \mathrm{LIM}\big((c_n)_{n\in\mathbb{N}}\big) = \mathrm{LIM}\big((\sigma - c_n)_{n\in\mathbb{N}}\big) \ge 0,$$

which implies that

$$\iota \le \mathrm{LIM}\big((c_n)_{n\in\mathbb{N}}\big) \le \sigma.$$

Thus, we have shown that (b) and (c) yield (1).

Note also that from (1) and (2) we obtain

$$\liminf_{n\to\infty} a_n \le \mathrm{LIM}\big((a_n)_{n\in\mathbb{N}}\big) \le \limsup_{n\to\infty} a_n, \qquad (a_n)_{n\in\mathbb{N}} \in \ell^\infty, \tag{3}$$

and consequently

$$\mathrm{LIM}((a_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} a_n, \qquad (a_n)_{n\in\mathbb{N}} \in c.$$

The Banach limit has found applications in various areas of mathematics. The classical results in the theory of it were obtained in [6,7]. Looking at the Banach limit as an invariant mean defined on a space of bounded sequences, i.e., bounded functions defined on a set of natural numbers, we recommend reading the papers of Badora, Ger, Páles [8], and Kania [9] for a generalization of the notion of the Banach limit to the case of vector-valued sequences (the first instance of such a generalization can probably be found in [10]). Let us also mention here that in the vector case, there are Banach spaces with no Banach limits (a typical example is c_0, that is, the space of all null sequences [11]). Further information on the Banach limit and its applications can be found in surveys [12–14].

In this work, we focus our attention on the stability of the Cauchy additive functional equation

$$F(x+y) = F(x) + F(y) \tag{4}$$

and the linear functional equation in a single variable

$$\Phi(f(x)) = g(x)\Phi(x) + h(x), \tag{5}$$

where f, g, h are given functions and the function Φ is unknown.

Because Equation (4) is very well-known, we only refer the reader to monographs [2,15,16] for further information on its solutions, applications, and stability. As for Equation (5), we discuss it in more detail.

Continuous solutions of Equation (5) were intensively studied in the 1950s and 1960s, and a summary of these investigations can be found in [17]. Further progress concerning mainly monotonic, differentiable, and analytic solutions as well as of bounded variation solutions of (5) was surveyed in [18]. To see the equation within dynamical context, one can consult [19,20]. Let us mention some applications of the linear functional equations in ergodic theory, probability theory (branching processes and doubly stochastic measures), and differential equations, which were presented in [17,18]. Now, we pay attention to a few important particular cases of Equation (5).

Equation (5) with $f(x) = x+1$, $g(x) = x$ and $h(x) = 0$ leads to the gamma functional equation

$$\Phi(x+1) = x\Phi(x), \tag{6}$$

which is useful in some characterizations of Euler's gamma function. One of them, the famous Bohr–Mollerup theorem, states that Euler's gamma function $\Gamma : (0,\infty) \to (0,\infty)$ is the only solution of Equation (6) such that $\Phi(1) = 1$ and $\log\Phi$ is convex (see, for instance, [17,18]). The history of various definitions and characterizations of this function (including a few with the use of (6)), also in the complex case, can be found in [21].

The next considerably important, especially in dynamical systems, special case of (5) is the cohomological equation

$$\Phi(f(x)) = \Phi(x) + h(x). \tag{7}$$

Some results with its monotonic solutions are described in books [17,18], whereas results with its smooth and analytic systems are reported in [20]. As for dynamical systems, Equation (7) has applications, among others, in smoothness of invariant measures and conjugacies, mixing properties of suspended flows and rigidity of group actions (see [22], where some criteria for the existence and regularity of solutions of the cohomological equation, under the general assumption that f is an accessible and partially hyperbolic diffeomorphism of a compact manifold, are also given). In Lyubich's survey [23], the cohomological equation is considered from the point of view of functional analysis and dynamical systems, but its role in other areas of mathematics is also mentioned.

Let us finally consider two more particular cases of (5), i.e., the Schröder functional equation

$$\Phi(f(x)) = s\Phi(x) \tag{8}$$

and the Abel functional equation

$$\Phi(f(x)) = \Phi(x) + 1. \tag{9}$$

They both appeared at the very beginning of complex dynamics (see [24] for the details) and have been studied in various settings since then. As a lot of information about solutions (in several classes) as well as applications (in dynamical systems and ergodic theory, probability theory, differential equations, and iteration theory) of these equations can be found, for instance, in monographs [17,18,20] and survey [19], we mention only some recent papers on this topic.

In [25], Equations (8) and (9) were applied to study one-parameter semigroups of holomorphic mappings, and in [26] to provide some criteria for the embeddability of an analytic function into a semigroup of analytic self-maps of the open unit disk.

On the other hand, monotonic solutions of Equation (8) were studied in Banach spaces in [27], whereas continuous and smooth solutions of this equation in the case of normed spaces were investigated in [28]. The Schröder equation in several variables was considered in [29–34], and its regularly varying solutions, among others, in [35]. Let us mention the connections between Equation (8) and a special class of Volterra integral Equation [36], invariant measures of some one-dimensional chaotic maps [37], median stable distributions [38], and iteration theory [39].

As for the Abel equation, its complex solutions were studied in [40,41], whereas in [42], Equation (9) was investigated from functional analytic and operator theoretical point of view.

2. Ulam Stability of the Linear Equation in a Single Variable

In this section, X always denotes a nonempty set and $f : X \to X$, $g, h : X \to \mathbb{R}$ are given functions with $g(X) \subset (0, \infty)$, unless clearly stated otherwise. Moreover, $\mathbb{R}_+$ stands for the set of nonnegative reals. We denote also by f^n, where n is a nonnegative integer, the nth iterate of f, i.e., $f^0 = id$ (the identity function) and $f^{n+1} = f \circ f^n$ (function composition).

The following definition depicts the type of the Ulam stability that we apply in this paper.

Definition 1. *Let $n \in \mathbb{N}$, $\mathcal{C} \subset {\mathbb{R}_+}^{X^n}$ and $\mathcal{D} \subset \mathbb{R}^X$ be nonempty, $\mathcal{T} : \mathcal{C} \to {\mathbb{R}_+}^X$, and $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{D} \to \mathbb{R}^{X^n}$. We say that the equation*

$$\mathcal{F}_1\varphi(x_1, \dots, x_n) = \mathcal{F}_2\varphi(x_1, \dots, x_n) \tag{10}$$

is $\mathcal{T}$—stable if for any $\varepsilon \in \mathcal{C}$ and $\varphi_0 \in \mathcal{D}$ satisfying the inequality

$$d\big(\mathcal{F}_1\varphi_0(x_1, \dots, x_n), \mathcal{F}_2\varphi_0(x_1, \dots, x_n)\big) \le \varepsilon(x_1, \dots, x_n), \qquad x_1, \dots, x_n \in X,$$

there exists a solution $\varphi \in \mathcal{D}$ of Equation (10) with

$$d\big(\varphi(x), \varphi_0(x)\big) \le \mathcal{T}\varepsilon(x), \qquad x \in X.$$

If $\mathcal{C}$ consists of all constant functions from ${\mathbb{R}_+}^{X^n}$, and all elements of the set $\mathcal{T}(\mathcal{C})$ are also only constant functions, then the $\mathcal{T}$—stability is usually called Hyers–Ulam (or Ulam–Hyers) stability. More information on such stability can be found in monographs [1–3] and surveys [43–45].

The first Ulam stability result for Equation (5) (written in a somewhat modified form) was proved by Baker [46] with a fixed point technique based on the Banach contraction principle. Some of its generalizations were published in [47] (Theorems 2.1 and 3.1) (for a particular case of (5)) and improved further by Trif [48] (Theorem 2.1) (cf. [49–54] for related results). Now, we recall Trif's result, because we refer to it in this section.

Theorem 2 ([48] (Theorem 2.1))**.** *Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $a : X \to \mathbb{F}$, $\chi : X \to V$ and $\eta : X \to \mathbb{R}_+$ be such that $0 \notin a(X)$ and*

$$\rho(x) := \sum_{j=0}^{\infty} \frac{\eta(f^j(x))}{\prod_{k=0}^{j} |a(f^k(x))|} < \infty, \qquad x \in X. \tag{11}$$

If $\varphi : X \to V$ *satisfies the inequality*

$$\|\varphi(f(x)) - a(x)\varphi(x) - \chi(x)\| \le \eta(x), \qquad x \in X, \tag{12}$$

then there is a unique solution $\widetilde{\varphi} : X \to V$ *to the equation*

$$\widetilde{\varphi}(f(x)) = a(x)\widetilde{\varphi}(x) + \chi(x), \qquad x \in X, \tag{13}$$

such that

$$\|\varphi(x) - \widetilde{\varphi}(x)\| \le \rho(x), \qquad x \in X. \tag{14}$$

Let us mention here that some studies of condition (11) can be found in [52].

Next, note that in the case where $V = \mathbb{R}$ and $\mathbb{F} = \mathbb{R}$, inequality (12) may be rewritten as

$$-\eta(x) \le \varphi(f(x)) - a(x)\varphi(x) - \chi(x) \le \eta(x), \qquad x \in X. \tag{15}$$

We show that in such case, an analogous result can be obtained with the Banach limit technique also if $a(X) \subset (0, \infty)$ and inequality (15) is replaced by

$$\xi(x) \le \varphi(f(x)) - a(x)\varphi(x) - \chi(x) \le \eta(x), \qquad x \in X \tag{16}$$

with any function $\xi : X \to \mathbb{R}$ such that $\xi(x) \le \eta(x)$ for $x \in X$ and the sequence $\big(\widehat{\xi}_n(x)\big)_{n\in\mathbb{N}}$, where

$$\widehat{\xi}_n(x) := \sum_{j=0}^{n-1} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} a(f^i(x))}, \qquad n \in \mathbb{N}, \tag{17}$$

is bounded for $x \in X$.

Moreover, it is not necessary to assume that η has nonnegative values and, analogously as for ξ, condition (11) can be replaced by the boundedness of each sequence $\big(\widehat{\eta}_n(x)\big)_{n\in\mathbb{N}}$, where

$$\widehat{\eta}_n(x) := \sum_{j=0}^{n-1} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} a(f^i(x))}, \qquad n \in \mathbb{N}, \tag{18}$$

which is a significantly weaker assumption for functions η that may take both negative and positive values.

The following generalization of the real scalar version of Theorem 2 is the main result of this section.

Theorem 3. *Assume that* $\xi, \eta : X \to \mathbb{R}$ *are such that the sequences* $\big(\widehat{\xi}_n(x)\big)_{n\in\mathbb{N}}$ *and* $\big(\widehat{\eta}_n(x)\big)_{n\in\mathbb{N}}$ *given by*

$$\widehat{\xi}_n(x) := \sum_{j=0}^{n-1} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}, \qquad \widehat{\eta}_n(x) := \sum_{j=0}^{n-1} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}, \qquad n \in \mathbb{N}, \tag{19}$$

are bounded for every $x \in X$. *Let* $\psi : X \to \mathbb{R}$ *be a function fulfilling the inequalities*

$$\xi(x) \le \psi(f(x)) - g(x)\psi(x) - h(x) \le \eta(x), \qquad x \in X. \tag{20}$$

Then, the sequence $(a_n(x))_{n\in\mathbb{N}}$, *given by*

$$a_n(x) = \frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} - \sum_{k=0}^{n-1} \frac{h(f^k(x))}{\prod_{j=0}^{k} g(f^j(x))}, \qquad n \in \mathbb{N},$$

is bounded for each $x \in X$ and the function $\Psi : X \to \mathbb{R}$, defined by

$$\Psi(x) = \mathrm{LIM}\left(\left(\frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} - \sum_{k=0}^{n-1} \frac{h(f^k(x))}{\prod_{j=0}^{k} g(f^j(x))}\right)_{n\in\mathbb{N}}\right), \qquad x \in X, \tag{21}$$

is a solution of Equation (5) such that

$$\widehat{\xi}(x) := \liminf_{k\to\infty} \widehat{\xi}_k(x) \le \Psi(x) - \psi(x) \le \limsup_{k\to\infty} \widehat{\eta}_k(x) =: \widehat{\eta}(x), \qquad x \in X. \tag{22}$$

Moreover, if

$$\inf_{n\in\mathbb{N}} \frac{\widehat{\eta}(f^n(x)) - \widehat{\xi}(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} = 0, \qquad x \in X, \tag{23}$$

then such Ψ is unique.

Proof. Clearly, (20) can be written as

$$\xi(x) + g(x)\psi(x) \le \psi(f(x)) - h(x) \le \eta(x) + g(x)\psi(x), \qquad x \in X. \tag{24}$$

Note that this is the case $n = 1$ of the following inequalities

$$\begin{aligned}\psi(x) + \widehat{\xi}_n(x) \le \frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} - \sum_{j=0}^{n-1} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}\\ \le \psi(x) + \widehat{\eta}_n(x), \qquad x \in X.\end{aligned} \tag{25}$$

We show by induction with respect to n that (25) holds for every $n \in \mathbb{N}$. Thus, assume that (25) is fulfilled for a given $n \in \mathbb{N}$ and replace x by $f^n(x)$ in (24). Then, for every $x \in X$, we have

$$\begin{aligned}\xi(f^n(x)) + g(f^n(x))\psi(f^n(x)) + h(f^n(x)) \le \psi(f^{n+1}(x))\\ \le \eta(f^n(x)) + g(f^n(x))\psi(f^n(x)) + h(f^n(x)),\end{aligned}$$

whence

$$\begin{aligned}\frac{\xi(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))} + \frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} + \frac{h(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))} \le \frac{\psi(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))}\\ \le \frac{\eta(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))} + \frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} + \frac{h(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))}.\end{aligned} \tag{26}$$

Next, by (26) and (25) written in the form

$$\begin{aligned}\psi(x) + \sum_{j=0}^{n-1} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} + \sum_{j=0}^{n-1} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} \le \frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))}\\ \le \psi(x) + \sum_{j=0}^{n-1} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} + \sum_{j=0}^{n-1} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))},\end{aligned}$$

we obtain

$$
\begin{aligned}
&\frac{\xi(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))} + \psi(x) + \sum_{j=0}^{n-1} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} \\
&\qquad + \sum_{j=0}^{n-1} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} + \frac{h(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))} \\
&\quad \leq \frac{\psi(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))} \\
&\quad \leq \frac{\eta(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))} + \psi(x) + \sum_{j=0}^{n-1} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} \\
&\qquad + \sum_{j=0}^{n-1} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} + \frac{h(f^n(x))}{\prod_{i=0}^{n} g(f^i(x))},
\end{aligned}
$$

which means that

$$
\begin{aligned}
\psi(x) + \sum_{j=0}^{n} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} + \sum_{j=0}^{n} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} &\leq \frac{\psi(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))} \\
&\leq \psi(x) + \sum_{j=0}^{n} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} + \sum_{j=0}^{n} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))},
\end{aligned}
$$

that is

$$
\psi(x) + \widehat{\xi}_{n+1}(x) \leq \frac{\psi(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))} - \sum_{j=0}^{n} \frac{h(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} \leq \psi(x) + \widehat{\eta}_{n+1}(x)
$$

for every $x \in X$. We thus conclude that the required inequalities also are valid for $n+1$, which completes the inductive proof.

According to (25) the sequence

$$
\left(\frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} - \sum_{k=0}^{n-1} \frac{h(f^k(x))}{\prod_{j=0}^{k} g(f^j(x))} \right)_{n\in\mathbb{N}}
$$

is bounded for every $x \in X$. This means that we can define a function $\Psi : X \to \mathbb{R}$ by (21).

It is easily seen that (3) yields

$$
\begin{aligned}
&\liminf_{n\to\infty} \left(\frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} - \sum_{k=0}^{n-1} \frac{h(f^k(x))}{\prod_{j=0}^{k} g(f^j(x))} \right) \leq \Psi(x) \\
&\qquad \leq \limsup_{n\to\infty} \left(\frac{\psi(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))} - \sum_{k=0}^{n-1} \frac{h(f^k(x))}{\prod_{j=0}^{k} g(f^j(x))} \right), \qquad x \in X.
\end{aligned}
$$

Hence, by (25), Ψ satisfies inequalities (22).

Next, (2) and the linearity of LIM imply that

$$
\begin{aligned}
\Psi(f(x)) &= \mathrm{LIM}\left(\left(\frac{\psi(f^{n+1}(x))}{\prod_{i=1}^{n} g(f^i(x))} - \sum_{k=1}^{n} \frac{h(f^k(x))}{\prod_{j=1}^{k} g(f^j(x))} \right)_{n\in\mathbb{N}} \right) \\
&= g(x)\mathrm{LIM}\left(\left(\frac{\psi(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))} - \sum_{k=0}^{n} \frac{h(f^k(x))}{\prod_{j=0}^{k} g(f^j(x))} \right)_{n\in\mathbb{N}} \right) + h(x) \\
&= g(x)\Psi(x) + h(x), \qquad x \in X.
\end{aligned}
$$

This means that Ψ fulfils Equation (5).

The uniqueness of Ψ remains to be demonstrated. To do this, assume that (23) holds, and $\Psi_0 : X \to \mathbb{R}$ is such that

$$\Psi_0(f(x)) = g(x)\Psi_0(x) + h(x), \qquad x \in X \tag{27}$$

and

$$\widehat{\xi}(x) \le \Psi_0(x) - \psi(x) \le \widehat{\eta}(x), \qquad x \in X. \tag{28}$$

Then

$$\begin{aligned} \widehat{\xi}(x) - \widehat{\eta}(x) \le \Psi_0(x) - \Psi(x) &= (\Psi_0(x) - \psi(x)) + (\psi(x) - \Psi(x)) \\ &\le \widehat{\eta}(x) - \widehat{\xi}(x), \qquad x \in X, \end{aligned}$$

whence

$$|\Psi_0(x) - \Psi(x)| \le \widehat{\eta}(x) - \widehat{\xi}(x), \qquad x \in X. \tag{29}$$

We show by induction that, for each $n \in \mathbb{N}$,

$$|\Psi_0(x) - \Psi(x)| \le \frac{\widehat{\eta}(f^n(x)) - \widehat{\xi}(f^n(x))}{\prod_{i=0}^{n-1} g(f^i(x))}, \qquad x \in X. \tag{30}$$

The case $n = 1$ is easy, because, then, in view of (29), (5) and (27) we have

$$g(x)\,|\Psi_0(x) - \Psi(x)| = |\Psi_0(f(x)) - \Psi(f(x))| \le \widehat{\eta}(f(x)) - \widehat{\xi}(f(x))$$

and consequently

$$|\Psi_0(x) - \Psi(x)| \le \frac{\widehat{\eta}(f(x)) - \widehat{\xi}(f(x))}{g(x)}.$$

Now, assume that (30) holds for a fixed $n \in \mathbb{N}$. Then, replacing x with $f(x)$ in (30), in view of (5) and (27), for each $x \in X$ we obtain

$$g(x)\,|\Psi_0(x) - \Psi(x)| = |\Psi_0(f(x)) - \Psi(f(x))| \le \frac{\widehat{\eta}(f^{n+1}(x)) - \widehat{\xi}(f^{n+1}(x))}{\prod_{i=0}^{n-1} g(f^{i+1}(x))},$$

which implies that

$$|\Psi_0(x) - \Psi(x)| \le \frac{\widehat{\eta}(f^{n+1}(x)) - \widehat{\xi}(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))}, \qquad x \in X.$$

In this way we have shown that (30) holds for every $n \in \mathbb{N}_0$, whence using (23), we conclude that $\Psi_0 = \Psi$, which ends the proof. □

Remark 1. *Let the sequences* $\left(\widehat{\xi}_n(x)\right)_{n\in\mathbb{N}}$ *and* $\left(\widehat{\eta}_n(x)\right)_{n\in\mathbb{N}}$*, given by* (19)*, be convergent for each* $x \in X$*. Then*

$$\widehat{\xi}(x) := \sum_{j=0}^{\infty} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}, \qquad \widehat{\eta}(x) := \sum_{j=0}^{\infty} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}, \qquad x \in X,$$

and consequently

$$\begin{aligned} \frac{\widehat{\xi}(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))} &= \sum_{j=n+1}^{\infty} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}, \\ \frac{\widehat{\eta}(f^{n+1}(x))}{\prod_{i=0}^{n} g(f^i(x))} &= \sum_{j=n+1}^{\infty} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} \end{aligned}$$

for any $x \in X$ and $n \in \mathbb{N}$, which means that condition (23) is valid. Therefore, in such a case, we have the uniqueness of Ψ in Theorem 3.

Theorem 3 yields the following simple observation:

Corollary 1. *Let $\varepsilon : X \to \mathbb{R}$ be such that the sequence $\left(\widehat{\varepsilon}_n(x)\right)_{n\in\mathbb{N}}$ given by*

$$\widehat{\varepsilon}_n(x) = \sum_{j=0}^{n-1} \frac{\varepsilon(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))}, \qquad n \in \mathbb{N} \tag{31}$$

is bounded for every $x \in X$. If a function $\psi : X \to \mathbb{R}$ satisfies the equation

$$\psi(f(x)) = g(x)\psi(x) + h(x) + \varepsilon(x), \qquad x \in X,$$

then there exists a unique solution $\Psi : X \to \mathbb{R}$ of Equation (5) such that

$$\liminf_{n\to\infty} \widehat{\varepsilon}_n(x) \le \Psi(x) - \psi(x) \le \limsup_{n\to\infty} \widehat{\varepsilon}_n(x), \qquad x \in X.$$

Proof. It is sufficient to use Theorem 3 with $\xi = \eta = \varepsilon$. □

The following remark shows that the assumption in Theorem 3 on the boundedness of the sequences defined by (19) cannot be omitted.

Remark 2. *Let $X \subset \mathbb{R}$, $f(x) = x$ and $g(x) = 1$ for $x \in X$, $h(X) \subset \mathbb{R}_+$ and $h(x_0) \neq 0$ for an $x_0 \in X$. Let, moreover, $\xi(x) = -h(x)$ and $\eta(x) = 0$ for $x \in X$. Then, every function $\psi : X \to \mathbb{R}$ fulfils*

$$\xi(x) \le \psi(x) - \psi(x) - h(x) \le \eta(x), \qquad x \in X,$$

but $\Psi(x_0) \neq \Psi(x_0) + h(x_0)$, which means that the set of solutions of the equation

$$\Psi(x) = \Psi(x) + h(x), \qquad x \in X$$

is empty. Note that in this case, for any $n \in \mathbb{N}$ and $x \in X$, we have

$$\widehat{\xi}_n(x) = n\xi(x), \qquad \widehat{\eta}_n(x) = 0,$$

so the sequence $\left(\widehat{\xi}_n(x_0)\right)_{n\in\mathbb{N}}$ is unbounded from below.

We can argue analogously with $h(X) \subset (-\infty, 0]$, $\eta(x) = -h(x)$ and $\xi(x) = 0$ for $x \in X$. Then the sequence $\left(\widehat{\eta}_n(x_0)\right)_{n\in\mathbb{N}}$ is unbounded from above.

The following two remarks provide examples of very simple applications of Theorem 3.

Remark 3. *Let $X = [1, \infty)$, $p, q \in \mathbb{R}_+$, $p \le q$, $\alpha, \beta \in (1, \infty)$, $\alpha^q < \beta$, $c, d, \delta \in (0, \infty)$, $c \le d$. Let $f(x) = \alpha x$, $g(x) = \beta$, $\xi(x) = cx^p$ and $\eta(x) = dx^q + \delta$ for $x \in X$. Then, for any $n \in \mathbb{N}$ and $x \in X$,*

$$\widehat{\xi}_n(x) = \sum_{j=0}^{n-1} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} = c\sum_{j=0}^{n-1} \frac{(\alpha^j x)^p}{\beta^{j+1}} = \frac{c}{\beta}\sum_{j=0}^{n-1} \frac{\alpha^{pj}x^p}{\beta^j},$$

$$\widehat{\eta}_n(x) = \sum_{j=0}^{n-1} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} = \frac{1}{\beta}\sum_{j=0}^{n-1} \frac{d\alpha^{qj}x^q + \delta}{\beta^j},$$

which means that

$$\lim_{n\to\infty} \widehat{\xi}_n(x) = \frac{cx^p}{\beta - \alpha^p}, \qquad \lim_{n\to\infty} \widehat{\eta}_n(x) = \frac{dx^q}{\beta - \alpha^q} + \frac{\delta}{\beta - 1}, \qquad x \in X.$$

Hence, by Theorem 3, for every function $\psi : X \to \mathbb{R}$ fulfilling

$$cx^p \le \psi(\alpha x) - \beta\psi(x) - h(x) \le dx^q + \delta, \qquad x \in X \tag{32}$$

there exists a solution $\Psi : X \to \mathbb{R}$ of the equation

$$\Psi(\alpha x) = \beta\Psi(x) + h(x)$$

such that

$$\widehat{\xi}(x) = \frac{cx^p}{\beta - \alpha^p} \le \Psi(x) - \psi(x) \le \widehat{\eta}(x) = \frac{dx^q}{\beta - \alpha^q} + \frac{\delta}{\beta - 1}, \qquad x \in X.$$

Furthermore, by Remark 1, such Ψ is unique.

Note that if, for instance, $p < q$ and $\beta = \alpha^q$, then (32) holds with $\psi(x) \equiv x^q$ and every function h satisfying the inequality

$$-cx^p \ge h(x) \ge -dx^q - \delta, \qquad x \in X.$$

Remark 4. *Let $m \in \mathbb{N}$ be odd, $X = \mathbb{R}$, $\alpha \in (1, \infty)$ and $d, \delta \in (0, \infty)$. Let $f(x) = -\alpha^m x$, $g(x) = \alpha$, $\xi(x) = d\sqrt[m]{x}$ and $\eta(x) = d\sqrt[m]{x} + \delta$ for $x \in X$. Then, for any $n \in \mathbb{N}$ and $x \in X$,*

$$\begin{aligned}
\widehat{\xi}_n(x) &= \sum_{j=0}^{n-1} \frac{\xi(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} = d\sum_{j=0}^{n-1} \frac{(-\alpha)^j \sqrt[m]{x}}{\alpha^{j+1}} \\
&= d\sum_{j=0}^{n-1} \frac{(-1)^j \sqrt[m]{x}}{\alpha} = d\,\frac{(1 + (-1)^{n-1})\sqrt[m]{x}}{2\alpha}, \\
\widehat{\eta}_n(x) &= \sum_{j=0}^{n-1} \frac{\eta(f^j(x))}{\prod_{i=0}^{j} g(f^i(x))} = \sum_{j=0}^{n-1} \frac{d(-\alpha)^j \sqrt[m]{x} + \delta}{\alpha^{j+1}} \\
&= d\,\frac{(1 + (-1)^{n-1})\sqrt[m]{x}}{2\alpha} + \sum_{j=0}^{n-1} \frac{\delta}{\alpha^{j+1}},
\end{aligned}$$

whence, by Theorem 3, for every function $\psi : X \to \mathbb{R}$ fulfilling

$$d\sqrt[m]{x} \le \psi(-\alpha^m x) - \alpha\psi(x) - h(x) \le d\sqrt[m]{x} + \delta, \qquad x \in X, \tag{33}$$

there exists a solution $\Psi : X \to \mathbb{R}$ of the equation

$$\Psi(-\alpha^m x) = \alpha\Psi(x) + h(x)$$

such that

$$\widehat{\xi}(x) \le \Psi(x) - \psi(x) \le \widehat{\eta}(x), \qquad x \in X, \tag{34}$$

where

$$\widehat{\xi}(x) = \begin{cases} 0 & \text{if } x \ge 0; \\ \dfrac{d\sqrt[m]{x}}{\alpha} & \text{if } x < 0; \end{cases} \qquad \widehat{\eta}(x) = \begin{cases} \dfrac{d\sqrt[m]{x}}{\alpha} + \dfrac{\delta}{\alpha - 1} & \text{if } x \ge 0; \\ \dfrac{\delta}{\alpha - 1} & \text{if } x < 0. \end{cases}$$

Furthermore,

$$\frac{\widehat{\eta}(f^n(x)) - \widehat{\xi}(f^n(x))}{\prod_{i=0}^{n-1} |g(f^i(x))|} = \Big(\frac{d(-\alpha)^n \sqrt[m]{x}}{\alpha} + \frac{\delta}{\alpha - 1}\Big)\frac{1}{\alpha^n}, \qquad x \ge 0, n \in \mathbb{N},$$

$$\frac{\widehat{\eta}(f^n(x)) - \widehat{\xi}(f^n(x))}{\prod_{i=0}^{n-1} |g(f^i(x))|} = \left(\frac{\delta}{\alpha - 1} - \frac{d(-\alpha)^n \sqrt[m]{x}}{\alpha}\right)\frac{1}{\alpha^n}, \qquad x < 0, n \in \mathbb{N},$$

whence, unfortunately, (23) does not hold, which means that we cannot deduce the uniqueness of Ψ from Theorem 3 and we do not know if such Ψ is unique in this situation.

It is easily seen that analogous reasonings are valid if $f(x) = \sqrt[m]{-\alpha}\, x$, $g(x) = \alpha$, $\xi(x) = dx^m$ and $\eta(x) = dx^m + \delta$ for $x \in X$.

3. Ulam Stability of the Cauchy Equation

In this section, we consider the stability of the Cauchy functional equation

$$f(x+y) = f(x) + f(y). \tag{35}$$

The first stability result for this equation was published by Hyers [55] in answer to a question of Ulam (cf. [56]). A few years later Aoki [57] extended Hyers' theorem in the following way (the Aoki result with $p = 0$ gives the Hyers outcome).

Theorem 4. *Let E_1 and E_2 be real normed spaces, E_2 be complete, $\varepsilon \geq 0$, $p \in [0,1)$ and $f\colon E_1 \to E_2$ be a mapping with*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \qquad x, y \in E_1. \tag{36}$$

Then, there is a unique additive (i.e., satisfying Equation (35)) mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p, \qquad x \in E_1. \tag{37}$$

Nearly thirty years later, Rassias [58] independently published a result resembling that of Aoki, but for linear mappings. Next, he noticed that a similar reasoning (as for Theorem 4) also works for $p < 0$, and Gajda [59] proved an analogous result for $p > 1$, at the same time providing an example that for $p = 1$, it is not possible.

In 1994, Găvruta [60] published a generalization of the Aoki and Rassias results replacing (36) with a more general inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y). \tag{38}$$

Namely, he proved the following.

Theorem 5. *Assume that $(H,+)$ is an abelian group, V is a Banach space, and $\phi : H^2 \to [0,\infty)$ fulfils*

$$\widetilde{\phi}(x,y) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty, \qquad x, y \in H.$$

If $f : H \to V$ satisfies (38) for any $x, y \in H$, then there is a unique additive $h : H \to V$ with $\|f(x) - h(x)\| \leq \widetilde{\phi}(x,x)$ for each $x \in H$.

We should also mention here that a result more general than Theorem 5 was obtained much earlier in [61].

We refer the reader to [1–3,62] for further information concerning the stability of Equation (35). Various pieces of information on solutions to this equation can be found in [15,16].

Below, using the Banach limit approach, similarly as in the previous section, we prove a generalization of Theorem 5, but only for functions taking real values. We use in it the notion of a square symmetric groupoid.

Let us recall that a groupoid $(G,\star)$ is square symmetric if

$$x^2 \star y^2 = (x \star y)^2, \qquad x, y \in G, \tag{39}$$

where $x^2 := x \star x$ (cf. Remark 5 and [63]). In some situations, for the sake of simplicity of notation (as in the subsequent theorem), it is convenient to denote the operation in the groupoid by the symbol $+$ (without assuming its commutativity), and then condition (39) can be rewritten as

$$2x + 2y = 2(x + y), \qquad x, y \in G, \tag{40}$$

where $2x := x + x$. It is easy to prove by induction that

$$2^n x + 2^n y = 2^n (x + y), \qquad x, y \in G, n \in \mathbb{N}_0, \tag{41}$$

where $2^0 x := x$, $2^{n+1} x := 2(2^n x + 2^n x)$ for $x \in G$ and $n \in \mathbb{N}_0$.

Theorem 6. *Let $(G, +)$ be a square symmetric groupoid, $D \subset G$ be nonempty, $2D := \{2x : x \in D\} \subset D$, $A, B : D^2 \to \mathbb{R}$ be such that*

$$\liminf_{n\to\infty} \frac{B(2^n x, 2^n y)}{2^n} = 0, \qquad \limsup_{n\to\infty} \frac{A(2^n x, 2^n y)}{2^n} = 0, \qquad x, y \in D, \tag{42}$$

and the sequences $(A_n(x))_{n\in\mathbb{N}}$ and $(B_n(x))_{n\in\mathbb{N}}$, where

$$A_n(x) = \sum_{j=0}^{n-1} \frac{A(2^j x, 2^j x)}{2^{j+1}}, \qquad B_n(x) = \sum_{j=0}^{n-1} \frac{B(2^j x, 2^j x)}{2^{j+1}}, \qquad n \in \mathbb{N}, x \in D, \tag{43}$$

be bounded for every $x \in D$. If $\psi : D \to \mathbb{R}$ satisfies

$$B(x, y) \le \psi(x + y) - \psi(y) - \psi(x) \le A(x, y), \qquad x, y \in D, x + y \in D, \tag{44}$$

then the sequence $\big(a_n(x)\big)_{n\in\mathbb{N}}$, where

$$a_n(x) := \frac{\psi(2^n x)}{2^n}, \qquad n \in \mathbb{N}, x \in D, \tag{45}$$

is bounded for every $x \in D$ and the function $\Psi : D \to \mathbb{R}$, given by

$$\Psi(x) := \mathrm{LIM}\big((a_n(x))_{n\in\mathbb{N}}\big), \qquad x \in D, \tag{46}$$

is a solution of the conditional Cauchy functional equation

$$\Psi(x + y) = \Psi(x) + \Psi(y), \qquad x, y \in D, x + y \in D, \tag{47}$$

with

$$\beta(x) := \liminf_{k\to\infty} B_k(x) \le \Psi(x) - \psi(x) \le \limsup_{k\to\infty} A_k(x) =: \alpha(x), \qquad x \in D. \tag{48}$$

Moreover, if

$$\inf_{n\in\mathbb{N}} \frac{\alpha(2^n x) - \beta(2^n x)}{2^n} = 0, \qquad x \in D, \tag{49}$$

then $\Psi : D \to \mathbb{R}$ is the unique solution to (47) such that (48) is valid.

Proof. Taking $x = y$ in (44), we get

$$B(x, x) \le \psi(2x) - 2\psi(x) \le A(x, x), \qquad x \in D. \tag{50}$$

According to Theorem 3 with $X = D$, $\xi(x) = B(x, x)$, $\eta(x) = A(x, x)$, $f(x) \equiv 2x$, $g(x) \equiv 2$ and $h(x) \equiv 0$, the sequence $\big(a_n(x)\big)_{n\in\mathbb{N}}$ defined by (45) is bounded for every $x \in D$, and the function $\Psi : D \to \mathbb{R}$, given by (46), fulfils inequalities (48).

Note that

$$\Psi(x+y)-\Psi(y)-\Psi(x)=\mathrm{LIM}\big((a_n(x+y)-a_n(y)-a_n(x))_{n\in\mathbb{N}}\big),\\ x,y\in D, x+y\in D, \tag{51}$$

and according to (44) and (41) we have

$$\begin{aligned}\frac{B(2^n x,2^n y)}{2^n}&\le a_n(x+y)-a_n(y)-a_n(x)\\&=\frac{\psi(2^n(x+y))}{2^n}-\frac{\psi(2^n y)}{2^n}-\frac{\psi(2^n x)}{2^n}\\&=\frac{\psi(2^n x+2^n y)}{2^n}-\frac{\psi(2^n y)}{2^n}-\frac{\psi(2^n x)}{2^n}\\&\le\frac{A(2^n x,2^n y)}{2^n},\qquad x,y\in D, x+y\in D, n\in\mathbb{N}.\end{aligned} \tag{52}$$

Hence, on account of (3),

$$\liminf_{n\to\infty}\frac{B(2^n x,2^n y)}{2^n}\le\Psi(x+y)-\Psi(y)-\Psi(x)\le\limsup_{n\to\infty}\frac{A(2^n x,2^n y)}{2^n},\\ x,y\in D, x+y\in D, \tag{53}$$

whence (42) implies that

$$\Psi(x+y)-\Psi(y)-\Psi(x)=0,\qquad x,y\in D, x+y\in D. \tag{54}$$

The uniqueness of Ψ remains to be demonstrated. To this end, assume that $\Psi_1,\Psi_2 : D\to\mathbb{R}$ are such that

$$\Psi_i(x+y)=\Psi_i(x)+\Psi_i(y),\qquad x,y\in D, x+y\in D, i=1,2, \tag{55}$$

and

$$\begin{aligned}\beta(x):=\liminf_{k\to\infty}\ B_k(x)&\le\Psi_i(x)-\psi(x)\\&\le\limsup_{k\to\infty}\ A_k(x)=:\alpha(x),\qquad x\in D, i=1,2.\end{aligned} \tag{56}$$

Then

$$\beta(x)-\alpha(x)\le\Psi_1(x)-\Psi_2(x)\le\alpha(x)-\beta(x),\qquad x\in D, \tag{57}$$

which means that

$$|\Psi_1(x)-\Psi_2(x)|\le\alpha(x)-\beta(x),\qquad x\in D. \tag{58}$$

Next, note that (55) gives

$$\Psi_i(2^n x)=2^n\Psi_i(x),\qquad x\in D, i=1,2, n\in\mathbb{N}, \tag{59}$$

where, replacing x by $2^n x$ in (58), we get

$$|\Psi_1(x)-\Psi_2(x)|\le\frac{\alpha(2^n x)-\beta(2^n x)}{2^n},\qquad x\in D, n\in\mathbb{N}, \tag{60}$$

which in view of (49) yields $\Psi_1=\Psi_2$. This completes the proof. □

Remark 5. *Clearly, every commutative semigroup is a square symmetric groupoid. If $c,d,e\in\mathbb{R}$ and $x\oplus y=cx+dy+e$ for $x,y\in\mathbb{R}$, then it is easy to check that $(\mathbb{R},\oplus)$ is another simple example of such a groupoid, which in general (depending on c and d) is neither commutative nor associative.*

Arguing analogously as above, we obtain a complementary version of Theorem 6. Before we state it, let us recall that we say that a groupoid $(G,+)$ is uniquely divisible by 2 provided for every $y \in G$ there exists a unique $x \in G$ with $x+x=y$; we denote such x by $2^{-1}y$ and define recurrently $2^{-n-1}y := 2^{-1}(2^{-n}y)$ for $n \in \mathbb{N}$. Note also that the square symmetric groupoid mentioned in Remark 5 is uniquely divisible by 2 if and only if $c+d \neq 0$.

Theorem 7. *Let $(G,+)$ be a square symmetric groupoid, uniquely divisible by 2; $D \subset G$ be nonempty; and $2^{-1}D := \{2^{-1}x : x \in D\} \subset D$, $A, B : D^2 \to \mathbb{R}$ be such that*

$$\liminf_{n\to\infty} 2^n B(2^{-n}x, 2^{-n}y) = 0, \qquad \limsup_{n\to\infty} 2^n A(2^{-n}x, 2^{-n}y) = 0, \qquad x, y \in D, \tag{61}$$

and the sequences $(A_n(x))_{n\in\mathbb{N}}$ and $(B_n(x))_{n\in\mathbb{N}}$, where

$$A_n(x) = \sum_{j=0}^{n-1} 2^j A(2^{-j-1}x, 2^{-j-1}x), \qquad B_n(x) = \sum_{j=0}^{n-1} 2^j B(2^{-j-1}x, 2^{-j-1}x), \qquad n \in \mathbb{N}, x \in D, \tag{62}$$

be bounded for every $x \in D$. If $\psi : D \to \mathbb{R}$ satisfies (44), then the sequence $\big(a_n(x)\big)_{n\in\mathbb{N}}$, where

$$a_n(x) := 2^n \psi(2^{-n}x), \qquad n \in \mathbb{N}, x \in D, \tag{63}$$

is bounded for every $x \in D$ and the function $\Psi : D \to \mathbb{R}$, given by (46), is a solution of Equation (47) satisfying the inequalities

$$\beta(x) := \liminf_{k\to\infty} B_k(x) \le \psi(x) - \Psi(x) \le \limsup_{k\to\infty} A_k(x) =: \alpha(x), \qquad x \in D. \tag{64}$$

Moreover, if

$$\inf_{n\in\mathbb{N}} 2^n \big(\alpha(2^{-n}x) - \beta(2^{-n}x)\big) = 0, \qquad x \in D, \tag{65}$$

then $\Psi : D \to \mathbb{R}$ is the unique solution to (47) such that (64) is valid.

Proof. As we have mentioned, the proof is very analogous to the proof of Theorem 6, but for the convenience of the readers, we provide some details of it.

Replacing x and y in (44) by $2^{-1}x$, we obtain

$$\begin{aligned} B(2^{-1}x, 2^{-1}x) &\le \psi(x) - 2\psi(2^{-1}x) \\ &\le A(2^{-1}x, 2^{-1}x), \qquad x \in D, \end{aligned} \tag{66}$$

which can be rewritten as

$$\begin{aligned} -\frac{1}{2}A(2^{-1}x, 2^{-1}x) &\le \psi(2^{-1}x) - \frac{1}{2}\psi(x) \\ &\le -\frac{1}{2}B(2^{-1}x, 2^{-1}x), \qquad x \in D. \end{aligned}$$

According to Theorem 3 with $X = D$,

$$\xi(x) = -\frac{1}{2}A(2^{-1}x, 2^{-1}x), \qquad \eta(x) = -\frac{1}{2}B(2^{-1}x, 2^{-1}x), \qquad x \in D,$$

$f(x) = 2^{-1}x$, $g(x) \equiv \frac{1}{2}$ and $h(x) \equiv 0$, the sequence $\big(a_n(x)\big)_{n\in\mathbb{N}}$ defined by (63) is bounded for every $x \in D$ and the function $\Psi : D \to \mathbb{R}$, given by (46), fulfils the inequalities

$$\liminf_{k\to\infty}(-A_k(x)) \le \Psi(x) - \psi(x) \le \limsup_{k\to\infty}(-B_k(x)), \qquad x \in D, \tag{67}$$

which implies (64).

Next, note that (51) holds, and, according to (44), we obtain

$$\begin{aligned} 2^n B(2^{-n}x, 2^{-n}y) &\le a_n(x+y) - a_n(y) - a_n(x) \\ &= 2^n \psi(2^{-n}(x+y)) - 2^n \psi(2^{-n}y) - 2^n \psi(2^{-n}x) \\ &\le 2^n A(2^{-n}x, 2^{-n}y), \qquad n \in \mathbb{N}, x, y \in D, x+y \in D. \end{aligned} \tag{68}$$

Hence

$$\begin{aligned} \liminf_{n\to\infty} 2^n B(2^{-n}x, 2^{-n}y) &\le \Psi(x+y) - \Psi(y) - \Psi(x) \\ &\le \limsup_{n\to\infty} 2^n A(2^{-n}x, 2^{-n}y), \qquad x, y \in D, x+y \in D, \end{aligned}$$

and (61) now shows that (54) is valid.

The uniqueness of Ψ remains to be proved. To do this, let us suppose that $\Psi_1, \Psi_2 : D \to \mathbb{R}$ are solutions of Equation (47) such that (48) holds.

Then, we obtain (57) and, consequently, (58). Now, replacing x by $2^{-n}x$ in (58), we see that

$$|\Psi_1(x) - \Psi_2(x)| \le 2^n \big(\alpha(2^{-n}x) - \beta(2^{-n}x)\big), \qquad x \in D, n \in \mathbb{N}, \tag{69}$$

whence (65) implies that $\Psi_1 = \Psi_2$. □

Remark 6. *Let $(G, +)$ be a groupoid, $D \subset G$ be nonempty and $c, \rho_1, \rho_2 \in \mathbb{R}$, $\rho_1 < \rho_2$. Let $\psi(x) = d(x) + h(x) + c$ for $x \in D$, where $h : D \to \mathbb{R}$ fulfils Equation (47), and $d : D \to \mathbb{R}$ be such that $d(D) \subset [\rho_1, \rho_2]$. Then*

$$\begin{aligned} \rho_1 - 2\rho_2 - c \le \psi(x+y) - \psi(y) - \psi(x) &= d(x+y) - d(y) - d(x) - c \\ &\le \rho_2 - 2\rho_1 - c, \qquad x, y \in D, x+y \in D \end{aligned} \tag{70}$$

and

$$\psi(x) - h(x) \in [\rho_1 + c, \rho_2 + c], \qquad x \in D. \tag{71}$$

Thus, we can see that families of functions considered in Theorems 6 and 7 are very large. Clearly, numerous other more sophisticated examples can be easily found.

Theorems 6 and 7 yield the following generalization of Theorem 4.

Theorem 8. *Let E_1 be a real normed space, $E_0 := E_1 \setminus \{0\}$, $\varepsilon \ge 0$, $\chi, \rho, p \in \mathbb{R}$, $p \ne 1$, $\chi \le \rho$ and $f\colon E_1 \to \mathbb{R}$ be a mapping with*

$$\chi(\|x\|^p + \|y\|^p) \le f(x+y) - f(x) - f(y) \le \rho(\|x\|^p + \|y\|^p), \qquad x, y \in E_0. \tag{72}$$

Then, there is a unique additive mapping $T\colon E_1 \to \mathbb{R}$ such that, in the case $p < 1$,

$$\frac{\chi}{1 - 2^{p-1}}\|x\|^p \le T(x) - f(x) \le \frac{\rho}{1 - 2^{p-1}}\|x\|^p, \qquad x \in E_0, \tag{73}$$

and, in the case $p > 1$,

$$\frac{\chi}{2^{p-1} - 1}\|x\|^p \le f(x) - T(x) \le \frac{\rho}{2^{p-1} - 1}\|x\|^p, \qquad x \in E_0. \tag{74}$$

Moreover, if f is continuous at a point, then T is continuous.

Proof. It is easy to check that, according to Theorems 6 and 7 (if $p < 1$, then we use Theorem 6; if $p > 1$, then Theorem 7 is applied), with $G = E_1$, $D = E_0$, $\psi = f_{|E_0}$ and $A(x,y) = \chi(\|x\|^p + \|y\|^p)$, $B(x,y) = \rho(\|x\|^p + \|y\|^p)$ for $x, y \in E_0$, there exists a unique solution $\Psi\colon E_0 \to \mathbb{R}$ to Equation (47), satisfying inequalities (48) ((64), respectively).

Define $T\colon E_1 \to \mathbb{R}$ by $T(x) = \Psi(x)$ for $x \in E_0$ and $T(0) = 0$. First, note that (48) ((64) respectively) is actually (73) ((74) respectively).

We are now able to prove that T is additive. In view of the definition of T, it is enough to show that $T(-x) = -T(x)$ for $x \in E_0$. To do this, fix an $x \in E_0$. Then

$$T(x) = \Psi(x) = \Psi(2x - x) = \Psi(2x) + \Psi(-x) = 2\Psi(x) + \Psi(-x) = 2T(x) + T(-x),$$

whence $T(-x) = -T(x)$, as required.

The uniqueness of T follows from the uniqueness of Ψ and the fact that $T(0) = 0$ for every additive function $T\colon E_1 \to \mathbb{R}$.

Finally, suppose that f is continuous at a point. Then, (73) ((74) respectively) implies that T is bounded on a neighborhood of that point. It is well-known that this yields the continuity of T (see, e.g., [15,16]). □

Remark 7. *Note that* (72) *gives*

$$|f(x+y) - f(x) - f(y)| \le \rho_0(\|x\|^p + \|y\|^p), \qquad x, y \in E_0, \tag{75}$$

where $\rho_0 := \max\{|\rho|, |\chi|\}$*. Thus, if* $p < 0$*, then every function* $f\colon E_1 \to \mathbb{R}$ *satisfying* (72) *has to be additive (see, e.g., ([62] Theorem 3.5)), i.e.,* $f(x+y) - f(x) - f(y) = 0$ *for* $x, y \in E_1$*. This means that functions* $f : E_1 \to \mathbb{R}$ *satisfying* (72) *with* $p < 0$ *exist only if* $\rho\chi \le 0$ *and they are additive.*

Author Contributions: Conceptualization, R.B., J.B. and K.C.; methodology, R.B., J.B. and K.C.; software, R.B., J.B. and K.C.; validation, R.B., J.B. and K.C.; formal analysis, R.B., J.B. and K.C.; investigation, R.B., J.B. and K.C.; resources, R.B., J.B. and K.C.; data curation, R.B., J.B. and K.C.; writing—original draft preparation, J.B.; writing—review and editing, R.B., J.B. and K.C.; visualization, R.B., J.B. and K.C.; supervision, J.B.; project administration, R.B., J.B. and K.C.; funding acquisition, J.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflicts of interest.

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