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Quadruple Roman Domination in Trees

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Abstract: This paper is devoted to the study of the quadruple Roman domination in trees, and it is a contribution to the Special Issue "Theoretical computer science and discrete mathematics" of Symmetry. For any positive integer k, a $[k]$-Roman dominating function ($[k]$-RDF) of a simple graph G is a function from the vertex set V of G to the set $\{0, 1, 2, \ldots, k+1\}$ if for any vertex $u \in V$ with $f(u) < k$, $\sum_{x \in N(u) \cup \{u\}} f(x) \geq |\{x \in N(u) : f(x) \geq 1\}| + k$, where $N(u)$ is the open neighborhood of u. The weight of a $[k]$-RDF is the value $\Sigma_{v \in V} f(v)$. The minimum weight of a $[k]$-RDF is called the $[k]$-Roman domination number $\gamma_{[kR]}(G)$ of G. In this paper, we establish sharp upper and lower bounds on $\gamma_{[4R]}(T)$ for nontrivial trees T and characterize extremal trees.

Keywords: quadruple Roman domination; Roman domination; trees

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1. Introduction

For notation and graph theory terminology, we in general follow Haynes et al. [1]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$ and let $n = n(G) = |V(G)|$ be the *order* of G. The *open neighborhood* of u is the set $N(u) = \{x \in V(G) \mid ux \in E(G)\}$ and the *closed neighborhood* of u is the set $N[u] = \{u\} \cup N(u)$. Let $d(u) = |N(u)|$ denote the *degree* of a vertex u of G. The number of vertices in distance 2 of u is denoted by $N_2(u)$. The *diameter* $diam(G)$ of G is the maximum distance among all pairs of vertices in G. A *leaf* is a vertex of degree one and a *support vertex* is a vertex adjacent to a leaf. For any tree T, let $\ell(T)$ and $s(T)$ denote the number of leaves and support vertices of T, respectively. For any vertex u in a rooted tree T, the subtree of T induced by u and its descendants is called the *maximal subtree* T_u at u. For any integer $s \geq 1$, a star $K_{1,s}$ with at most $s-1$ of its edges subdivided is a *wounded spider*, and we call a vertex of degree s the *head vertex* and call the leaves at distance two from the head vertex the *foot vertices*. For any integer $s \geq 2$, a star $K_{1,s}$ with all its edges subdivided is a *healthy spider* S_{s+1}. We let K_n and C_n be the complete graph and the cycle of order n, respectively.

A *dominating* set of a graph G is a set $U \subseteq V(G)$ if $N[U] = V(G)$. The minimum cardinality of a dominating set of G is called the *domination number* $\gamma(G)$ of G, and a *γ-set* of G is a dominating set of G of cardinality $\gamma(G)$. For a real-valued function $f : V(G) \rightarrow \mathbb{R}$ and $H \subseteq V(G)$, let $f(H) = \Sigma_{x \in H} f(x)$, and $\omega(f) = f(V(G))$ is called the *weight* of f.

ReVelle [2] and Stewart [3] introduced independently the Roman domination, which is defined formally by Cockayne et al. [4] as follows. A *Roman dominating function* (RDF) on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has a neighbor u with $f(u) = 2$. The minimum weight of an RDF on G is called the *Roman domination number* of a graph G, denoted by $\gamma_R(G)$. An RDF on G with weight $\gamma_R(G)$ is a *$\gamma_R(G)$-function*. For more details on Roman domination, the readers may refer to two chapter books [5,6] and survey papers [7–9].

A generalization of the Roman domination number was introduced by Ahanghar et al. in [10]. For any integer $k \geq 1$, let $S := \{0, 1, 2, \ldots, k+1\}$. For any function $f : V(G) \longrightarrow S$, the *active neighborhood* of a vertex u of G is the set $\mathrm{AN}(u) = \{x \in N(u) : f(x) \geq 1\}$. A *[k]-Roman dominating function* ([k]-RDF) is a function $f : V(G) \longrightarrow S$ if each vertex $u \in V(G)$ with $f(u) < k$ satisfies that $\sum_{x \in N[u]} f(x) \geq |\mathrm{AN}(u)| + k$. The minimum weight of a $[k]$-RDF on G is called the *[k]-Roman domination number* $\gamma_{[kR]}(G)$ of G. A $[k]$-RDF on G with weight $\gamma_{[kR]}(G)$ is a *$\gamma_{[kR]}(G)$-function*. The case $k = 1$ is the usual Roman domination, the case $k = 2$ is called double Roman domination and were studied in several papers [11–20] while the case $k = 3$ is called the triple Roman domination and were studied in several papers [10,21]. The case $k = 4$ is called *quadruple Roman domination* and were studied in [22].

To better understand the definition, we give an example. A regular icosahedron is a convex polyhedron with 20 faces, 30 edges and 12 vertices. It is one of the five Platonic solids, and the one with the most faces. The skeleton of the icosahedron forms a graph (see Figure 1). It is shown in [23] that the regular icosahedron graph is symmetric. It is not hard to see that $[k]$-Roman domination number of icosahedra is $2k + 2$ for $k = 1, 2, 3, 4$.

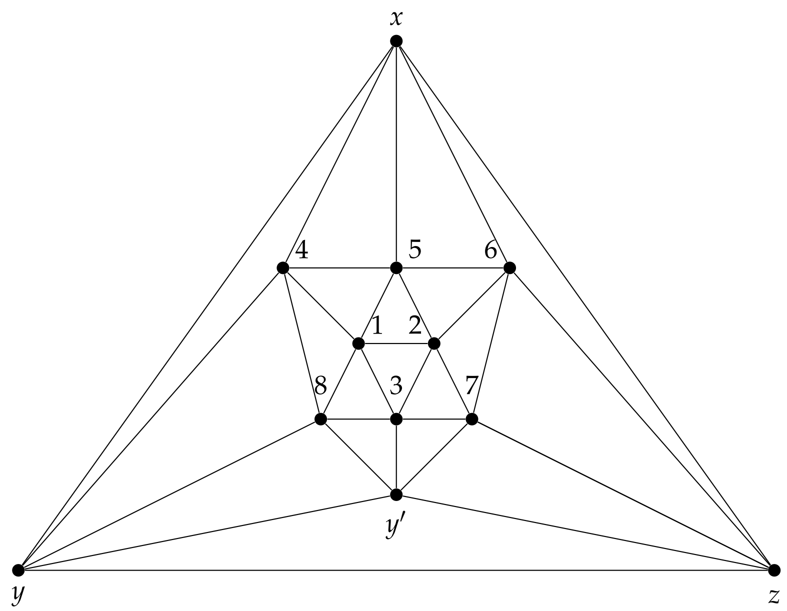

Figure 1. The regular icosahedron graph.

Every function $f : V(G) \to \{0, 1, 2, \ldots, k\}$ can be represented by the ordered partition $(V_0, V_1, \ldots, V_k)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2, \ldots, k\}$.

In this paper, we first prove that $\frac{5(n+2-\ell(T))}{3} \leq \gamma_{[4R]}(T) \leq \frac{4n+s(T)}{2}$ and we characterize the trees achieving these bounds. Then we present bounds on $\gamma_{[4R]}(T)$ in terms of (Roman) domination number and we give a characterization of extremal trees. The following results will be used in this paper.

Proposition 1 ([22]). *For any graph G, there exists a $\gamma_{[4R]}(G)$-function such that $f(x) \neq 1$ for any $x \in V(G)$.*

Let $\mathcal{T}$ be the family of all trees that can be obtained from $k \geq 1$ paths $P_4^i := v_1^i v_2^i v_3^i v_4^i$, where $1 \leq i \leq k$, by adding $k - 1$ edges incident with the v_2^i's such that they induce a connected subgraph.

Proposition 2 ([22]). *For any tree T with order $n \geq 3$, $\gamma_{[4R]}(T) \leq \dfrac{9n}{4}$, and equality holds if and only if $T \in \mathcal{T}$.*

2. Bound on $\gamma_{[4R]}(T)$

In this section, we establish bounds on the quadruple Roman domination in trees. First, we improve the upper bound of Proposition 2 for trees. Let $\mathcal{F}$ be the family of all trees that can be built from $k_1 \geq 0$ paths $P_4^i = v_1^i v_2^i v_3^i v_4^i$ and $k_2 \geq 0$ healthy spiders S_4^i with vertex set $V(S_4^i) = \{z^i, w_1^i, w_2^i, w_3^i, w_4^i, z_1^i, z_2^i, z_3^i, z_4^i\}$ and edge set $E(S_4^i) = \{z^i w_j^i, w_j^i z_j^i \mid 1 \leq j \leq 4\}$, where $k_1 + k_2 \geq 1$, by adding $k_1 + k_2 - 1$ edges between $v_2^1, \ldots, v_2^{k_1}, z^1, z^2, \ldots, z^{k_2}$ to connect the graph (for instance see Figure 2).

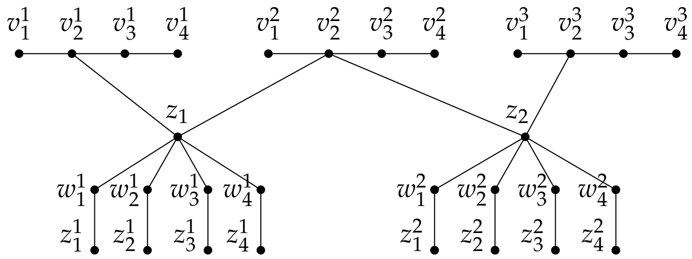

Figure 2. A tree $T \in \mathcal{T}$ in which $k_1 = 3$ and $k_2 = 2$

Lemma 1. If $T \in \mathcal{F}$, then $\gamma_{[4R]}(T) = \frac{4n+s(T)}{2}$.

Proof. Let $T \in \mathcal{F}$ be obtained from $k_1 \geq 0$ paths P_4 and $k_2 \geq 0$ healthy spiders S_4. If H is an induced subgraph of T that is isomorphic to one copy of P_4, then each [4]-RDF on T assigns a weight of at least 9 to H. On the other hand, if H is an induced subgraph of T that is isomorphic to one copy of healthy spider S_4, then each [4]-RDF on T assigns a weight of at least 20 to H. By the definition of $\mathcal{F}$, T has k_1 disjoint copies of P_4 and k_2 disjoint copies of S_4. Hence, $\gamma_{[4R]}(T) \geq 9k_1 + 20k_2 = \frac{4n+s(T)}{2}$.

Now define g on T by $g(v_2^i) = 5, g(v_4^i) = 4$, for $1 \leq i \leq k_1$, $g(w_1^j) = g(w_2^j) = g(w_3^j) = g(w_4^j) = 5$ for $1 \leq j \leq k_2$ and $g(x) = 0$ otherwise, is a [4]-RDF on T of weight $9k_1 + 20k_2 = \frac{4n+s(T)}{2}$. As a result, we have $\gamma_{[4R]}(T) = \frac{4n+s(T)}{2}$. $\square$

In the sequel, if T is a tree and T' is a subtree of T, then $T - T'$ denotes the graph $T - V(T')$.

Theorem 1. *For any tree T with order $n \geq 3$,*

$$\gamma_{[4R]}(T) \leq \frac{4n + s(T)}{2}.$$

Equality holds if and only if $T \in \mathcal{F}$.

Proof. The proof is by induction on $|V(T)| = n$. The statement is trivial for $n = 3$. Suppose that $n \geq 4$ and assume that the statement is true for any tree of order less than n. If $diam(T) = 2$, then statement is trivial. Let $diam(T) = 3$. Clearly T is a double star. Thus, $s(T) = 2$ and hence $\gamma_{[4R]}(T) \leq \frac{4n+s(T)}{2}$ with equality if and only if $T = P_4$. So, in the following we may assume that $diam(T) \geq 4$. Let $u_1 u_2 \ldots u_k$ be a diametral path in T satisfying that $d(u_2)$ is as large as possible, where $k \geq 5$. Root T at u_k. First suppose that $d(u_2) \geq 3$. Let f be a $\gamma_{[4R]}(T')$-function, where $T' = T - u_1$. Since $d(u_2) \geq 3, s(T) = s(T')$ and $f(N_{T'}(u_2) - \{u_3\}) \geq 4$. Define the function $h : V(T) \longrightarrow S$ by $h(u_2) = 5, h(x) = 0$ for each $x \in N_T(u_2) - \{u_3\}$ and $h(x) = f(x)$ otherwise. Obviously h is a [4]-RDF on T and $\gamma_{[4R]}(T) \leq \omega(g) \leq \omega(f) + 1 = \gamma_{[4R]}(T') + 1$. By the induction hypothesis, we obtain $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T') + 1 \leq \frac{4(n-1)+s(T')}{2} + 1 = \frac{4n+s(T)-2}{2} < \frac{4n+s(T)}{2}$, as desired. Suppose that $d(u_2) = 2$. By the choice of diametral path, we have that the degree of each child of u_3 with depth 1, is 2.

If $d(u_3) = 2$, then clearly any $\gamma_{[4R]}(T - T_{u_3})$-function can be extended to a [4]-RDF on T by assigning 0 to u_1, u_3 and 5 to u_2, and so by the induction hypothesis, $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_3}) + 5 \leq \frac{4(n-3)+s(T-T_{u_3})}{2} + 5 < \frac{4n+s(T)}{2}$. Assume that $d(u_3) \geq 3$. If u_3 is a strong support vertex or u_3 is a support vertex and $d(u_3) \geq 4$, then by the induction hypothesis, $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_2}) + 4 \leq \frac{4(n-2)+s(T-T_{u_2})}{2} + 4 < \frac{4n+s(T)}{2}$, as desired.

Case 1. u_3 is a support vertex of degree 3.

If $diam(G) = 4$, then by assigning 5 to u_3, 4 to all leaves at distance 2 from u_3 and 0 to remaining vertices we obtain a [4]-RDF on T with weight 13, implying that $\gamma_{[4R]}(T) \leq 13 < \frac{4n+s(T)}{2}$. Thus, let $diam(G) \geq 5$. If $d(u_4) = 2$, then by assigning 5 to u_3, 4 to u_1 and 0 to other vertices of T_{u_4}, a $\gamma_{[4R]}(T - T_{u_4})$-function is extended to a $\gamma_{[4R]}(T)$-function and so by induction hypothesis, $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_4}) + 9 \leq \frac{4(n-5)+s(T')}{2} + 9 < \frac{4n+s(T)}{2}$. Now assume that $d(u_4) \geq 3$, it is obvious that $s(T - T_{u_3}) = s(T) - 2$ and as above $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_3}) + 9$. By induction hypothesis on $T - T_{u_3}$, we have

$$\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T') + 9 \leq \frac{4(n-4) + s(T')}{2} + 9 = \frac{4n + s(T)}{2}, \tag{1}$$

as desired.

Let the equality $\gamma_{[4R]}(T) = \frac{4n+s(T)}{2}$ hold. Then we must have equality throughout in Equation (1). In particular, we must have $\gamma_{[4R]}(T') = \frac{4(n-4)+s(T')}{2}$. By the induction hypothesis, $T \in \mathcal{F}$. Hence T' can be built from $k_1 \geq 0$ copies $P_4^i = v_1^i v_2^i v_3^i v_4^i$ and $k_2 \geq 0$ copies S_4^j by adding $k_1 + k_2 - 1$ edges between $v_2^1, \ldots, v_2^{k_1}, z^1, z^2, \ldots, z^{k_2}$ to connect the graph. Without loss of generality, we may assume that $u_4 \in V(P_4^{k_1}) \cup V(S_4^{k_2})$. First let $u_4 \in V(P_4^{k_1})$. If u_4 is a leaf of T', then there exists a $\gamma_{[4R]}(T')$-function that assigns 4 to u_4 and 5 to the other support vertex of $P_4^{k_1}$, and the function h defined by $h(u_3) = 5, h(u_1) = 4, h(u_4) = h(u_2) = h(w) = 0$ and $h(x) = f(x)$ otherwise, is a [4]-RDF on T and so $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_3}) + 5 \leq \frac{4(n-4)+s(T)-1}{2} + 5 < \frac{4n+s(T)}{2}$ which is impossible. Hence u_4 is a support vertex. If $k_1 + k_2 = 1$, then $T \in \mathcal{F}$. Suppose that $k_1 + k_2 \geq 2$. If $u_4 = v_3^{k_1}$, then the function h defined by $h(u_3) = h(v_2^i) = 5$ for $1 \leq i \leq k_1 - 1$, $h(v_4^i) = 4$ for $1 \leq i \leq k_1$, $h(v_1^{k_1}) = h(u_1) = 4$, $h(z^j) = h(z_1^j) = h(z_2^j) = h(z_3^j) = h(z_4^j) = 4$ for $1 \leq j \leq k_2$ and $h(x) = 0$ otherwise, is a [4]-RDF of T and so $\gamma_{[4R]}(T) < \frac{4n+s(T)}{2}$, a contradiction. Thus, $u_4 = v_2^{k_1}$ and so $T \in \mathcal{F}$.

Now let $u_4 \in V(S_4^{k_2})$. If u_4 is a leaf of T', say $u_4 = z_4^{k_2}$, then the function g defined on T by $g(v_2^i) = 5, g(v_4^i) = 4$, for $1 \leq i \leq k_1$, $g(w_1^j) = g(w_2^j) = g(w_3^j) = g(w_4^j) = 5$ for $1 \leq j \leq k_2 - 1$, $g(z^{k_2}) = 5, g(z_1^{k_2}) = g(z_2^{k_2}) = g(z_3^{k_2}) = 4$, $g(u_3) = 5, g(u_1) = 4$ and $g(x) = 0$ otherwise, is a quadruple Roman dominating function of T of weight less than $\frac{4n+s(T)}{2}$ which is a contradiction. If u_4 is a support vertex of T', say $u_4 = w_4^{k_2}$, then the function g defined on T by $g(v_2^i) = 5, g(v_4^i) = 4$, for $1 \leq i \leq k_1$, $g(w_1^j) = g(w_2^j) = g(w_3^j) = g(w_4^j) = 5$ for $1 \leq j \leq k_2 - 1$, $g(z_4^{k_2}) = 4, g(w_1^{k_2}) = g(w_2^{k_2}) = g(w_3^{k_2}) = 5$, $g(u_3) = 5, g(u_1) = 4$ and $g(x) = 0$ otherwise, is a quadruple Roman dominating function of T of weight less than $\frac{4n+s(T)}{2}$ which is a contradiction. Thus, $u_4 = z^k$ and so $T \in \mathcal{F}$

Case 2. u_3 is not a support vertex and $d(u_3) \geq 4$.

It is clear that T_{u_3} is a healthy spider with at least three feet. We first assume that $d(u_3) \geq 6$, then $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_2}) + 4$ and applying the fact $s(T - T_{u_2}) = s(T) - 1$ and the induction hypothesis, we have $\gamma_{[4R]}(T) < \frac{4n+s(T)}{2}$. Now, we distinguish two situations.

Subcase 2.1. $d(u_3) = 4$.

First let $d(u_4) = 2$ and let $T' = T - T_{u_4}$. Then $s(T - T_{u_4}) \leq s(T) - 2$ and every $\gamma_{[4R]}(T - T_{u_4})$-function can be extended to a $\gamma_{[4R]}(T)$-function by assigning 5 to u_3, 4 to leaves of T_{u_4} except u_4 and 0 to remaining vertices. Thus, by induction hypothesis,

$$\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_4}) + 17 \leq \frac{4(n-8) + s(T) - 2}{2} + 17 \leq \frac{4n + s(T)}{2}.$$

Let the equality $\gamma_{[4R]}(T) = \frac{4n+s(T)}{2}$ hold. Then we must have equality throughout in Equation (1). In particular, we must have $\gamma_{[4R]}(T') = \frac{4(n-4)+s(T')}{2}$ and that u_5 is a leaf in T'. If $|V(T')| = 2$, then clearly $\gamma_{[4R]}(T) < \frac{4n+s(T)}{2}$, which is impossible. By the induction hypothesis, we have $T' \in \mathcal{F}$. Clearly T' has a $\gamma_{[4R]}(T')$-function h assigning 4 to u_5. Now h can be extended to a [4]-RDF on T by assigning 4 to u_3 and any leaf of T_{u_4} at distance 2 from u_3 and 0 to remaining vertices, implying that $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_4}) + 16 < \frac{4n+s(T)}{2}$ which is a contradiction.

Assume now that $d(u_4) \geq 3$. Then $s(T - T_{u_3}) = s(T) - 3$ and each $\gamma_{[4R]}(T - T_{u_3})$-function can be extended to a $\gamma_{[4R]}(T)$-function by assigning 5 to support vertices in T_{u_3} and 0 to remaining vertices. Thus, by induction hypothesis, $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_3}) + 15 \leq \frac{4(n-7)+s(T-T_{u_3})}{2} + 15 < \frac{4n+s(T)}{2}$, as desired.

Subcase 2.2. $d(u_3) = 5$.

Assume first that $d(u_4) = 2$ and let $T' = T - T_{u_4}$. Then $s(T - T_{u_4}) \leq s(T) - 3$ and each $\gamma_{[4R]}(T - T_{u_4})$-function can be extended to a $\gamma_{[4R]}(T)$-function by assigning 5 to u_3, 4 to leaves of T_{u_4} except u_4 and 0 to remaining vertices. Thus, $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_4}) + 21$. Therefore, by induction hypothesis,

$$\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_4}) + 21 \leq \frac{4(n-10) + s(T) - 3}{2} + 21 < \frac{4n + s(T)}{2}.$$

Assume now that $d(u_4) \geq 3$. Then $s(T - T_{u_3}) = s(T) - 4$ and each $\gamma_{[4R]}(T - T_{u_3})$-function can be extended to a $\gamma_{[4R]}(T)$-function by assigning 5 to support vertices in T_{u_3} and 0 to the remaining vertices. Thus, by induction hypothesis,

$$\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_3}) + 20 \leq \frac{4(n-9) + s(T) - 4}{2} + 20 \leq \frac{4n + s(T)}{2}. \quad (2)$$

Let the equality $\gamma_{[4R]}(T) = \frac{4n+s(T)}{2}$ hold. Let $T' = T - T_{u_3}$. Then we must have equality throughout in Equation (2). Moreover, we must have $\gamma_{[4R]}(T') = \frac{4(n-9)+s(T)-4}{2}$ and that u_4 is not a leaf of T'. By the induction hypothesis, $T' \in \mathcal{F}$. As a result, T' can be built from $k_1 \geq 0$ copies $P_4^i = v_1^i v_2^i v_3^i v_4^i$ and $k_2 \geq 0$ copies S_4^j by adding $k_1 + k_2 - 1$ edges between $v_2^1, \ldots, v_2^{k_1}, z^1, z^2, \ldots, z^{k_2}$ to connect the graph. Without loss of generality, we may assume that $u_4 \in V(P_4^{k_1}) \cup V(S_4^{k_2})$. First let $u_4 \in V(P_4^{k_1})$. Then u_4 is a support vertex. If $k_1 + k_2 = 1$, then $T \in \mathcal{F}$. Let $k_1 + k_2 \geq 2$. If $u_4 = v_3^{k_1}$, then the function h defined by $h(v_1^{k_1}) = 4$, $h(v_2^i) = 5$ for each $1 \leq i \leq k_1 - 1$, $h(v_4^i) = 4$ for $1 \leq i \leq k_1$, $h(x) = 4$ for $x \in L(T_{u_3}) \cup \{u_3\}$, $h(z^j) = h(z_1^j) = h(z_2^j) = h(z_3^j) = h(z_4^j) = 4$ for each $1 \leq j \leq k_2$, and $h(x) = 0$ otherwise, is a [4]-RDF on T and so $\gamma_{[4R]}(T) < \frac{4n+s(T)}{2}$, a contradiction. Thus, $u_4 = v_2^{k_1}$ and so $T \in \mathcal{F}$.

Now let $u_4 \in V(S_4^{k_2})$. If u_4 is a support vertex of T', say $u_4 = w_4^{k_2}$, then the function h defined on T by $h(v_2^i) = 5, h(v_4^i) = 4$, for $1 \leq i \leq k_1$, $h(w_1^j) = h(w_2^j) = h(w_3^j) = h(w_4^j) = 5$ for $1 \leq j \leq k_2 - 1$, $h(z_4^{k_2}) = 4, h(w_1^{k_2}) = h(w_2^{k_2}) = h(w_3^{k_2}) = 5$, $h(x) = 4$ for $x \in L(T_{u_3}) \cup \{u_3\}$ and $h(x) = 0$ otherwise, is a quadruple Roman dominating function of T of weight less than $\frac{4n+s(T)}{2}$ which is a contradiction. Thus, $u_4 = z^{k_2}$ and so $T \in \mathcal{F}$.

Case 3. u_3 is not a support vertex and $d(u_3) = 3$.

If $diam(T) = 4$, then assigning 5 to support vertices of T and 0 to other vertices, introduces a [4]-RDF on T, implying that $\gamma_{[4R]}(T) \leq 15 < \frac{4n+s(T)}{2}$, as desired. Let $diam(T) \geq 5$, then $s(T - T_{u_3}) \leq s(T) - 1$, and each $\gamma_{[4R]}(T - T_{u_3})$-function can be extended to a $\gamma_{[4R]}(T)$-function by assigning 5 to support vertices in T_{u_3} and 0 to the remaining vertices. Thus, by induction hypothesis, $\gamma_{[4R]}(T) \leq \gamma_{[4R]}(T - T_{u_3}) + 10 \leq \frac{4(n-5)+s(T-T_{u_3})}{2} + 10 < \frac{4n+s(T)}{2}$, as desired. □

Now, we provide a lower bound on $\gamma_{[4R]}(T)$ as follows.

Theorem 2. *For any tree T with order $n \geq 2$,*

$$\gamma_{[4R]}(T) \geq \frac{5(n + 2 - \ell(T))}{3}.$$

Equality holds if $T = K_{1,s}$, where $s \geq 2$.

Proof. The proof is by induction on n. If $n = 2$, then $\gamma_{[4R]}(T) = 5 > \frac{10}{3} = \frac{5(2+n-l(T))}{3}$. If $n = 3$, then $\gamma_{[4R]}(T) = 5 = \frac{5(n(T)+2-l(T))}{3}$.

Suppose that $n \geq 4$ and assume that the statement is true for any tree of order less than n. If $diam(T) = 2$, then $\gamma_{[4R]}(T) = 5 = \frac{5(n(T)+2-l(T))}{3}$. If $diam(T) = 3$, then $\gamma_{[4R]}(T) \geq 9 > \frac{5(n(T)+2-l(T))}{3} = \frac{20}{3}$. Let $diam(T) \geq 4$ and $u_1u_2 \ldots u_k$ be a diametral path of T such that $d(u_2)$ is as large as possible. We root T at u_k. Now let f be a $\gamma_{[4R]}(T)$-function. If $d(u_2) \geq 3$, then $f(u_2) = 5$ and so f restricted to $T - u_1$ is a [4]-RDF. As a result, by induction hypothesis, $\gamma_{[4R]}(T) \geq \frac{5(n(T-u_1)+2-(l(T)-1))}{3} = \frac{5(n+2-l(T))}{3}$. Now, let $d(u_2) = 2$. By the choice of diametral path, the degree of every child of u_3 with depth 1, is two.

Case 1. u_3 is a strong support vertex or u_3 has a child w with depth one different from u_2.

If u_3 is a strong support vertex, then as above we have $\gamma_{[4R]}(T) \geq \frac{5(n+2-l(T))}{3}$. Assume that w is a child of u_3 with depth one different from u_2. Suppose that $f(w) \geq f(u_2)$. Let $T' = T - T_{u_2}$. Clearly $f(u_1) + f(u_2) \geq 4$ and the function f, restricted to T' is a [4]-RDF of T', implying that $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') + 4$. By the above inequality and the induction hypothesis, $\gamma_{[4R]}(T) \geq \frac{5(n(T')+2-l(T'))}{3} + 4 = \frac{5(n(T)-2+2-(l(T)-1))}{3} + 4 > \frac{5(n+2-l(T))}{3}$, as desired.

Case 2. $d(u_3) = 3$ and u_3 is a support vertex.

Considering Case 1, we can assume that u_3 is a support vertex. Assume that w is a leaf adjacent to u_3. Let $T' = T - T_{u_2}$. Obviously, $f(V(T_{u_3})) \geq 9$. Without loss of generality, we may assume that $f(u_3) = 5$ and $f(u_1) = 4$. Now the result follows as in Case 1.

Case 3. $d(u_3) = 2$.

Let $T' = T - T_{u_3}$. Without loss of generality, we may assume that $f(u_2) = 5$ and $f(u_3) = f(u_1) = 0$. Hence the function f restricted to T_{u_3} is a [4]-RDF of T', implying that $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') + 5$. As a result, by the induction hypothesis, $\gamma_{[4R]}(T) = \frac{5((n-3)+2-(l(T)-1))}{3} + 5 > \frac{5(n+2-l(T))}{3}$, and this completes the proof. □

3. Quadruple Roman Domination vs. Domination and Roman Domination in Trees

In this section, we first prove that $4\gamma(T) + 1 \leq \gamma_{[4R]}(T) \leq 4\gamma_R(T) - 3$ and then we present a characterization of trees for which $\gamma_{[4R]}(T) = 4\gamma(T) + 1$ and $\gamma_{[4R]}(T) + 3 = 4\gamma_R(T)$ hold.

Theorem 3. *For any nontrivial tree T, $\gamma_{[4R]}(T) \geq 4\gamma(T) + 1$.*

Proof. The proof is by induction on $n = |V(T)|$. If T is a star, then the function by assigning 5 to the central vertex and 0 to every leaf, is [4]-RDF on T and hence $\gamma_{[4R]}(T) = 5$. Observe that $\gamma(T) = 1$. As a result, $\gamma_{[4R]}(T) = 4\gamma(T) + 1$. If $diam(T) = 3$, then T is the double star $S_{r,s}$ where $1 \leq r \leq s$ and so $\gamma(S_{r,s}) = 2$ and $\gamma_{[4R]}(S_{r,s}) = 9$ if $r = 1$ and $\gamma_{[4R]}(S_{r,s}) = 10$ if $r \geq 2$. As a result, $\gamma_{[4R]}(S_{r,s}) \geq 4\gamma(S_{r,s}) + 1$. Suppose next that $diam(T) \geq 4$. Please note that $n \geq 5$.

Suppose that the statement is true for any nontrivial tree of order less than n. We first assume that T has a strong support vertex v adjacent to a leaf u. We observe that if each [4]-RDF f on T assigns a value less than 5 to v, then $\sum_{x \in X \cup \{v\}} f(x) \geq 6$, where $X = \{x : x \text{ is a leaf adjacent to } v\}$, Thus, each $\gamma_{[4R]}(T)$-function assigns 5 to v and 0 to its leaf neighbors. Let $T' = T - u$. Please note that v is a support vertex of T'. Thus, $\gamma(T) = \gamma(T')$ and $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T')$. On the other hand, T' is nontrivial tree and so by induction hypothesis, $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') \geq 4\gamma(T') + 1 = 4\gamma(T) + 1$. Suppose now that each support vertex is adjacent to exactly one leaf. Choose two leaves u and v of T satisfying that the distance between u and v equals the diameter of T. Now root the tree T at vertex u. Let w be the unique vertex adjacent to v, x be the parent of w and y be the parent of x. Observe that each child of w is a leaf in T. Please note that there are no strong support vertices in T. Thus, $d(w) = 2$. We consider the following cases based on $d(x)$.

Case 1. $d(x) \geq 3$.

Let $T' = T - T_w$. Since any leaf or its support vertex must be in any dominating set, we may assume that X is a γ-set of T that contains all the support vertices of T. Let X' be the restriction of X on T'. Clearly, $w \in X$ and $v \notin X$. By the choice of v, each child of x is a leaf or a support adjacent to a unique leaf. Thus, x or a child of x is in X. Please note that $d(x) \geq 3$. Then x is dominated by one vertex of X'. As a result, $\gamma(T') \leq |X'| = |X| - 1 = \gamma(T) - 1$, implying that $\gamma(T') + 1 \leq \gamma(T)$. Moreover, each dominating set of T' can be extended to a dominating set of T be adding w. Thus, $\gamma(T) \leq \gamma(T') + 1$. As a result, $\gamma(T) = \gamma(T') + 1$.

Let f be a $\gamma_{[4R]}(T)$-function such that $f(V(T_w))$ is minimum. Now, consider the following subcases:

Subcase 1. x is a support vertex.

Let z be a leaf adjacent to x. If $f(x) = 0$, then each leaf adjacent to x must be assigned 4 and each child of x with depth 1 is assigned a 5 under f. Let h be a function which assigns 5 to x, 0 to each child of x, 4 to the leaves at distance two from x in T_x and $f(a)$ for $a \in T - T_x$. Clearly h is a [4]-RDF on T with $\omega(h) \leq \omega(f)$ and $h(V(T_w)) < f(V(T_w))$, a contradiction to the choice of f. Hence $f(x) \neq 0$, so we can suppose that $f(x) = 5$, $f(y) = 0$ for each leaf y adjacent with x, $f(v) = 4$ and $f(w) = 0$. This implies that the restriction of f on T' is a [4]-RDF on T' and so $\gamma_{[4R]}(T') \leq \omega(f) - 4 = \gamma_{[4R]}(T) - 4$. By the induction hypothesis, $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') + 4 \geq 4\gamma(T') + 1 + 4 = 4(\gamma(T') + 1) + 1 = 4\gamma(T) + 1$.

Subcase 2. x is not a support vertex.

Clearly every child of x is a support vertex with degree 2. If $f(x) = 0$, then each support vertex adjacent to x is assigned 5. Please note that $d(x) \geq 3$. Then the restriction of f on T' is a [4]-RDF of T' with $f(V(T')) = \omega(f) - 5$. Therefore similar to above, $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') + 4 \geq 4\gamma(T') + 1 + 4 = 4(\gamma(T') + 1) + 1 = 4\gamma(T) + 1$, as desired. If $f(x) \neq 0$, then $f(w) + f(v) \geq 4$. Define the function $h : V(T') \longrightarrow S$ by $h(x) = \min\{f(x) + f(v) + f(w) - 4, 5\}$ and $h(a) = f(a)$ for $a \in V(T') - \{x\}$. Clearly h is a [4]-RDF of T' and hence $\gamma_{[4R]}(T') \leq \omega(h) = \omega(f) - 4 = \gamma_{[4R]}(T) - 4$. Applying our inductive hypothesis, $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') + 4 \geq 4\gamma(T') + 1 + 4 = 4(\gamma(T') + 1) + 1 = 4\gamma(T) + 1$, as desired.

Case 2. $d(x) = 2$.

Consider $T' = T - T_x$. Since each γ-set of T' can be extended to a dominating set of T by adding w, $\gamma(T) \leq \gamma(T') + 1$. Observe that we can choose a dominating set X of T that does not contain x and $|X \cap \{v, w, x\}|$ is as small as possible. Thus, $X - \{w\}$ is a dominating set of T'. So $\gamma(T') \leq |X| - 1 \leq \gamma(T) - 1$. As a result, $\gamma(T) = \gamma(T') + 1$.

Let f be a $\gamma_{[4R]}(T)$-function such that $f(V(T_x))$ is minimum and $f(w)$ is as large as possible. Assume that y is the parent of x. Clearly $f(v) \in \{0,4\}$ and $f(w) \in \{0,5\}$. Furthermore, $f(v) = f(x) = 0$, $f(w) = 5$, or $f(w) = 0, f(v) = 4$ and $f(x) \neq 0$. Please note that when f is restricted to T_x, the latter has a large weight. We next prove that this case is not occur. Since $f(v) = 4$ and $f(x) + f(v) \geq 6$, $f(x) \geq 2$. Define the function $h : V(T) \longrightarrow S$ by $h(y) = \min\{f(x) + f(y) - 1, 5\}, h(w) = 5, h(v) = h(x) = 0$ and $h(a) = f(a)$ otherwise. Then h is a [4]-RDF on T with $h(V(T_x)) < f(V(T_x))$, a contradiction to the choice of f. Therefore $f(v) = f(x) = 0$, $f(w) = 5$ and hence the function f restricted to T' is a [4]-RDF on T'. Then $\gamma_{[4R]}(T') \leq \gamma_{[4R]}(T) - 5$. Applying the induction hypothesis, $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') + 5 \geq 4\gamma(T') + 1 + 5 = 4\gamma(T) + 2 > 4\gamma(T) + 1$. □

Let $\mathcal{T}$ denote the family of trees $T_{n,j}$ with order n obtained from a star S by subdividing j edges exactly once, where $j \geq 0$ and $n > 2j + 1$. The tree $T_{9,3}$ is shown in Figure 3. The head vertex of a star S is the central vertex of S.

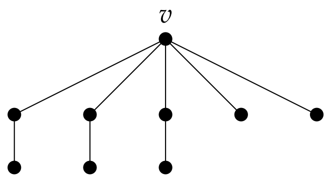

Figure 3. The tree $T_{9,3}$, with head vertex v.

Proposition 3. *If $T \in \mathcal{T}$, then $\gamma_{[4R]}(T) = 4\gamma(T) + 1$.*

Proof. If T is a star, then the assertion is trivial. Suppose next that T is not a star. So $T = T_{n,j}$ for some $n \geq 2$ where $n > 2j + 1$ and $j > 0$. If $j = 1$ then result is obvious. Hence, we may assume that $j > 1$. With the assumption, we have $\gamma(T) = j + 1$. Observe that the function f that assigns 5 to the head vertex, 4 to each of the j leaves at distance 2 from the head, and 0 to all other vertices is a [4]-RDF of T, and so $\gamma_{[4R]}(T) \leq 4j + 5$. By Theorem 3, $\gamma_{[4R]}(T) \geq 4\gamma(T) + 1 = 4(j + 1) + 1 = 4j + 5$. So $\gamma_{[4R]}(T) = 4j + 5$. As a result, $\gamma_{[4R]}(T) = 4\gamma(T) + 1$. □

Theorem 4. *For any nontrivial tree T with order n, $\gamma_{[4R]}(T) = 4\gamma(T) + 1$ if and only if $T \in \mathcal{T}$.*

Proof. The sufficiency follows from Proposition 3. Next, we show the necessity.

Suppose that $\gamma_{[4R]}(T) = 4\gamma(T) + 1$. The proof is by induction on n. If $2 \leq n \leq 4$, then clearly $T \in \{P_2, P_3, P_4\} \subseteq \mathcal{T}$. Suppose that $n \geq 5$ and that the statement is true for any nontrivial tree of order less than n. If T is a star, then clearly $T \in \mathcal{T}$, as desired. Let $diam(T) \geq 3$.

Assume that $diam(T) = 3$, and so T is a double star $S_{r,s}$, where $r \geq s \geq 1$. If $s \geq 2$, then $\gamma_{[4R]}(T) = 10$ and $\gamma(T) = 2$, a contradiction to the fact that $\gamma_{[4R]}(T) = 4\gamma(T) + 1$. Therefore $s = 1$ and $T = S_{r,1} = T_{r+3,1} \in \mathcal{T}$. Assume now that $diam(T) \geq 4$. Root the tree T at some vertex r at the end of a longest path in T. Necessary, r is a leaf. Let t be a vertex that has the maximum distance from r. Necessary, t is a leaf. Assume that u is the parent of t, v is the parent of u and w is the parent of v. Since t has the maximum distance from r, each child of u is a leaf.

We shall prove that $d(u) = 2$. Suppose, to the contrary, that $d(u) \geq 3$. Let $T' = T - t$. Please note that u is a support vertex of T. Then $\gamma(T) = \gamma(T')$ and $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T')$. It follows from Theorem 3 that $\gamma_{[4R]}(T) \geq \gamma_{[4R]}(T') \geq 4\gamma(T') + 1 = 4\gamma(T) + 1$. Moreover, since $\gamma_{[4R]}(T) = 4\gamma(T) + 1$, $\gamma_{[4R]}(T') = 4\gamma(T') + 1$. Applying the induction hypothesis, $T' \in \mathcal{T}$. It follows that $T' = T_{n-1,j}$ where v is the head of T' and $j \geq 2$. As a result, $\gamma_{[4R]}(T) = \gamma_{[4R]}(T') + 1 = 4\gamma(T) + 2$, a contradiction.

So $d(u) = 2$. Similarly, each child of v with depth 1 has degree two. We now claim v is a support vertex. Suppose, to the contrary, that v is not a support vertex.

Case 1. $d(v) \geq 3$.

Let u' be a support vertex of T distinct of u, adjacent to v. Then $d(u') = 2$. Let $T' = T - T_u$. Since each $\gamma(T')$-set can be extended to a dominating set of T by adding u, we have $\gamma(T) \leq \gamma(T') + 1$. Additionally, if X is a $\gamma(T)$-set without leaves, then $u, u' \in X$ and so $X - \{u\}$ is a dominating set of T'. Thus, $\gamma(T') \leq \gamma(T) - 1$ and consequently $\gamma(T') + 1 = \gamma(T)$. Now assume that f is a $\gamma_{[4R]}(T)$-function and f' is the restriction of f on T'. If $f(v) = 0$, then $f(u) = f(u') = 5$ and $f(t) = 0$. Thus, f' is a [4]-RDF of T' and so $\gamma_{[4R]}(T') \leq \omega(f') = \omega(f) - 5 = \gamma_{[4R]}(T) - 5$. Therefore by Proposition 3, $4\gamma(T') + 1 \leq \gamma_{[4R]}(T') \leq \gamma_{[4R]}(T) - 5 = 4\gamma(T) - 4 = 4\gamma(T')$, a contradiction. Now assume that $f(v) \neq 0$. Please note that $f(v) \geq 2$. Then by Proposition 1, we have $f(u) = 0$ and $f(t) = 4$ and so f' is a [4]-RDF of T' and $\omega(f') = \omega(f) - 4$. Thus, $\gamma_{[4R]}(T') + 4 \leq \gamma_{[4R]}(T)$. Since $\gamma_{[4R]}(T') \geq 4\gamma(T') + 1$ and $\gamma(T) = \gamma(T') + 1$, we have $\gamma_{[4R]}(T) = 4\gamma(T) + 1 = 4\gamma(T') + 1 + 4 \leq \gamma_{[4R]}(T') + 4 \leq \gamma_{[4R]}(T)$. Hence, we must have equality throughout this inequality chain. In particular, $\gamma_{[4R]}(T') = 4\gamma(T') + 1$. By the induction hypothesis, $T' \in \mathcal{T}$ where v is the head of T' and so is a support vertex, contradicting the assumption that v is not a support vertex.

Case 2. $d(v) = 2$.

Let $T' = T - T_v$, since each $\gamma(T')$-set can be extended to a dominating set of T by adding u, we have $\gamma(T') + 1 \geq \gamma(T)$. Additionally, if X is a $\gamma(T)$-set with no leaves, then $u \in X$. If $v \in X$, then $\{w\} \cup (X - \{v, u\})$ is a dominating set of T'. If $v \notin X$, then $X - \{u\}$ is a dominating set of T'. Thus, $\gamma(T') \leq \gamma(T) - 1$ and consequently $\gamma(T') + 1 = \gamma(T)$. Let f be a $\gamma_{[4R]}(T)$-function and f' be the restriction of f to T'. Please note that $f(u) + f(t) \geq 4$. If $f(v) \neq 0$, then the function $h : V(T) \longrightarrow S$ defined by $h(w) = \min\{f(v) + f(w) - 1, 5\}, h(v) = 0, h(u) = 5, h(t) = 0$ and $h(x) = f(x)$ otherwise, is a [4]-RDF on T with weight at most $\gamma_{[4R]}(T) - 5$. Thus, we may assume that $f(v) = 0$, then $f(u) = 5$ and $f(t) = 0$. This implies that f' is a [4]-RDF of T' with weight $\omega(f) - 5$, so $\gamma_{[4R]}(T') \leq \gamma_{[4R]}(T) - 5$. Since $\gamma_{[4R]}(T) = 4\gamma(T) + 1 = 4\gamma(T') + 1 + 4$, we have

$$1 + 4\gamma(T') \leq \gamma_{[4R]}(T') \leq \gamma_{[4R]}(T) - 5 = 4\gamma(T) + 1 - 5 = 4\gamma(T') + 4 - 4 = 4\gamma(T'),$$

a contradiction. Therefore, our claim is true, hence v is a support vertex.

Let $T' = T - T_u$, f be a $\gamma_{[4R]}(T)$-function and let f' be the restriction of f on T'. Since v is a support vertex, we may assume without loss of generality that $f(v) = 5, f(u) = 0$ and $f(t) = 4$. Thus, f' is a [4]-RDF of T' and so $\gamma_{[4R]}(T') \leq \omega(f') = \omega(f) - 4 = \gamma_{[4R]}(T) - 4$. Moreover, observe that $\gamma(T) = \gamma(T') + 1$. Therefore, it follows from Proposition 3 that $\gamma_{[4R]}(T) = 1 + 4\gamma(T) = 4\gamma(T') + 5 \leq \gamma_{[4R]}(T') + 4 \leq \gamma_{[4R]}(T)$.

Thus, we must have equality throughout the above inequality chain. Please note that $\gamma_{[4R]}(T') = 1 + 4\gamma(T')$. By the induction hypothesis, $T' \in \mathcal{T}$. If v is not the head of T', then w is the head of T', then the function $h : V(T) \to S$ defined by $h(w) = h(v) = 5$, $h(x) = 0$ for $x \in (N(w) \cup N(v)) \setminus \{w, v\}$ and $h(t) = h(y) = 4$ for $y \in N_2(w) \setminus N(v)$. It is not difficult to verify that $\gamma_{[4R]}(T) \geq 4\gamma(T) + 2$, a contradiction. Hence v is the head of T' and so $T \in \mathcal{T}$. □

Proposition 4. *For any nontrivial tree T, $\gamma_{[4R]}(T) + 3 \leq 4\gamma_R(T)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(T)$-function such that $|V_2|$ is as large as possible. Since T is nontrivial, we may assume that $V_2 \neq \emptyset$. Clearly the function $h : V(T) \longrightarrow S$ defined by $h(x) = 0$ for $x \in V_0$, $h(x) = 4$ for $x \in V_1$ and $h(x) = 5$ for $x \in V_2$, is a [4]-RDF on T with $\omega(h) = 4|V_1| + 5|V_2|$. Thus

$$4\gamma_R(T) = 4|V_1| + 8|V_2| \geq 4|V_1| + 5|V_2| + 3|V_2| \geq \gamma_{[4R]}(T) + 3|V_2| \geq \gamma_{[4R]}(T) + 3, \quad (3)$$

as desired. □

Now we give a characterization of all trees attaining the equality of Proposition 4.

Theorem 5. *For any nontrivial tree T, $\gamma_{[4R]}(T) + 3 = 4\gamma_R(T)$ if and only if $T \in \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then it follows from Theorem 3 that $\gamma_{[4R]}(T) = 4\gamma(T) + 1$, and since $\gamma_R(T) = \gamma(T) + 1$, $\gamma_{[4R]}(T) + 3 = 4\gamma_R(T)$. We next prove the necessity.

Suppose that $\gamma_{[4R]}(T) + 3 = 4\gamma_R(T)$ and that $f = (V_0, V_1, V_2)$ is a $\gamma_R(T)$-function such that $|V_2|$ is as large as possible. Then all inequalities occurring in Equation (3) must be equalities. In particular we must have $|V_2| = 1$. On the other hand, we deduce from the choice of f that V_1 is an independent set. Assume that $V_2 = \{v\}$. Clearly every vertex of V_0 is adjacent to v. Since V_1 is independent, each vertex in V_1 is at distance 2 from v, and since $|V_2|$ is maximized, any vertex of V_0 can be adjacent to at most one vertex of V_1. It follows that T is a spider. If T is a healthy spider with at least two feet, then by assigning 4 to v and each leaf of T, and 0 to remaining vertices, we obtain a [4]-RDF of T and so $\gamma_{[4R]}(T) \le 4d(v) + 4 \le 4\gamma_R(T) + 2$, a contradiction. As a result, T is a wounded spider and so $T \in \mathcal{T}$. $\square$

4. Conclusions

The main objective of this paper was to study the quadruple Roman domination number in graphs. We focused on trees and we presented lower and upper bounds on the quadruple Roman domination number of trees and characterized all extremal trees. For further works, one can find Nordhaus–Gaddum-type inequalities for quadruple Roman domination number in graphs.

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