Codazzi Tensors and the Quasi-Statistical Structure Associated with Affine Connections on Three-Dimensional Lorentzian Lie Groups

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Abstract: In this paper, we classify three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Bott connections, canonical connections and Kobayashi–Nomizu connections are Codazzi tensors associated with these connections. We also classify three-dimensional Lorentzian Lie groups with the quasi-statistical structure associated with Bott connections, canonical connections and Kobayashi–Nomizu connections.

Keywords: Codazzi tensors; Bott connections; canonical connections; Kobayashi–Nomizu connections; the quasi-statistical structure

1. Introduction

In [1], Andrzej and Shen studied some geometric and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold. They also introduced Codazzi tensors associated with any linear connections. Bourguignon obtained the results of this type and gave the proof of the existence of such a strong constraint of the tensor on the curvature operator in [2]. In [3], Dajczer and Tojeiro found the correspondence between the Ribaucour transformation of a submanifold and Codazzi tensor exchanged with its second fundamental form. In [4], the authors defined a Codazzi tensor on conformally symmetric space, and characterized the Einstein manifold and constant sectional curvature manifold by inequalities between certain functions of this tensor.

In [5], Merton and Gabe discussed the classification of Codazzi tensors with exactly two eigenfunctions on a Riemannian manifold of three or more dimensions. In [6], Blaga and Nannicini considered the statistical structure on a smooth manifold with a torsion-free affine connection, and they also gave the definition of the quasi-statistical structure, which is the generalization of the statistical structure. Wang gave algebraic Ricci solitons and affine Ricci solitons associated with canonical connections and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups, respectively in [7,8]. In [2,9], the authors gave the definition of the Bott connection.

In [1], Andrzej and Shen showed that the existence of nontrivial Codazzi tensors on Riemannian manifolds induces some geometric and topological results. We believe that our study of affine Codazzi tensors will induce some geometric and topological results in affine geometry, and we are prepared to study this question in the future. The classification of Class B manifolds: \( \left( \nabla^X_Y \rho \right)(Y, Z) = \left( \nabla^Y_Z \rho \right)(X, Z) \) in three-dimensional Lie groups is given by Calvaruso in [10]. Our research shows that the Ricci tensors of Bott connections, canonical connections and Kobayashi–Nomizu connections are Codazzi tensors can be used as an affine parallel to the above results in [10]. Class B manifolds are widely used in differential geometry classifications, and one can find more examples in [11]. We believe our results are useful for the classification of affine Lie groups.
the statistical structure on a smooth manifold with a torsion-free affine connection, and they also gave the definition of the quasi-statistical structure, which is the generalization of the statistical structure in [6]. Blaga and Nannicini proved that any quasi-statistical structure on $M$, defined by a symmetric or skew-symmetric tensor, induces the generalized quasi-statistical structures and the generalized dual quasi-statistical connection on $TM \oplus T^*M$ in [12]. In this paper, we classify three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Bott connections, canonical connections and Kobayashi–Nomizu connections are Codazzi tensors associated with these connections. We also classify three-dimensional Lorentzian Lie groups with the quasi-statistical structure associated with Bott connections, canonical connections and Kobayashi–Nomizu connections.

In Section 2, we classify three-dimensional Lorentzian Lie groups on which Ricci tensors and Kobayashi–Nomizu connections are Codazzi tensors associated with Bott connections. In Section 3, we classify three-dimensional Lorentzian Lie groups with the quasi-statistical structure associated with Bott connections. In Section 4, we classify three-dimensional Lorentzian Lie groups on which Ricci tensors associated with canonical connections and Kobayashi–Nomizu connections are Codazzi tensors associated with these connections. We also classify three-dimensional Lorentzian Lie groups with the quasi-statistical structure associated with Bott connections, canonical connections and Kobayashi–Nomizu connections.

2. Codazzi Tensors Associated with Bott Connections on Three-Dimensional Lorentzian Lie Groups

Let $\{G_i\}_{i=1,...,7}$ denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric $g$ and having Lie algebra $\{g_i\}_{i=1,...,7}$ and let $\nabla^L$ be the Levi–Civita connection of $G_i$. Next, we recall the definition of the Bott connection $\nabla^B$. Let $M$ be a smooth manifold, and let $TM = \text{span} \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$, then take the distribution $D = \text{span} \{\tilde{e}_1, \tilde{e}_2\}$ and $D^\perp = \text{span} \{\tilde{e}_3\}$.

The definition of the Bott connection $\nabla^B$ is given as follows: (see [2,9])

$$\nabla^B_XY = \begin{cases} 
\pi_D(\nabla^L_XY), & X, Y \in \Gamma^\infty(D) \\
\pi_D([X, Y]), & X \in \Gamma^\infty(D^\perp), Y \in \Gamma^\infty(D) \\
\pi_{D^\perp}([X, Y]), & X \in \Gamma^\infty(D), Y \in \Gamma^\infty(D^\perp) \\
\pi_{D^\perp}(\nabla^L_XY), & X, Y \in \Gamma^\infty(D^\perp), 
\end{cases}$$

(1)

where $\pi_D$ (resp. $\pi_{D^\perp}$) is the projection on $D$ (resp. $D^\perp$).

We define

$$R^B(X, Y)Z = \nabla^B_X\nabla^B_YZ - \nabla^B_Y\nabla^B_XZ - \nabla^B_{[X, Y]}Z.$$  

(2)

The Ricci tensor of $(G_i, g)$ associated with the Bott connection $\nabla^B$ is defined by

$$\rho^B(X, Y) = -g(R^B(X, \tilde{e}_1)Y, \tilde{e}_1) - g(R^B(Y, \tilde{e}_2)X, \tilde{e}_2) + g(R^B(X, \tilde{e}_3)Y, \tilde{e}_3),$$

(3)

where $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a pseudo-orthonormal basis, with $\tilde{e}_3$ timelike.

Let

$$\tilde{\rho}^B(X, Y) = \frac{\rho^B(X, Y) + \rho^B(Y, X)}{2}.$$ 

(4)

Let $\omega$ be a $(0,2)$ tensor field, then we define:

$$(\nabla_X\omega)(Y, Z) := X[\omega(Y, Z)] - \omega(\nabla_XY, Z) - \omega(Y, \nabla_XZ),$$

(5)

for arbitrary vector fields $X, Y, Z$. 

Definition 1. ([11], p. 17) Let $M$ be a smooth manifold endowed with a linear connection $\nabla$, and the tensor fields $\omega$ is called a Codazzi tensor on $(M, \nabla)$, if it satisfies
\[
 f(X, Y, Z) = (\nabla_X \omega)(Y, Z) - (\nabla_Y \omega)(X, Z) = 0,
\]
where $f$ is $C^\infty(M)$-linear for $X, Y, Z$.

Proposition 1. The tensor $\omega$ is called a Codazzi tensor on $(M, \nabla)$ if and only if
\[
 f(X, Y, Z) = -f(Y, X, Z)
\]
Then, we have that $\omega$ is a Codazzi tensor on $(G_i, \nabla)_{i=1,2,3}$ if and only if the following three equations hold:
\[
\begin{align*}
 f(\tilde{e}_1, \tilde{e}_2, \tilde{e}_j) &= 0 \\
 f(\tilde{e}_1, \tilde{e}_3, \tilde{e}_j) &= 0 \\
 f(\tilde{e}_2, \tilde{e}_3, \tilde{e}_j) &= 0,
\end{align*}
\]
where $1 \leq j \leq 3$.

2.1. Codazzi Tensors of $G_1$

By [13], we have the following Lie algebra of $G_1$ which satisfies
\[
\begin{align*}
 [\tilde{e}_1, \tilde{e}_2] &= a\tilde{e}_1 - b\tilde{e}_3, \quad [\tilde{e}_1, \tilde{e}_3] = -a\tilde{e}_1 - b\tilde{e}_2, \quad [\tilde{e}_2, \tilde{e}_3] = b\tilde{e}_1 + a\tilde{e}_2 + a\tilde{e}_3, \quad a \neq 0,
\end{align*}
\]
where $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a pseudo-orthonormal basis, with $\tilde{e}_3$ timelike.

Lemma 1. The Bott connection $\nabla^B$ of $G_1$ is given by
\[
\begin{align*}
 \nabla^B_{\tilde{e}_1}\tilde{e}_1 &= -a\tilde{e}_2, \quad \nabla^B_{\tilde{e}_1}\tilde{e}_2 = a\tilde{e}_1, \quad \nabla^B_{\tilde{e}_1}\tilde{e}_3 = 0, \\
 \nabla^B_{\tilde{e}_2}\tilde{e}_1 &= 0, \quad \nabla^B_{\tilde{e}_2}\tilde{e}_2 = 0, \quad \nabla^B_{\tilde{e}_2}\tilde{e}_3 = a\tilde{e}_3, \\
 \nabla^B_{\tilde{e}_3}\tilde{e}_1 &= a\tilde{e}_1 + b\tilde{e}_2, \quad \nabla^B_{\tilde{e}_3}\tilde{e}_2 = -b\tilde{e}_1 - a\tilde{e}_2, \quad \nabla^B_{\tilde{e}_3}\tilde{e}_3 = 0.
\end{align*}
\]

Lemma 2. The curvature $R^B$ of the Bott connection $\nabla^B$ of $(G_1, g)$ is given by
\[
\begin{align*}
 R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_1 &= a\beta\tilde{e}_1 + (a^2 + \beta^2)\tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_2 = -(a^2 + \beta^2)\tilde{e}_1 - a\beta\tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_3 = 0, \\
 R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_1 &= -3a^2\tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_2 = -a^2\tilde{e}_1, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_3 = a\beta\tilde{e}_3, \\
 R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_1 &= -a^2\tilde{e}_1, \quad R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_2 = a^2\tilde{e}_2, \quad R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_3 = -a^2\tilde{e}_3.
\end{align*}
\]

By (3), we have
\[
\begin{align*}
 \rho^B(\tilde{e}_1, \tilde{e}_1) &= -a^2 - \beta^2, \quad \rho^B(\tilde{e}_1, \tilde{e}_2) = a\beta, \quad \rho^B(\tilde{e}_1, \tilde{e}_3) = -a\beta, \\
 \rho^B(\tilde{e}_2, \tilde{e}_1) &= a\beta, \quad \rho^B(\tilde{e}_2, \tilde{e}_2) = -(a^2 + \beta^2), \quad \rho^B(\tilde{e}_2, \tilde{e}_3) = a^2, \\
 \rho^B(\tilde{e}_3, \tilde{e}_1) &= \rho^B(\tilde{e}_3, \tilde{e}_2) = \rho^B(\tilde{e}_3, \tilde{e}_3) = 0.
\end{align*}
\]

Then,
\[
\begin{align*}
 \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_1) &= -(a^2 + \beta^2), \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_2) = a\beta, \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_3) = -\frac{a\beta}{2}, \\
 \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_1) &= -(a^2 + \beta^2), \quad \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_2) = a^2, \quad \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_3) = 0.
\end{align*}
\]
By (5), we have
\[
(\nabla^{B}_{\tilde{e}_{1}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{1}) = 0, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{1}, \tilde{e}_{1}) = 0, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{2}) = -2\alpha^{2} \beta,
\]
\[
(\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{2}) = 0, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{3}) = \frac{\alpha^{2} \beta}{2}, \quad (\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{1}, \tilde{e}_{3}) = \frac{\alpha^{2} \beta}{2},
\]
\[
(\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{3}, \tilde{e}_{1}) = \frac{\alpha^{3}}{2}, \quad (\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{1}, \tilde{e}_{3}) = 2\alpha \beta, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{3}, \tilde{e}_{2}) = \frac{\alpha^{2} \beta}{2},
\]
\[
(\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{1}, \tilde{e}_{2}) = 0, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{3}, \tilde{e}_{3}) = 0, \quad (\nabla^{B}_{\tilde{e}_{1}} \bar{\rho}^{B})(\tilde{e}_{1}, \tilde{e}_{3}) = 0,
\]
\[
(\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{3}, \tilde{e}_{1}) = \frac{\alpha^{2} \beta}{2}, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{1}) = 0, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{3}, \tilde{e}_{2}) = -\frac{\alpha^{2} \beta}{2},
\]
\[
(\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{2}) = -2\alpha^{2}, \quad (\nabla^{B}_{\tilde{e}_{3}} \bar{\rho}^{B})(\tilde{e}_{3}, \tilde{e}_{3}) = 0, \quad (\nabla^{B}_{\tilde{e}_{2}} \bar{\rho}^{B})(\tilde{e}_{2}, \tilde{e}_{3}) = \frac{\alpha}{2}(\alpha^{2} - \beta^{2}).
\]  

Then, if $\bar{\rho}^{B}$ is a Codazzi tensor on $(G_{1}, \nabla^{B})$, by (6) and (7), we have the following three equations:
\[
\begin{cases}
\alpha \beta = 0 \\
\alpha = 0 \\
\alpha(\alpha^{2} - \beta^{2}) = 0.
\end{cases}
\]  

By solving (15), we get $\alpha = 0$, and there is a contradiction. So,

**Theorem 1.** $\bar{\rho}^{B}$ is not a Codazzi tensor on $(G_{1}, \nabla^{B})$.

### 2.2. Codazzi Tensors of $G_{2}$

By [13], we have the following Lie algebra of $G_{2}$ which satisfies
\[
[\tilde{e}_{1}, \tilde{e}_{2}] = \gamma \tilde{e}_{2} - \beta \tilde{e}_{3}, \quad [\tilde{e}_{1}, \tilde{e}_{3}] = -\beta \tilde{e}_{2} - \gamma \tilde{e}_{3}, \quad [\tilde{e}_{2}, \tilde{e}_{3}] = \alpha \tilde{e}_{1}, \quad \gamma \neq 0,
\]
where $\tilde{e}_{1}, \tilde{e}_{2}, \tilde{e}_{3}$ is a pseudo-orthonormal basis, with $\tilde{e}_{3}$ timelike.

**Lemma 3.** The Bott connection $\nabla^{B}$ of $G_{2}$ is given by
\[
\begin{align*}
\nabla^{B}_{\tilde{e}_{1}} \tilde{e}_{1} &= 0, \quad \nabla^{B}_{\tilde{e}_{1}} \tilde{e}_{2} = 0, \quad \nabla^{B}_{\tilde{e}_{1}} \tilde{e}_{3} = -\gamma \tilde{e}_{3}, \\
\nabla^{B}_{\tilde{e}_{2}} \tilde{e}_{1} &= -\gamma \tilde{e}_{2}, \quad \nabla^{B}_{\tilde{e}_{2}} \tilde{e}_{2} = \gamma \tilde{e}_{1}, \quad \nabla^{B}_{\tilde{e}_{2}} \tilde{e}_{3} = 0, \\
\nabla^{B}_{\tilde{e}_{3}} \tilde{e}_{1} &= \beta \tilde{e}_{2}, \quad \nabla^{B}_{\tilde{e}_{3}} \tilde{e}_{2} = -\alpha \tilde{e}_{1}, \quad \nabla^{B}_{\tilde{e}_{3}} \tilde{e}_{3} = 0.
\end{align*}
\]  

**Lemma 4.** The curvature $R^{B}$ of the Bott connection $\nabla^{B}$ of $(G_{2}, g)$ is given by
\[
\begin{align*}
R^{B}(\tilde{e}_{1}, \tilde{e}_{2})\tilde{e}_{1} &= (\beta^{2} + \gamma^{2})\tilde{e}_{2}, \quad R^{B}(\tilde{e}_{1}, \tilde{e}_{2})\tilde{e}_{2} = -\gamma^{2} + \alpha \beta)\tilde{e}_{1}, \quad R^{B}(\tilde{e}_{1}, \tilde{e}_{2})\tilde{e}_{3} = 0, \\
R^{B}(\tilde{e}_{1}, \tilde{e}_{3})\tilde{e}_{1} &= 0, \quad R^{B}(\tilde{e}_{1}, \tilde{e}_{3})\tilde{e}_{2} = \gamma(\alpha - \beta)\tilde{e}_{1}, \quad R^{B}(\tilde{e}_{1}, \tilde{e}_{3})\tilde{e}_{3} = 0, \\
R^{B}(\tilde{e}_{2}, \tilde{e}_{3})\tilde{e}_{1} &= \gamma(\beta - \alpha)\tilde{e}_{1}, \quad R^{B}(\tilde{e}_{2}, \tilde{e}_{3})\tilde{e}_{2} = \gamma(\alpha - \beta)\tilde{e}_{2}, \quad R^{B}(\tilde{e}_{2}, \tilde{e}_{3})\tilde{e}_{3} = \alpha \gamma \tilde{e}_{3}.
\end{align*}
\]  

By (3), we have
\[
\begin{align*}
\rho^{B}(\tilde{e}_{1}, \tilde{e}_{1}) &= -\beta^{2} + \gamma^{2}, \quad \rho^{B}(\tilde{e}_{1}, \tilde{e}_{2}) = 0, \quad \rho^{B}(\tilde{e}_{1}, \tilde{e}_{3}) = 0, \\
\rho^{B}(\tilde{e}_{2}, \tilde{e}_{1}) &= 0, \quad \rho^{B}(\tilde{e}_{2}, \tilde{e}_{2}) = -\gamma^{2} + \alpha \beta, \quad \rho^{B}(\tilde{e}_{2}, \tilde{e}_{3}) = -\alpha \gamma, \\
\rho^{B}(\tilde{e}_{3}, \tilde{e}_{1}) &= \rho^{B}(\tilde{e}_{3}, \tilde{e}_{2}) = \rho^{B}(\tilde{e}_{3}, \tilde{e}_{3}) = 0.
\end{align*}
\]
Then,
\[ \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_1) = - (\beta^2 + \gamma^2), \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_2) = 0, \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_3) = 0, \]
\[ \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_2) = - (\gamma^2 + \alpha \beta), \quad \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_3) = \frac{-\alpha \gamma}{2}, \quad \tilde{\rho}^B(\tilde{e}_3, \tilde{e}_3) = 0. \quad (20) \]

By (5), we have
\[ (\nabla^B_{\tilde{e}_1}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = 0, \quad (\nabla^B_{\tilde{e}_2}\tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla^B_{\tilde{e}_2}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_2) = 0, \]
\[ (\nabla^B_{\tilde{e}_2}\tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_2) = \gamma (\beta^2 - \alpha \beta), \quad (\nabla^B_{\tilde{e}_1}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_3) = - \frac{\alpha \gamma^2}{2}, \quad (\nabla^B_{\tilde{e}_2}\tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_3) = - \frac{\alpha \gamma^2}{2}, \]
\[ (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_2) = 0, \quad (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_3) = \frac{\alpha \beta \gamma}{2}, \]
\[ (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_2) = - \frac{\alpha \gamma^2}{2}, \quad (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = \gamma^2 (\beta - \alpha), \quad (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_1) = 0, \]
\[ (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_3) = 0, \quad (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla^B_{\tilde{e}_3}\tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_2) = 0. \quad (21) \]

Then, if \( \tilde{\rho}^B \) is a Codazzi tensor on \( G_2, \nabla^B \), by (6) and (7), we have the following three equations:
\[
\begin{align*}
\gamma (\beta - \alpha) &= 0 \\
\alpha \beta \gamma &= 0 \\
\gamma (\alpha - 2 \beta) &= 0.
\end{align*}
\quad (22)\]

By solving (22), we get \( \alpha = \beta = 0 \), and this condition does not hold. Therefore,

**Theorem 2.** \( \tilde{\rho}^B \) is not a Codazzi tensor on \( (G_2, \nabla^B) \).

2.3. Codazzi Tensors of \( G_3 \)

By [13], we have the following Lie algebra of \( G_3 \) which satisfies
\[ [\tilde{e}_1, \tilde{e}_2] = - \gamma \tilde{e}_3, \quad [\tilde{e}_1, \tilde{e}_3] = - \beta \tilde{e}_2, \quad [\tilde{e}_2, \tilde{e}_3] = \alpha \tilde{e}_1, \quad (23) \]

where \( \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \) is a pseudo-orthonormal basis, with \( \tilde{e}_3 \) timelike.

**Lemma 5.** The Bott connection \( \nabla^B \) of \( G_3 \) is given by
\[ \nabla^B_{\tilde{e}_1} \tilde{e}_1 = 0, \quad \nabla^B_{\tilde{e}_1} \tilde{e}_2 = 0, \quad \nabla^B_{\tilde{e}_1} \tilde{e}_3 = - \gamma \tilde{e}_3, \]
\[ \nabla^B_{\tilde{e}_2} \tilde{e}_1 = 0, \quad \nabla^B_{\tilde{e}_2} \tilde{e}_2 = 0, \quad \nabla^B_{\tilde{e}_2} \tilde{e}_3 = 0, \]
\[ \nabla^B_{\tilde{e}_3} \tilde{e}_1 = \beta \tilde{e}_2, \quad \nabla^B_{\tilde{e}_3} \tilde{e}_2 = - \alpha \tilde{e}_1, \quad \nabla^B_{\tilde{e}_3} \tilde{e}_3 = 0. \quad (24) \]

**Lemma 6.** The curvature \( R^B \) of the Bott connection \( \nabla^B \) of \( (G_3, \tilde{g}) \) is given by
\[ R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_2 = \beta \gamma \tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_2 = - \alpha \gamma \tilde{e}_1, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_3 = 0, \]
\[ R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_1 = 0, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_2 = 0, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_3 = 0, \]
\[ R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_1 = 0, \quad R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_2 = 0, \quad R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_3 = 0. \quad (25) \]
By (3), we have
\begin{align*}
\rho^B(\tilde e_1, \tilde e_1) &= -\beta \gamma, \quad \rho^B(\tilde e_1, \tilde e_2) = 0, \quad \rho^B(\tilde e_1, \tilde e_3) = 0, \\
\rho^B(\tilde e_2, \tilde e_1) &= 0, \quad \rho^B(\tilde e_2, \tilde e_2) = -\alpha \gamma, \quad \rho^B(\tilde e_2, \tilde e_3) = 0, \\
\rho^B(\tilde e_3, \tilde e_1) &= \rho^B(\tilde e_3, \tilde e_2) = \rho^B(\tilde e_3, \tilde e_3) = 0. \tag{26}
\end{align*}

Then,
\begin{align*}
\tilde \rho^B(\tilde e_1, \tilde e_1) &= -\beta \gamma, \quad \tilde \rho^B(\tilde e_1, \tilde e_2) = \rho^B(\tilde e_1, \tilde e_3) = 0, \\
\tilde \rho^B(\tilde e_2, \tilde e_2) &= -\alpha \gamma, \quad \tilde \rho^B(\tilde e_2, \tilde e_3) = \rho^B(\tilde e_3, \tilde e_3) = 0. \tag{27}
\end{align*}

By (5), we have
\begin{align*}
(\nabla^B_{\tilde e_1} \rho^B)(\tilde e_2, \tilde e_j) &= (\nabla^B_{\tilde e_1} \rho^B)(\tilde e_1, \tilde e_j) = (\nabla^B_{\tilde e_1} \rho^B)(\tilde e_3, \tilde e_j) = 0, \\
(\nabla^B_{\tilde e_1} \rho^B)(\tilde e_1, \tilde e_j) &= (\nabla^B_{\tilde e_1} \rho^B)(\tilde e_3, \tilde e_j) = 0. \tag{28}
\end{align*}

where $1 \leq j \leq 3$.

Then, we obtain

**Theorem 3.** $\tilde \rho^B$ is a Codazzi tensor on $(G_3, \nabla^B)$.

2.4. Codazzi Tensors of $G_4$

By [13], we have the following Lie algebra of $G_4$ which satisfies
\begin{equation}
[\tilde e_1, \tilde e_2] = -\tilde e_2 + (2\eta - \beta)\tilde e_3, \quad \eta = \pm 1, \quad [\tilde e_1, \tilde e_3] = -\beta \tilde e_2 + \tilde e_3, \quad [\tilde e_2, \tilde e_3] = a \tilde e_1, \tag{29}
\end{equation}
where $\tilde e_1, \tilde e_2, \tilde e_3$ is a pseudo-orthonormal basis, with $\tilde e_3$ timelike.

**Lemma 7.** The Bott connection $\nabla^B$ of $G_4$ is given by
\begin{align*}
\nabla^B_{\tilde e_1} \tilde e_1 &= 0, \quad \nabla^B_{\tilde e_1} \tilde e_2 = 0, \quad \nabla^B_{\tilde e_1} \tilde e_3 = \tilde e_3, \\
\nabla^B_{\tilde e_2} \tilde e_1 &= \tilde e_2, \quad \nabla^B_{\tilde e_2} \tilde e_2 = -\tilde e_1, \quad \nabla^B_{\tilde e_2} \tilde e_3 = 0, \\
\nabla^B_{\tilde e_3} \tilde e_1 &= \beta \tilde e_2, \quad \nabla^B_{\tilde e_3} \tilde e_2 = -a \tilde e_1, \quad \nabla^B_{\tilde e_3} \tilde e_3 = 0. \tag{30}
\end{align*}

**Lemma 8.** The curvature $R^B$ of the Bott connection $\nabla^B$ of $(G_4, g)$ is given by
\begin{align*}
R^B(\tilde e_1, \tilde e_2)\tilde e_1 &= (\beta - \eta)\tilde e_2, \quad R^B(\tilde e_1, \tilde e_2)\tilde e_2 = (2\alpha \eta - \alpha \beta - 1)\tilde e_1, \quad R^B(\tilde e_1, \tilde e_2)\tilde e_3 = 0, \\
R^B(\tilde e_1, \tilde e_3)\tilde e_1 &= 0, \quad R^B(\tilde e_1, \tilde e_3)\tilde e_2 = (\alpha - \beta)\tilde e_1, \quad R^B(\tilde e_1, \tilde e_3)\tilde e_3 = 0, \\
R^B(\tilde e_2, \tilde e_3)\tilde e_1 &= (\alpha - \beta)\tilde e_1, \quad R^B(\tilde e_2, \tilde e_3)\tilde e_2 = (\beta - \alpha)\tilde e_2, \quad R^B(\tilde e_2, \tilde e_3)\tilde e_3 = -a \tilde e_3. \tag{31}
\end{align*}

By (3), we have
\begin{align*}
\rho^B(\tilde e_1, \tilde e_1) &= - (\beta - \eta)^2, \quad \rho^B(\tilde e_1, \tilde e_2) = 0, \quad \rho^B(\tilde e_1, \tilde e_3) = 0, \\
\rho^B(\tilde e_2, \tilde e_1) &= (2\alpha \eta - \alpha \beta - 1), \quad \rho^B(\tilde e_2, \tilde e_2) = \alpha, \quad \rho^B(\tilde e_2, \tilde e_3) = 0, \\
\rho^B(\tilde e_3, \tilde e_1) &= \rho^B(\tilde e_3, \tilde e_2) = \rho^B(\tilde e_3, \tilde e_3) = 0. \tag{32}
\end{align*}

Then,
\begin{align*}
\tilde \rho^B(\tilde e_1, \tilde e_1) &= - (\beta - \eta)^2, \quad \tilde \rho^B(\tilde e_1, \tilde e_2) = 0, \quad \tilde \rho^B(\tilde e_1, \tilde e_3) = 0, \\
\tilde \rho^B(\tilde e_2, \tilde e_2) &= (2\alpha \eta - \alpha \beta - 1), \quad \tilde \rho^B(\tilde e_2, \tilde e_3) = \frac{\alpha}{2}, \quad \tilde \rho^B(\tilde e_3, \tilde e_3) = 0. \tag{33}
\end{align*}
By (5), we have
\[
(\nabla_{\tilde{e}_1} B) (\tilde{e}_2, \tilde{e}_1) = 0, \quad (\nabla_{\tilde{e}_2} B) (\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_2, \tilde{e}_2) = \alpha \beta + 2\beta \eta - 2\alpha \eta - \beta^2,
\]
\[
(\nabla_{\tilde{e}_1} B) (\tilde{e}_1, \tilde{e}_2) = -\frac{\alpha}{2}, \quad (\nabla_{\tilde{e}_1} B) (\tilde{e}_2, \tilde{e}_3) = -\frac{\alpha}{2}, \quad (\nabla_{\tilde{e}_1} B) (\tilde{e}_1, \tilde{e}_3) = 0,
\]
\[
(\nabla_{\tilde{e}_2} B) (\tilde{e}_3, \tilde{e}_1) = 0, \quad (\nabla_{\tilde{e}_2} B) (\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla_{\tilde{e}_2} B) (\tilde{e}_3, \tilde{e}_2) = -\frac{\alpha}{2},
\]
\[
(\nabla_{\tilde{e}_3} B) (\tilde{e}_1, \tilde{e}_2) = \beta - \alpha, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_1, \tilde{e}_3) = -\frac{\alpha \beta}{2},
\]
\[
(\nabla_{\tilde{e}_3} B) (\tilde{e}_2, \tilde{e}_1) = \beta - \alpha, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_2, \tilde{e}_2) = \beta - \alpha, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_3, \tilde{e}_2) = 0,
\]
\[
(\nabla_{\tilde{e}_3} B) (\tilde{e}_2, \tilde{e}_2) = 0, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_3} B) (\tilde{e}_3, \tilde{e}_3) = 0. \quad (34)
\]

Then, if \( B^\beta \) is a Codazzi tensor on \( (G_4, \nabla^B) \), by (6) and (7), we have the following three equations:

\[
\begin{cases}
(\beta - \eta)^2 + 2\alpha \eta - \alpha \beta - 1 = 0 \\
\alpha - 2\beta = 0 \\
\alpha \beta = 0.
\end{cases} \quad (35)
\]

By solving (35), it turns out that

**Theorem 4.** \( \hat{B}^\beta \) is a Codazzi tensor on \( (G_4, \nabla^B) \) if and only if \( \alpha = \beta = 0 \).

2.5. Codazzi Tensors of \( G_5 \)

By [13], we have the following Lie algebra of \( G_5 \) which satisfies
\[
[\tilde{e}_1, \tilde{e}_2] = 0, \quad [\tilde{e}_1, \tilde{e}_3] = \alpha \tilde{e}_1 + \beta \tilde{e}_2, \quad [\tilde{e}_2, \tilde{e}_3] = \gamma \tilde{e}_1 + \delta \tilde{e}_2, \quad \alpha + \delta \neq 0, \quad \alpha \gamma + \beta \delta = 0, \quad (36)
\]
where \( \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \) is a pseudo-orthonormal basis, with \( \tilde{e}_3 \) timelike.

**Lemma 9.** The Bott connection \( \nabla^B \) of \( G_5 \) is given by
\[
\nabla_{\tilde{e}_1}^B \tilde{e}_1 = 0, \quad \nabla_{\tilde{e}_2}^B \tilde{e}_2 = 0, \quad \nabla_{\tilde{e}_3}^B \tilde{e}_3 = 0,
\]
\[
\nabla_{\tilde{e}_1}^B \tilde{e}_1 = 0, \quad \nabla_{\tilde{e}_2}^B \tilde{e}_2 = 0, \quad \nabla_{\tilde{e}_3}^B \tilde{e}_3 = 0,
\]
\[
\nabla_{\tilde{e}_3}^B \tilde{e}_1 = -\alpha \tilde{e}_1 - \beta \tilde{e}_2, \quad \nabla_{\tilde{e}_3}^B \tilde{e}_2 = -\gamma \tilde{e}_1 - \delta \tilde{e}_2, \quad \nabla_{\tilde{e}_3}^B \tilde{e}_3 = 0. \quad (37)
\]

**Lemma 10.** The curvature \( R^B \) of the Bott connection \( \nabla^B \) of \( (G_5, \mathfrak{g}) \) is given by
\[
R^B(\tilde{e}_i, \tilde{e}_j) \tilde{e}_k = 0, \quad (38)
\]
for any \( (i, j, k) \).

By (3), we have
\[
\rho^B(\tilde{e}_i, \tilde{e}_j) = 0, \quad (39)
\]
then,
\[
\hat{\rho}^B(\tilde{e}_i, \tilde{e}_j) = 0, \quad (40)
\]
for any pairs \( (i, j) \).
By (5), we have
\[
(\nabla_{\tilde{e}_i} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_1) = (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = 0,
\]
\[
(\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_1) = (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_1) = (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = 0,
\]
(41)
where \(1 \leq j \leq 3\).

So,

**Theorem 5.** \(\tilde{\rho}^B\) is a Codazzi tensor on \((G_5, \nabla^B)\).

2.6. Codazzi Tensors of \(G_6\)

By [13], we have the following Lie algebra of \(G_6\) which satisfies
\[
[\tilde{e}_1, \tilde{e}_2] = \alpha \tilde{e}_2 + \beta \tilde{e}_3, \quad [\tilde{e}_1, \tilde{e}_3] = \gamma \tilde{e}_2 + \delta \tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha \gamma - \beta \delta = 0, \quad (42)
\]
where \(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\) is a pseudo-orthonormal basis, with \(\tilde{e}_3\) timelike.

**Lemma 11.** The Bott connection \(\nabla^B\) of \(G_6\) is given by
\[
\nabla^B_{\tilde{e}_i} \tilde{e}_1 = 0, \quad \nabla^B_{\tilde{e}_i} \tilde{e}_2 = 0, \quad \nabla^B_{\tilde{e}_3} \tilde{e}_3 = \delta \tilde{e}_3,
\]
\[
\nabla^B_{\tilde{e}_2} \tilde{e}_1 = -\alpha \tilde{e}_2, \quad \nabla^B_{\tilde{e}_2} \tilde{e}_2 = \alpha \tilde{e}_1, \quad \nabla^B_{\tilde{e}_2} \tilde{e}_3 = 0,
\]
\[
\nabla^B_{\tilde{e}_1} \tilde{e}_1 = -\gamma \tilde{e}_2, \quad \nabla^B_{\tilde{e}_3} \tilde{e}_2 = 0, \quad \nabla^B_{\tilde{e}_3} \tilde{e}_3 = 0. \quad (43)
\]

**Lemma 12.** The curvature \(R^B\) of the Bott connection \(\nabla^B\) of \((G_6, g)\) is given by
\[
R^B(\tilde{e}_1, \tilde{e}_2) \tilde{e}_1 = (\alpha^2 + \beta \gamma) \tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_2) \tilde{e}_2 = -\alpha^2 \tilde{e}_1, \quad R^B(\tilde{e}_1, \tilde{e}_2) \tilde{e}_3 = 0,
\]
\[
R^B(\tilde{e}_1, \tilde{e}_3) \tilde{e}_1 = \gamma (\alpha + \delta) \tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_3) \tilde{e}_2 = -\alpha \gamma \tilde{e}_1, \quad R^B(\tilde{e}_1, \tilde{e}_3) \tilde{e}_3 = 0,
\]
\[
R^B(\tilde{e}_2, \tilde{e}_3) \tilde{e}_1 = -\alpha \gamma \tilde{e}_1, \quad R^B(\tilde{e}_2, \tilde{e}_3) \tilde{e}_2 = \alpha \gamma \tilde{e}_2, \quad R^B(\tilde{e}_2, \tilde{e}_3) \tilde{e}_3 = 0. \quad (44)
\]

By (3), we have
\[
\rho^B(\tilde{e}_1, \tilde{e}_1) = - (\alpha^2 + \beta \gamma), \quad \rho^B(\tilde{e}_1, \tilde{e}_2) = \rho^B(\tilde{e}_1, \tilde{e}_3) = 0,
\]
\[
\rho^B(\tilde{e}_2, \tilde{e}_1) = 0, \quad \rho^B(\tilde{e}_2, \tilde{e}_2) = -\alpha^2, \quad \rho^B(\tilde{e}_2, \tilde{e}_3) = 0,
\]
\[
\rho^B(\tilde{e}_3, \tilde{e}_1) = \rho^B(\tilde{e}_3, \tilde{e}_2) = \rho^B(\tilde{e}_3, \tilde{e}_3) = 0. \quad (45)
\]

Then,
\[
\tilde{\rho}^B(\tilde{e}_1, \tilde{e}_1) = - (\alpha^2 + \beta \gamma), \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_2) = \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_3) = 0,
\]
\[
\tilde{\rho}^B(\tilde{e}_2, \tilde{e}_1) = -\alpha^2, \quad \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_3) = 0, \quad \tilde{\rho}^B(\tilde{e}_3, \tilde{e}_3) = 0. \quad (46)
\]

By (5), we have
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_1) = (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = 0,
\]
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_2) = 0, \quad (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_2) = 0, \quad (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_1, \tilde{e}_2) = 0,
\]
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_1) = (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_1) = (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_1) = 0,
\]
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = -\alpha^2 \gamma, \quad (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = 0, \quad (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_1) = 0,
\]
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_2) = 0, \quad (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_2) = 0, \quad (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_2) = 0,
\]
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_3) = -\alpha^2 \gamma, \quad (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_2, \tilde{e}_3) = 0,
\]
\[
(\nabla_{\tilde{e}_1} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_2} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_3} \tilde{\rho}^B)(\tilde{e}_3, \tilde{e}_3) = 0. \quad (47)
\]
Then, if $\tilde{\rho}^B$ is a Codazzi tensor on $(G_6, \nabla^B)$, by (6) and (7), we have the following two equations:

\[
\begin{aligned}
\alpha\beta\gamma &= 0 \\
\alpha\gamma &= 0.
\end{aligned} \tag{48}
\]

By solving (48), we obtain

**Theorem 6.** $\tilde{\rho}^B$ is a Codazzi tensor on $(G_6, \nabla^B)$ if and only if

\[(1)\alpha = \beta = 0, \quad \delta \neq 0; \]

\[(2)\alpha \neq 0, \quad \gamma = \beta\delta = 0.\]

2.7. Codazzi Tensors of $G_7$

By [13], we have the following Lie algebra of $G_7$ which satisfies

\[
[\tilde{e}_1, \tilde{e}_2] = -a\tilde{e}_1 - \beta\tilde{e}_2 - \beta\tilde{e}_3, \quad [\tilde{e}_1, \tilde{e}_3] = a\tilde{e}_1 + \beta\tilde{e}_2 + \beta\tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3] = \gamma\tilde{e}_1 + \delta\tilde{e}_2 + \delta\tilde{e}_3, \quad a + \delta \neq 0, \quad \alpha\gamma = 0, \tag{49}
\]

where $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a pseudo-orthonormal basis, with $\tilde{e}_3$ timelike.

**Lemma 13.** The Bott connection $\nabla^B$ of $G_7$ is given by

\[
\begin{aligned}
\nabla_{\tilde{e}_1}^B \tilde{e}_1 &= a\tilde{e}_2, \quad \nabla_{\tilde{e}_1}^B \tilde{e}_2 = -a\tilde{e}_1, \quad \nabla_{\tilde{e}_1}^B \tilde{e}_3 = \beta\tilde{e}_3, \\
\nabla_{\tilde{e}_2}^B \tilde{e}_1 &= \beta\tilde{e}_2, \quad \nabla_{\tilde{e}_2}^B \tilde{e}_2 = -\beta\tilde{e}_1, \quad \nabla_{\tilde{e}_2}^B \tilde{e}_3 = \delta\tilde{e}_3, \\
\nabla_{\tilde{e}_3}^B \tilde{e}_1 &= -a\tilde{e}_1 - \beta\tilde{e}_2, \quad \nabla_{\tilde{e}_3}^B \tilde{e}_2 = -\gamma\tilde{e}_1 - \delta\tilde{e}_2, \quad \nabla_{\tilde{e}_3}^B \tilde{e}_3 = 0. \tag{50}
\end{aligned}
\]

**Lemma 14.** The curvature $R^B$ of the Bott connection $\nabla^B$ of $(G_7, g)$ is given by

\[
\begin{aligned}
R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_1 &= a^2\tilde{e}_2 - a\beta\tilde{e}_1, \quad R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_2 = -(a^2 + \beta^2 + \beta\gamma)\tilde{e}_1 - \beta\delta\tilde{e}_2, \\
R^B(\tilde{e}_1, \tilde{e}_2)\tilde{e}_3 &= \beta(a - \delta)\tilde{e}_3, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_1 = a(2\beta + \gamma)\tilde{e}_1 + (a\delta - 2a^2)\tilde{e}_2, \\
R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_2 &= (a\delta + \beta^2 + \beta\gamma)\tilde{e}_1 + (\beta\delta - a\beta - a\gamma)\tilde{e}_2, \quad R^B(\tilde{e}_1, \tilde{e}_3)\tilde{e}_3 = -\beta(a + \delta)\tilde{e}_3, \\
R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_1 &= (\beta^2 + \beta\gamma + a\delta)\tilde{e}_1 + (\beta\delta - a\beta - a\gamma)\tilde{e}_2, \\
R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_2 &= (2\beta\delta + \delta\gamma + a\gamma - \alpha\beta)\tilde{e}_1 + (\alpha^2 - \beta^2 - \beta\gamma)\tilde{e}_2, \quad R^B(\tilde{e}_2, \tilde{e}_3)\tilde{e}_3 = -(\beta\gamma + \delta^2)\tilde{e}_3. \tag{51}
\end{aligned}
\]

By (3), we have

\[
\begin{aligned}
\rho^B(\tilde{e}_1, \tilde{e}_1) &= -a^2, \quad \rho^B(\tilde{e}_1, \tilde{e}_2) = \beta\delta, \quad \rho^B(\tilde{e}_1, \tilde{e}_3) = \beta(a + \delta), \\
\rho^B(\tilde{e}_2, \tilde{e}_1) &= -a\beta, \quad \rho^B(\tilde{e}_2, \tilde{e}_2) = -(a^2 + \beta^2 + \beta\gamma), \quad \rho^B(\tilde{e}_2, \tilde{e}_3) = (\beta\gamma + \delta^2), \\
\rho^B(\tilde{e}_3, \tilde{e}_1) &= \beta(a + \delta), \quad \rho^B(\tilde{e}_3, \tilde{e}_2) = \delta(a + \delta), \quad \rho^B(\tilde{e}_3, \tilde{e}_3) = 0. \tag{52}
\end{aligned}
\]

Then,

\[
\begin{aligned}
\tilde{\rho}^B(\tilde{e}_1, \tilde{e}_1) &= -a^2, \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_2) = \frac{\beta(\delta - a)}{2}, \quad \tilde{\rho}^B(\tilde{e}_1, \tilde{e}_3) = \delta(a + \delta), \\
\tilde{\rho}^B(\tilde{e}_2, \tilde{e}_2) &= -(a^2 + \beta^2 + \beta\gamma), \quad \tilde{\rho}^B(\tilde{e}_2, \tilde{e}_3) = \delta^2 + \frac{\beta\gamma + a\delta}{2}, \quad \tilde{\rho}^B(\tilde{e}_3, \tilde{e}_3) = 0. \tag{53}
\end{aligned}
\]

By (5), we have
\((\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_2, \tilde{e}_1) = \alpha (\beta^2 + \beta \gamma), \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_1, \tilde{e}_1) = \beta^2(\alpha - \delta), \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_2, \tilde{e}_2) = \alpha \beta (\delta - \alpha), \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_1, \tilde{e}_2) = \beta^2(\beta + \gamma), \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_2, \tilde{e}_3) = \alpha^2 \beta + \frac{\alpha \beta \delta - \beta^2 \gamma}{2} - \beta \delta^2, \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_1, \tilde{e}_3) = -(2 \beta \delta^2 + \frac{\beta^2 \gamma + 3 \alpha \beta \delta}{2}), \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_1) = -(\alpha \beta^2 + \beta^2 \delta + \alpha \delta^2 + \frac{\alpha \beta \gamma + \alpha^2 \delta}{2}), \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_1, \tilde{e}_2) = \beta^2 \delta - \alpha \beta^2 - 2 \alpha^3, \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_2) = \alpha^2 \beta + \frac{\alpha \beta \delta - \beta^2 \gamma}{2} - \beta \delta^2 - \frac{\beta^2 \gamma}{2}, \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_1, \tilde{e}_3) = \beta^2 \delta - 3 \alpha^2 \beta - \beta^3 - \beta^2 \gamma - \alpha^2 \gamma, \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_3) = -(2 \beta \delta^2 + \frac{3 \alpha \beta \delta + \beta^2 \gamma}{2}), \quad (\nabla_{\tilde{e}_1}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_3) = -(\alpha \beta^2 + \beta^2 \delta + 2 \alpha^2 \delta + 2 \beta \delta^2), \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla_{\tilde{e}_2}^{B} \tilde{\rho}^{B})(\tilde{e}_3, \tilde{e}_3) = \beta \alpha^2 \gamma + \delta^3 + \frac{3 \beta \delta \gamma + \alpha \delta^2}{2}. \quad (54)

Then, if \(\tilde{\rho}^{B}\) is a Codazzi tensor on \((G_7, \nabla^{B})\), by (6) and (7), we have the following nine equations:

\[
\begin{align*}
\beta(\alpha \gamma + \beta \delta) &= 0 \\
\beta(\alpha \delta - \alpha^2 - \beta^2 - \beta \gamma) &= 0 \\
\beta(\alpha + \delta) &= 0 \\
4 \alpha^3 - 4 \beta^2 \delta - 2 \alpha \delta^2 - (\alpha \beta \gamma + \alpha^2 \delta) &= 0 \\
5 \alpha^2 \beta + \alpha \beta \delta + \beta^2 \gamma - 3 \beta \delta^2 + 2 (\alpha^2 \gamma + \beta^3) &= 0 \\
\beta(2 \alpha^2 + 6 \alpha \delta + 2 \delta^2 + \beta \gamma) &= 0 \\
3 \alpha^2 \beta - 3 \alpha \beta \delta + \beta^2 \gamma - 5 \beta \delta^2 + 2 (\beta^3 + \alpha^2 \gamma) &= 0 \\
\beta \delta \gamma - \alpha \delta^2 + 2 (\alpha \beta \gamma + 2 \alpha \delta^2 + \alpha \beta^2 + 3 \beta^2 \delta - \delta^3) &= 0 \\
2 (\alpha \beta \gamma + \delta^3) + 3 \beta \delta \gamma + \alpha \delta^2 &= 0.
\end{align*}
\]  

(55)

By solving (55), we obtain \(\alpha = \delta = 0\), and the result is wrong. So,

**Theorem 7.** \(\tilde{\rho}^{B}\) is not a Codazzi tensor on \((G_7, \nabla^{B})\).

3. Quasi-Statistical Structure Associated with Bott Connections on Three-Dimensional Lorentzian Lie Groups

The torsion tensor of \((G_7, g, \nabla^{B})\) is defined by

\[T^{B}(X, Y) = \nabla_{\nabla_{X}^{B}Y - \nabla_{Y}^{B}X - [X, Y]}.\]  

(56)

Next, we recall the quasi-statistical structure.

**Definition 2** [6]. Let \(M\) be a smooth manifold endowed with a linear connection \(\nabla\), and a tensor field \(\omega\). Then \((M, \nabla, \omega)\) is called a quasi-statistical structure, if it satisfies

\[
\tilde{f}(X, Y, Z) = (\nabla_{X}^{B} \omega)(Y, Z) - (\nabla_{Y}^{B} \omega)(X, Z) + \omega(T(X, Y), Z) = 0,
\]

(57)

where \(\tilde{f}\) is \(C^{\infty}(M)\)-linear for \(X, Y, Z\).
Proposition 2. \((M, \nabla, \omega)\) is called a quasi-statistical structure and is called a Codazzi tensor on \((M, \nabla)\) if and only if
\[
\tilde{f}(X, Y, Z) = -\tilde{f}(Y, X, Z).
\] (58)

Then we have that \((G_i, \nabla^B, \omega_i)_{i=1,2,...,7}\) is a quasi-statistical structure if and only if the following three equations hold:
\[
\begin{align*}
\tilde{f}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_j) &= 0, \\
\tilde{f}(\tilde{e}_1, \tilde{e}_3, \tilde{e}_j) &= 0, \\
\tilde{f}(\tilde{e}_2, \tilde{e}_3, \tilde{e}_j) &= 0,
\end{align*}
\] (59)
where \(1 \leq j \leq 3\).

For \((G_1, \nabla^B)\), we have
\[
T^B(\tilde{e}_1, \tilde{e}_2) = \beta \tilde{e}_3, \quad T^B(\tilde{e}_1, \tilde{e}_3) = T^B(\tilde{e}_2, \tilde{e}_3) = 0,
\] (60)
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = -\frac{\alpha \beta^2}{2}, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{\alpha \beta^2}{2}, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = 0, \quad \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\] (61)
where \(1 \leq j \leq 3\).

Then, if \((G_1, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure, by (57) and (58), we have the following three equations:
\[
\begin{align*}
\alpha \beta &= 0, \\
\alpha &= 0, \\
\alpha (\alpha^2 - \beta^2) &= 0.
\end{align*}
\] (62)

By solving (62), we obtain \(\alpha = 0\), and there is a contradiction. So,

Theorem 8. \((G_1, \nabla^B, \tilde{\rho}^B)\) is not a quasi-statistical structure.

For \((G_2, \nabla^B)\), we have
\[
T^B(\tilde{e}_1, \tilde{e}_2) = \beta \tilde{e}_3, \quad T^B(\tilde{e}_1, \tilde{e}_3) = T^B(\tilde{e}_2, \tilde{e}_3) = 0,
\] (63)
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = 0, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = -\frac{\alpha \beta \gamma}{2}, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\] (64)
where \(1 \leq j \leq 3\).

Then, if \((G_2, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure, by (57) and (58), we have the following two equations:
\[
\begin{align*}
\alpha \beta \gamma &= 0, \\
\gamma (\alpha - 2\beta) &= 0.
\end{align*}
\] (65)

By solving (65), it turns out that

Theorem 9. \((G_2, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure if and only if \(\alpha = \beta = 0, \quad \gamma \neq 0\).
For \((G_3, \nabla^B)\), we have
\[
T^B(\tilde{e}_1, \tilde{e}_2) = \gamma \tilde{e}_3, \quad T^B(\tilde{e}_1, \tilde{e}_3) = T^B(\tilde{e}_2, \tilde{e}_3) = 0, \quad (66)
\]
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) = \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0, \quad (67)
\]
where \(1 \leq j \leq 3\).

Similarly, we can obtain

**Theorem 10.** \((G_3, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure.

For \((G_4, \nabla^B)\), we have
\[
T^B(\tilde{e}_1, \tilde{e}_2) = (\beta - 2\eta)\tilde{e}_3, \quad T^B(\tilde{e}_1, \tilde{e}_3) = T^B(\tilde{e}_2, \tilde{e}_3) = 0, \quad (68)
\]
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = 0, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{\alpha(\beta - 2\eta)}{2}, \quad \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \quad (69)
\]
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0, \quad (70)
\]
where \(1 \leq j \leq 3\).

Then, if \((G_4, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure, by (57) and (58), we have the following three equations:
\[
\begin{cases}
\beta^2 - 2\beta\eta + \alpha = 0 \\
\alpha - 2\beta = 0 \\
\alpha \beta = 0.
\end{cases} \quad (71)
\]

By solving (70), it turns out that

**Theorem 11.** \((G_4, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure if and only if \(\alpha = \beta = 0\).

For \((G_5, \nabla^B)\), we have
\[
T^B(\tilde{e}_1, \tilde{e}_2) = T^B(\tilde{e}_1, \tilde{e}_3) = T^B(\tilde{e}_2, \tilde{e}_3) = 0, \quad (72)
\]
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) = \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0, \quad (73)
\]
where \(1 \leq j \leq 3\).

Similarly, we obtain

**Theorem 12.** \((G_5, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure.

For \((G_6, \nabla^B)\), we have
\[
T^B(\tilde{e}_1, \tilde{e}_2) = -\beta\tilde{e}_3, \quad T^B(\tilde{e}_1, \tilde{e}_3) = T^B(\tilde{e}_2, \tilde{e}_3) = 0, \quad (74)
\]
\[
\tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = \tilde{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = 0, \quad (75)
\]
where \(1 \leq j \leq 3\).

Then, if \((G_6, \nabla^B, \tilde{\rho}^B)\) is a quasi-statistical structure, by (57) and (58), we have the following two equations:
\[
\begin{cases}
\alpha \beta \gamma = 0 \\
\alpha \gamma = 0.
\end{cases} \quad (76)
\]
By solving (75), we get

**Theorem 13.** \((G_6, \nabla^B, \bar{\rho}^B)\) is a quasi-statistical structure if and only if

\[
\begin{align*}
(1) & \alpha = \beta = 0, \quad \delta \neq 0; \\
(2) & \alpha \neq 0, \quad \gamma = \beta \delta = 0.
\end{align*}
\]

For \((G_7, \nabla^B)\), we have

\[
\begin{align*}
\bar{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2)) &= \beta \tilde{e}_3, \quad T^B(\tilde{e}_1, \tilde{e}_2) = T^B(\tilde{e}_2, \tilde{e}_3) = 0, \quad \text{(76)} \\
\bar{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) &= \beta^2(\alpha + \delta), \quad \bar{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \beta \delta^2 + \frac{\alpha \beta \delta + \beta^2 \gamma}{2}, \quad \bar{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \quad \text{(77)} \\
\bar{\rho}^B(T^B(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) &= \tilde{\rho}^B(T^B(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\end{align*}
\]

where \(1 \leq j \leq 3\).

Then, if \((G_7, \nabla^B, \bar{\rho}^B)\) is a quasi-statistical structure, by (57) and (58), we have the following nine equations:

\[
\begin{align*}
\beta(\alpha \gamma + \alpha \beta + 2 \beta \delta) &= 0 \\
2\beta(\alpha \delta - \alpha^2 - \beta^2 - \beta \gamma + \delta^2) + \beta^2 \gamma + \alpha \beta \delta &= 0 \\
\beta(\alpha + \delta)^2 &= 0 \\
4\alpha^3 - 4\beta^2 \delta - 2\alpha \delta^2 - \alpha(\beta \gamma + \alpha \delta) &= 0 \\
5\alpha^2 \beta + \alpha \beta \delta + \beta^2 \gamma - 3\beta \delta^2 + 2\alpha^2 \gamma + 2\beta^3 &= 0 \\
2\alpha \delta^3 + 3\alpha \beta \delta + \beta^2 \gamma &= 0 \\
3\alpha^2 \beta - 3\alpha \beta \delta + \beta^2 \gamma - 5\beta \delta^2 + 2\beta^3 + 3\alpha^2 \gamma &= 0 \\
\beta \delta \gamma - \alpha \delta^2 + 2\alpha \beta \gamma + 4\alpha^2 \delta + 2\alpha \beta^2 + 6\beta^2 \delta - 2\delta^3 &= 0 \\
2\alpha \beta \gamma + 2\delta^3 + 3\beta \delta \gamma + \alpha \delta^2 &= 0.
\end{align*}
\]

By solving (78), we get \(\alpha = \delta = 0\), and this condition does not hold. So,

**Theorem 14.** \((G_7, \nabla^B, \bar{\rho}^B)\) is not a quasi-statistical structure.

4. Codazzi Tensors Associated with Canonical Connections and Kobayashi–Nomizu Connections on Three-Dimensional Lorentzian Lie Groups

By [14], we define canonical connections and Kobayashi–Nomizu connections as follows:

\[
\nabla^L_X Y = \nabla^L_X Y - \frac{1}{2}(\nabla^L_X J) J Y, \quad \text{(79)}
\]

\[
\nabla^L_X Y = \nabla^L_X Y - \frac{1}{4}(\nabla^L_X J) J X - (\nabla^L_X J) J X, \quad \text{(80)}
\]

where \(J\) is a product structure on \(\{G_i\}_{i=1,2,\ldots,7}\) by \(J\tilde{e}_1 = \tilde{e}_1, J\tilde{e}_2 = \tilde{e}_2, J\tilde{e}_3 = -\tilde{e}_3\).
4.1. Codazzi Tensors of $G_1$

**Lemma 15 ([7]).** The canonical connection $\nabla^c$ of $G_1$ is given by

\[
\begin{align*}
\nabla^c_{\tilde{e}_1}\tilde{e}_1 &= -\alpha\tilde{e}_2, \quad \nabla^c_{\tilde{e}_2}\tilde{e}_2 = \alpha\tilde{e}_1, \quad \nabla^c_{\tilde{e}_1}\tilde{e}_3 = 0, \\
\nabla^c_{\tilde{e}_2}\tilde{e}_1 &= \nabla^c_{\tilde{e}_1}\tilde{e}_2 = \nabla^c_{\tilde{e}_2}\tilde{e}_3 = 0, \\
\nabla^c_{\tilde{e}_3}\tilde{e}_1 &= \frac{\beta}{2}\tilde{e}_2, \quad \nabla^c_{\tilde{e}_3}\tilde{e}_2 = -\frac{\beta}{2}\tilde{e}_1, \quad \nabla^c_{\tilde{e}_3}\tilde{e}_3 = 0.
\end{align*}
\]  

(81)

Then,

\[
\begin{align*}
\bar{\rho}^c(\tilde{e}_1, \tilde{e}_1) &= -(\alpha^2 + \frac{\beta^2}{2}), \quad \bar{\rho}^c(\tilde{e}_1, \tilde{e}_2) = 0, \quad \bar{\rho}^c(\tilde{e}_1, \tilde{e}_3) = \frac{\alpha\beta}{4}, \\
\bar{\rho}^c(\tilde{e}_2, \tilde{e}_2) &= -(\alpha^2 + \frac{\beta^2}{2}), \quad \bar{\rho}^c(\tilde{e}_2, \tilde{e}_3) = \frac{\alpha^2}{2}, \quad \bar{\rho}^c(\tilde{e}_3, \tilde{e}_3) = 0.
\end{align*}
\]  

(82)

By (5), we have

\[
\begin{align*}
(\nabla^c_{\tilde{e}_1}\bar{\rho}^c)(\tilde{e}_2, \tilde{e}_1) &= (\nabla^c_{\tilde{e}_2}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = (\nabla^c_{\tilde{e}_1}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_2) = 0, \\
(\nabla^c_{\tilde{e}_2}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_2) &= 0, \quad (\nabla^c_{\tilde{e}_2}\bar{\rho}^c)(\tilde{e}_2, \tilde{e}_2) = -\frac{\alpha\beta}{4}, \quad (\nabla^c_{\tilde{e}_2}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_3) = -\frac{\alpha\beta}{4}, \\
(\nabla^c_{\tilde{e}_1}\bar{\rho}^c)(\tilde{e}_3, \tilde{e}_1) &= \frac{\alpha^3}{2}, \quad (\nabla^c_{\tilde{e}_1}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla^c_{\tilde{e}_1}\bar{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = \frac{\alpha^2\beta}{4}, \\
(\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_2) &= (\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_1, \tilde{e}_3) = -\frac{\alpha^2\beta}{4}, \\
(\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_3, \tilde{e}_1) &= (\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_2, \tilde{e}_2) = 0, \quad (\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_3, \tilde{e}_2) = 0, \\
(\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_2, \tilde{e}_2) &= -2\alpha^2, \quad (\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla^c_{\tilde{e}_3}\bar{\rho}^c)(\tilde{e}_2, \tilde{e}_3) = \frac{\alpha^2\beta^2}{8}.
\end{align*}
\]  

(83)

Then, if $\bar{\rho}^c$ is a Codazzi tensor on $(G_1, \nabla^c)$, by (6) and (7), we have the following two equations:

\[
\begin{align*}
\alpha &= 0, \\
\alpha\beta &= 0.
\end{align*}
\]  

(84)

By solving (84), we get $\alpha = 0$, and the result is not true. So,

**Theorem 15.** $\bar{\rho}^c$ is not a Codazzi tensor on $(G_1, \nabla^c)$.

**Lemma 16 ([7]).** The Kobayashi–Nomizu connection $\nabla^k$ of $G_1$ is given by

\[
\begin{align*}
\nabla^k_{\tilde{e}_1}\tilde{e}_1 &= -\alpha\tilde{e}_2, \quad \nabla^k_{\tilde{e}_2}\tilde{e}_2 = \alpha\tilde{e}_1, \quad \nabla^k_{\tilde{e}_3}\tilde{e}_3 = 0, \\
\nabla^k_{\tilde{e}_2}\tilde{e}_1 &= 0, \quad \nabla^k_{\tilde{e}_1}\tilde{e}_2 = 0, \quad \nabla^k_{\tilde{e}_3}\tilde{e}_3 = \alpha\tilde{e}_3, \\
\nabla^k_{\tilde{e}_3}\tilde{e}_1 &= \alpha\tilde{e}_1 + \beta\tilde{e}_2, \quad \nabla^k_{\tilde{e}_3}\tilde{e}_2 = -\beta\tilde{e}_1 - \alpha\tilde{e}_2, \quad \nabla^k_{\tilde{e}_3}\tilde{e}_3 = 0.
\end{align*}
\]  

(85)

Then,

\[
\begin{align*}
\rho^k(\tilde{e}_1, \tilde{e}_1) &= -(\alpha^2 + \beta^2), \quad \rho^k(\tilde{e}_1, \tilde{e}_2) = \alpha\beta, \quad \rho^k(\tilde{e}_1, \tilde{e}_3) = -\frac{\alpha\beta}{2}, \\
\rho^k(\tilde{e}_2, \tilde{e}_2) &= -(\alpha^2 + \beta^2), \quad \rho^k(\tilde{e}_2, \tilde{e}_3) = \frac{\alpha^2}{2}, \quad \rho^k(\tilde{e}_3, \tilde{e}_3) = 0.
\end{align*}
\]  

(86)
By (5), we have

\[
(\nabla_{\tilde{c}_1} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_1) = 0, \quad (\nabla_{\tilde{c}_2} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_1) = 0, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_2) = -2\alpha^2 \tilde{\rho},
\]

\[
(\nabla_{\tilde{c}_1} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_2) = 0, \quad (\nabla_{\tilde{c}_2} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_3) = \frac{\alpha^2 \beta}{2}, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_3) = \frac{\alpha^2 \beta}{2},
\]

\[
(\nabla_{\tilde{c}_1} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_1) = \frac{\alpha^3}{2}, \quad (\nabla_{\tilde{c}_2} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_1) = 2\alpha^3, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_3, \tilde{c}_2) = \frac{\alpha^2 \beta}{2},
\]

\[
(\nabla_{\tilde{c}_2} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_2) = 0, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_3) = 0, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_1, \tilde{c}_3) = 0,
\]

\[
(\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_3, \tilde{c}_1) = \frac{\alpha^2 \beta}{2}, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_1) = 0, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_3, \tilde{c}_2) = -\frac{\alpha^3}{2},
\]

\[
(\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_2) = -2\alpha^3, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_3, \tilde{c}_3) = 0, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^k)(\tilde{c}_2, \tilde{c}_3) = \frac{\alpha}{2}(\alpha^2 - \beta^2). \quad (87)
\]

Then, if $\tilde{\rho}^k$ is a Codazzi tensor on $(G_1, \nabla^k)$, by (6) and (7), we have the following three equations:

\[
\begin{aligned}
\mathbf{a} \beta &= 0 \\
\alpha &= 0 \\
\alpha(\beta^2 - \alpha^2) &= 0.
\end{aligned} \quad (88)
\]

By solving (88), we get $\alpha = 0$, and the result does not match the condition. So,

**Theorem 16.** $\tilde{\rho}^k$ is not a Codazzi tensor on $(G_1, \nabla^k)$.

4.2. Codazzi Tensors of $G_2$

**Lemma 17 ([7]).** The canonical connection $\nabla^c$ of $G_2$ is given by

\[
\begin{aligned}
\nabla^c_{\tilde{c}_1} \tilde{c}_1 &= 0, \quad \nabla^c_{\tilde{c}_1} \tilde{c}_2 = \alpha \tilde{c}_1, \quad \nabla^c_{\tilde{c}_1} \tilde{c}_3 = 0, \\
\nabla^c_{\tilde{c}_2} \tilde{c}_1 &= -\gamma \tilde{c}_2, \quad \nabla^c_{\tilde{c}_2} \tilde{c}_2 = \gamma \tilde{c}_1, \quad \nabla^c_{\tilde{c}_2} \tilde{c}_3 = 0, \\
\nabla^c_{\tilde{c}_3} \tilde{c}_1 &= \frac{\alpha}{2} \tilde{c}_2, \quad \nabla^c_{\tilde{c}_3} \tilde{c}_2 = -\frac{\alpha}{2} \tilde{c}_1, \quad \nabla^c_{\tilde{c}_3} \tilde{c}_3 = 0.
\end{aligned} \quad (89)
\]

Then,

\[
\tilde{\rho}^c(\tilde{c}_1, \tilde{c}_1) = -\left(\gamma^2 + \frac{\alpha \beta}{2}\right), \quad \tilde{\rho}^c(\tilde{c}_1, \tilde{c}_2) = 0, \quad \tilde{\rho}^c(\tilde{c}_1, \tilde{c}_3) = 0,
\]

\[
\tilde{\rho}^c(\tilde{c}_2, \tilde{c}_2) = -\left(\gamma^2 + \frac{\alpha \beta}{2}\right), \quad \tilde{\rho}^c(\tilde{c}_2, \tilde{c}_3) = \gamma \left(\frac{\beta}{2} - \frac{\alpha}{4}\right), \quad \tilde{\rho}^c(\tilde{c}_3, \tilde{c}_3) = 0. \quad (90)
\]

By (5), we have

\[
\begin{aligned}
(\nabla_{\tilde{c}_1} \tilde{\rho}^c)(\tilde{c}_2, \tilde{c}_1) &= (\nabla_{\tilde{c}_2} \tilde{\rho}^c)(\tilde{c}_1, \tilde{c}_1) = (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_2, \tilde{c}_2) = 0, \\
(\nabla_{\tilde{c}_2} \tilde{\rho}^c)(\tilde{c}_1, \tilde{c}_2) &= (\nabla_{\tilde{c}_2} \tilde{\rho}^c)(\tilde{c}_2, \tilde{c}_3) = 0, \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_1, \tilde{c}_3) = \gamma^2 \left(\frac{\alpha}{4} - \frac{\beta}{2}\right), \\
(\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_3, \tilde{c}_1) &= (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_1, \tilde{c}_1) = (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_3, \tilde{c}_2) = 0, \\
(\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_1, \tilde{c}_2) &= (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_3, \tilde{c}_3) = (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_1, \tilde{c}_3) = \frac{\alpha \gamma}{4} \left(\frac{\alpha}{2} - \beta\right), \\
(\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_3, \tilde{c}_1) &= \gamma^2 \left(\frac{\beta}{2} - \frac{\alpha}{4}\right), \quad (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_2, \tilde{c}_1) = (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_3, \tilde{c}_2) = 0, \\
(\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_2, \tilde{c}_2) &= (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_3, \tilde{c}_3) = (\nabla_{\tilde{c}_3} \tilde{\rho}^c)(\tilde{c}_2, \tilde{c}_3) = 0. \quad (91)
\end{aligned}
\]
Then, if $\hat{\rho}^c$ is a Codazzi tensor on $(G_2, \nabla^c)$, by (6) and (7), we have the following two equations:

$$\begin{align*}
\gamma (\alpha - 2\beta) &= 0, \\
\alpha \gamma (\alpha - 2\beta) &= 0.
\end{align*}$$

(92)

By solving (92), we obtain

**Theorem 17.** $\hat{\rho}^c$ is a Codazzi tensor on $(G_2, \nabla^c)$ if and only if $\gamma \neq 0, \quad \alpha = 2\beta$.  

**Lemma 18 ([7]).** The Kobayashi–Nomizu connection $\nabla^k$ of $G_2$ is given by

$$\begin{align*}
\nabla^k e_1 e_1 &= 0, \quad \nabla^k e_2 e_2 = -\gamma e_2, \\
\nabla^k e_2 e_1 &= 0, \\
\nabla^k e_3 e_1 &= -\gamma e_3, \\
\nabla^k e_3 e_2 &= \gamma e_1, \\
\nabla^k e_3 e_3 &= 0.
\end{align*}$$

(93)

Then,

$$\begin{align*}
\hat{\rho}^k (e_1, e_1) &= - (\gamma^2 + \beta^2), \\
\hat{\rho}^k (e_1, e_2) &= 0, \\
\hat{\rho}^k (e_2, e_2) &= - \frac{\alpha \gamma}{2}, \\
\hat{\rho}^k (e_2, e_3) &= 0.
\end{align*}$$

(94)

By (5), we have

$$\begin{align*}
(\nabla^k e_1 \hat{\rho}^k)(e_2, e_1) &= 0, \\
(\nabla^k e_2 \hat{\rho}^k)(e_1, e_1) &= 0, \\
(\nabla^k e_2 \hat{\rho}^k)(e_2, e_2) &= 0, \\
(\nabla^k e_3 \hat{\rho}^k)(e_1, e_1) &= 0, \\
(\nabla^k e_3 \hat{\rho}^k)(e_2, e_2) &= 0, \\
(\nabla^k e_3 \hat{\rho}^k)(e_3, e_3) &= 0, \\
(\nabla^k e_3 \hat{\rho}^k)(e_1, e_2) &= 0, \\
(\nabla^k e_3 \hat{\rho}^k)(e_2, e_1) &= 0, \\
(\nabla^k e_3 \hat{\rho}^k)(e_3, e_1) &= 0.
\end{align*}$$

(95)

Then, if $\hat{\rho}^k$ is a Codazzi tensor on $(G_2, \nabla^k)$, by (6) and (7), we have the following three equations:

$$\begin{align*}
\beta \gamma (\alpha - \beta) &= 0, \\
\gamma (\alpha - 2\beta) &= 0, \\
\alpha \beta \gamma &= 0.
\end{align*}$$

(96)

The above equation induces the following theorem:

**Theorem 18.** $\hat{\rho}^k$ is a Codazzi tensor on $(G_2, \nabla^k)$ if and only if $\gamma \neq 0, \quad \alpha = \beta = 0$.  

4.3. Codazzi Tensors of $G_3$

**Lemma 19 ([7]).** The canonical connection $\nabla^c$ of $G_3$ is given by
\[ \nabla^c_{\bar{e}_i} \bar{e}_1 = 0, \quad \nabla^c_{\bar{e}_i} \bar{e}_2 = 0, \quad \nabla^c_{\bar{e}_i} \bar{e}_3 = 0, \]
\[ \nabla^c_{\bar{e}_j} \bar{e}_1 = 0, \quad \nabla^c_{\bar{e}_j} \bar{e}_2 = 0, \quad \nabla^c_{\bar{e}_j} \bar{e}_3 = 0, \]
\[ \nabla^c_{\bar{e}_k} \bar{e}_1 = m_3 \bar{e}_2, \quad \nabla^c_{\bar{e}_k} \bar{e}_2 = -m_3 \bar{e}_1, \quad \nabla^c_{\bar{e}_k} \bar{e}_3 = 0, \]

where
\[ m_1 = \frac{\alpha - \beta - \gamma}{2}, \quad m_2 = \frac{\alpha - \beta + \gamma}{2}, \quad m_3 = \frac{\alpha + \beta - \gamma}{2}. \]
\[ \nabla^k_1 \tilde{e}_1 = 0, \quad \nabla^k_2 \tilde{e}_2 = 0, \quad \nabla^k_3 \tilde{e}_3 = 0, \]
\[ \nabla^k_2 \tilde{e}_1 = \tilde{e}_2, \quad \nabla^k_2 \tilde{e}_2 = -\tilde{e}_1, \quad \nabla^k_2 \tilde{e}_3 = 0, \]
\[ \nabla^k_3 \tilde{e}_1 = n_3 \tilde{e}_2, \quad \nabla^k_3 \tilde{e}_2 = -n_3 \tilde{e}_1, \quad \nabla^k_3 \tilde{e}_3 = 0. \tag{105} \]

where
\[ n_1 = \frac{\alpha}{2} + \eta - \beta, \quad n_2 = \frac{\alpha}{2} - \eta, \quad n_3 = \frac{\alpha}{2} + \eta. \tag{106} \]

Then,
\[ \tilde{\rho}(\tilde{e}_1, \tilde{e}_1) = (2\eta - \beta) n_3 - 1, \quad \tilde{\rho}(\tilde{e}_1, \tilde{e}_2) = 0, \quad \tilde{\rho}(\tilde{e}_1, \tilde{e}_3) = 0, \]
\[ \tilde{\rho}(\tilde{e}_2, \tilde{e}_2) = (2\eta - \beta) n_3 - 1, \quad \tilde{\rho}(\tilde{e}_2, \tilde{e}_3) = \frac{\beta - n_3}{2}, \quad \tilde{\rho}(\tilde{e}_3, \tilde{e}_3) = 0. \tag{107} \]

By (5), we have
\[ (\nabla^k_1 \tilde{\rho})(\tilde{e}_2, \tilde{e}_1) = (\nabla^k_2 \tilde{\rho})(\tilde{e}_2, \tilde{e}_1) = (\nabla^k_3 \tilde{\rho})(\tilde{e}_2, \tilde{e}_1) = 0, \]
\[ (\nabla^k_2 \tilde{\rho})(\tilde{e}_1, \tilde{e}_2) = (\nabla^k_1 \tilde{\rho})(\tilde{e}_1, \tilde{e}_2) = (\nabla^k_3 \tilde{\rho})(\tilde{e}_1, \tilde{e}_2) = \beta + n_3, \]
\[ (\nabla^k_3 \tilde{\rho})(\tilde{e}_3, \tilde{e}_1) = (\nabla^k_1 \tilde{\rho})(\tilde{e}_3, \tilde{e}_1) = (\nabla^k_2 \tilde{\rho})(\tilde{e}_3, \tilde{e}_1) = 0, \]
\[ (\nabla^k_1 \tilde{\rho})(\tilde{e}_3, \tilde{e}_2) = \beta - n_3, \quad (\nabla^k_2 \tilde{\rho})(\tilde{e}_2, \tilde{e}_1) = (\nabla^k_3 \tilde{\rho})(\tilde{e}_2, \tilde{e}_1) = 0, \]
\[ (\nabla^k_3 \tilde{\rho})(\tilde{e}_2, \tilde{e}_2) = 0, \quad (\nabla^k_2 \tilde{\rho})(\tilde{e}_3, \tilde{e}_3) = (\nabla^k_3 \tilde{\rho})(\tilde{e}_2, \tilde{e}_3) = 0. \tag{108} \]

Then, if \( \tilde{\rho} \) is a Codazzi tensor on \((G_4, \nabla^c)\), by (6) and (7), we have the following two equations:
\[
\begin{cases}
\beta - n_3 = 0 \\
n_3 (\beta - n_3) = 0.
\end{cases} \tag{109}
\]

By solving (109), we get

**Theorem 21.** \( \tilde{\rho} \) is a Codazzi tensor on \((G_4, \nabla^c)\) if and only if \( \frac{\alpha}{2} + \eta - \beta = 0 \).

**Lemma 22 ([7]).** The Kobayashi–Nomizu connection \( \nabla^k \) of \( G_4 \) is given by
\[ \nabla^k_1 \tilde{e}_1 = 0, \quad \nabla^k_2 \tilde{e}_2 = 0, \quad \nabla^k_3 \tilde{e}_3 = \tilde{e}_3, \]
\[ \nabla^k_1 \tilde{e}_2 = \tilde{e}_2, \quad \nabla^k_2 \tilde{e}_2 = -\tilde{e}_1, \quad \nabla^k_2 \tilde{e}_3 = 0, \]
\[ \nabla^k_3 \tilde{e}_1 = (n_3 - n_1) \tilde{e}_2, \quad \nabla^k_3 \tilde{e}_2 = -(n_2 + n_3) \tilde{e}_1, \quad \nabla^k_3 \tilde{e}_3 = 0. \tag{110} \]

where
\[ n_1 = \frac{\alpha}{2} + \eta - \beta, \quad n_2 = \frac{\alpha}{2} - \eta, \quad n_3 = \frac{\alpha}{2} + \eta. \tag{111} \]

Then,
\[ \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_1) = -[1 + (\beta - 2\eta)(n_3 - n_1)], \quad \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_2) = 0, \quad \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_3) = 0, \]
\[ \tilde{\rho}^k(\tilde{e}_2, \tilde{e}_2) = [1 + (\beta - 2\eta)(n_2 + n_3)], \quad \tilde{\rho}^k(\tilde{e}_2, \tilde{e}_3) = \frac{\alpha + n_3 - n_1 - \beta}{2}, \quad \tilde{\rho}^k(\tilde{e}_3, \tilde{e}_3) = 0. \tag{112} \]
By (5), we have

\begin{align*}
(\nabla^k_{\tilde{e}_1}\tilde{\rho}^k)(\tilde{e}_2, \tilde{e}_1) &= (\nabla^k_{\tilde{e}_2}\tilde{\rho}^k)(\tilde{e}_1, \tilde{e}_1) = (\nabla^k_{\tilde{e}_1}\tilde{\rho}^k)(\tilde{e}_2, \tilde{e}_2) = 0, \\
(\nabla^k_{\tilde{e}_2}\tilde{\rho}^k)(\tilde{e}_1, \tilde{e}_1) &= (\nabla^k_{\tilde{e}_1}\tilde{\rho}^k)(\tilde{e}_2, \tilde{e}_2) = (n_1 + n_2)(\beta - 2\eta), \\
(\nabla^k_{\tilde{e}_3}\tilde{\rho}^k)(\tilde{e}_2, \tilde{e}_3) &= n_1 + \beta - \alpha - n_3, \\
(\nabla^k_{\tilde{e}_1}\tilde{\rho}^k)(\tilde{e}_1, \tilde{e}_3) &= n_1 + \beta - \alpha - n_3, \\
(\nabla^k_{\tilde{e}_2}\tilde{\rho}^k)(\tilde{e}_1, \tilde{e}_1) &= (\nabla^k_{\tilde{e}_3}\tilde{\rho}^k)(\tilde{e}_1, \tilde{e}_1) = 0, \\
(\nabla^k_{\tilde{e}_1}\tilde{\rho}^k)(\tilde{e}_3, \tilde{e}_1) &= n_1 + \beta - \alpha - n_3, \\
(\nabla^k_{\tilde{e}_2}\tilde{\rho}^k)(\tilde{e}_3, \tilde{e}_1) &= -(n_1 + n_2), \\
(\nabla^k_{\tilde{e}_3}\tilde{\rho}^k)(\tilde{e}_3, \tilde{e}_1) &= (n_1 + n_2), \\
(\nabla^k_{\tilde{e}_2}\tilde{\rho}^k)(\tilde{e}_2, \tilde{e}_1) &= -(n_1 + n_2), \\
(\nabla^k_{\tilde{e}_3}\tilde{\rho}^k)(\tilde{e}_2, \tilde{e}_1) &= 0, \\
(\nabla^k_{\tilde{e}_3}\tilde{\rho}^k)(\tilde{e}_3, \tilde{e}_2) &= (\nabla^k_{\tilde{e}_3}\tilde{\rho}^k)(\tilde{e}_3, \tilde{e}_2) = (\nabla^k_{\tilde{e}_1}\tilde{\rho}^k)(\tilde{e}_3, \tilde{e}_2) = 0. \\
\end{align*}

(113)

Then, if \(\tilde{\rho}^k\) is a Codazzi tensor on \((G_4, \nabla^k)\), by (6) and (7), we have the following three equations:

\begin{align*}
&\begin{cases}
(2\eta - \beta)(n_1 + n_2) = 0 \\
3n_1 + \beta - \alpha - n_3 + 2n_2 = 0 \\
(n_3 - n_1)\alpha + n_3 - n_1 - \beta = 0.
\end{cases}
\end{align*}

(114)

The above equations induce the following theorem:

**Theorem 22.** \(\tilde{\rho}^k\) is a Codazzi tensor on \((G_4, \nabla^k)\) if and only if \(\alpha = \beta = 0\).

### 4.5. Codazzi Tensors of \(G_5\)

**Lemma 23 ([7]).** The canonical connection \(\nabla^c\) of \(G_5\) is given by

\begin{align*}
\nabla^c_{\tilde{e}_1}\tilde{e}_1 &= 0, \\
\nabla^c_{\tilde{e}_1}\tilde{e}_2 &= 0, \\
\nabla^c_{\tilde{e}_1}\tilde{e}_3 &= 0, \\
\nabla^c_{\tilde{e}_2}\tilde{e}_1 &= 0, \\
\nabla^c_{\tilde{e}_2}\tilde{e}_2 &= 0, \\
\nabla^c_{\tilde{e}_2}\tilde{e}_3 &= 0, \\
\nabla^c_{\tilde{e}_3}\tilde{e}_1 &= \frac{T - \beta}{2}\tilde{e}_1, \\
\nabla^c_{\tilde{e}_3}\tilde{e}_2 &= \frac{T - \beta}{2}\tilde{e}_1, \\
\nabla^c_{\tilde{e}_3}\tilde{e}_3 &= 0.
\end{align*}

(115)

Then,

\begin{align*}
\tilde{\rho}^c(\tilde{e}_1, \tilde{e}_1) &= \tilde{\rho}^c(\tilde{e}_1, \tilde{e}_2) = \tilde{\rho}^c(\tilde{e}_1, \tilde{e}_3) = 0, \\
\tilde{\rho}^c(\tilde{e}_2, \tilde{e}_2) &= \tilde{\rho}^c(\tilde{e}_2, \tilde{e}_3) = \tilde{\rho}^c(\tilde{e}_3, \tilde{e}_3) = 0.
\end{align*}

(116)

By (5), we have

\begin{align*}
(\nabla^c_{\tilde{e}_1}\tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_1) &= (\nabla^c_{\tilde{e}_1}\tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = (\nabla^c_{\tilde{e}_1}\tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_1) = 0, \\
(\nabla^c_{\tilde{e}_2}\tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_1) &= (\nabla^c_{\tilde{e}_2}\tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_1) = (\nabla^c_{\tilde{e}_3}\tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_1) = 0.
\end{align*}

(117)

where \(1 \leq j \leq 3\).

Then, we get

**Theorem 23.** \(\tilde{\rho}^c\) is a Codazzi tensor on \((G_5, \nabla^c)\).

**Lemma 24 ([7]).** The Kobayashi–Nomizu connection \(\nabla^k\) of \(G_5\) is given by

\begin{align*}
\nabla^k_{\tilde{e}_1}\tilde{e}_1 &= 0, \\
\nabla^k_{\tilde{e}_1}\tilde{e}_2 &= 0, \\
\nabla^k_{\tilde{e}_1}\tilde{e}_3 &= 0, \\
\nabla^k_{\tilde{e}_2}\tilde{e}_1 &= 0, \\
\nabla^k_{\tilde{e}_2}\tilde{e}_2 &= 0, \\
\nabla^k_{\tilde{e}_2}\tilde{e}_3 &= 0, \\
\nabla^k_{\tilde{e}_3}\tilde{e}_1 &= -\alpha\tilde{e}_1 - \beta\tilde{e}_2, \\
\nabla^k_{\tilde{e}_3}\tilde{e}_2 &= -\gamma\tilde{e}_1 - \delta\tilde{e}_2, \\
\nabla^k_{\tilde{e}_3}\tilde{e}_3 &= 0.
\end{align*}

(118)
Then,
\[ \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_1) = \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_2) = \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_3) = 0, \]
\[ \tilde{\rho}^k(\tilde{e}_2, \tilde{e}_2) = \tilde{\rho}^k(\tilde{e}_2, \tilde{e}_3) = \tilde{\rho}^k(\tilde{e}_3, \tilde{e}_3) = 0. \]  \hspace{1cm} (119)

Then, we get

**Theorem 24.** \( \tilde{\rho}^k \) is a Codazzi tensor on \((G_5, \nabla^k)\).

4.6. Codazzi Tensors of \( G_6 \)

**Lemma 25 ([7]).** The canonical connection \( \nabla^c \) of \( G_6 \) is given by
\[ \begin{align*}
\nabla^c_1 \tilde{e}_1 &= 0, \quad \nabla^c_1 \tilde{e}_2 = a \tilde{e}_1, \quad \nabla^c_1 \tilde{e}_3 = 0, \\
\nabla^c_2 \tilde{e}_1 &= -a \tilde{e}_2, \quad \nabla^c_2 \tilde{e}_2 = a \tilde{e}_1, \quad \nabla^c_2 \tilde{e}_3 = 0, \\
\nabla^c_3 \tilde{e}_1 &= \frac{\beta - \gamma}{2} \tilde{e}_2, \quad \nabla^c_3 \tilde{e}_2 = -\frac{\beta - \gamma}{2} \tilde{e}_1, \quad \nabla^c_3 \tilde{e}_3 = 0. 
\end{align*} \]  \hspace{1cm} (120)

Then,
\[ \begin{align*}
\tilde{\rho}^c(\tilde{e}_1, \tilde{e}_1) &= \frac{1}{2} \beta (\beta - \gamma) - a^2, \quad \tilde{\rho}^c(\tilde{e}_1, \tilde{e}_2) = 0, \quad \tilde{\rho}^c(\tilde{e}_1, \tilde{e}_3) = 0, \\
\tilde{\rho}^c(\tilde{e}_2, \tilde{e}_2) &= \frac{1}{2} \beta (\beta - \gamma) - a^2, \quad \tilde{\rho}^c(\tilde{e}_2, \tilde{e}_3) = \frac{1}{2} [ -a \gamma + \frac{1}{2} \delta (\beta - \gamma) ], \quad \tilde{\rho}^c(\tilde{e}_3, \tilde{e}_3) = 0. 
\end{align*} \]  \hspace{1cm} (121)

By (5), we have
\[ \begin{align*}
(\nabla^c_1 \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_1) &= 0, \quad (\nabla^c_1 \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla^c_1 \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_2) = 0, \\
(\nabla^c_2 \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_2) &= 0, \quad (\nabla^c_2 \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_3) = 0, \quad (\nabla^c_2 \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_3) = \frac{a}{2} [ -a \gamma + \frac{1}{2} \delta (\beta - \gamma) ], \\
(\nabla^c_2 \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_1) &= 0, \quad (\nabla^c_2 \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = 0, \quad (\nabla^c_2 \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_2) = 0, \\
(\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_2) &= 0, \quad (\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_3) = \frac{\gamma - \beta}{4} [ -a \gamma + \frac{1}{2} \delta (\beta - \gamma) ], \\
(\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_1) &= \frac{a}{2} [ -a \gamma + \frac{1}{2} \delta (\beta - \gamma) ], \quad (\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_1) = 0, \quad (\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_2) = 0, \\
(\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_2) &= 0, \quad (\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = 0, \quad (\nabla^c_3 \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_3) = 0. 
\end{align*} \]  \hspace{1cm} (122)

Then, if \( \tilde{\rho}^c \) is a Codazzi tensor on \((G_6, \nabla^c)\), by (6) and (7), we have the following two equations:
\[ \begin{align*}
\alpha [ -2 a \gamma + \delta (\beta - \gamma) ] &= 0, \\
(\beta - \gamma) [ -2 a \gamma + \delta (\beta - \gamma) ] &= 0. 
\end{align*} \]  \hspace{1cm} (123)

By solving (123), we get

**Theorem 25.** \( \tilde{\rho}^c \) is a Codazzi tensor on \((G_6, \nabla^c)\) if and only if
\[ \begin{align*}
(1) a &= \beta = \gamma = 0, \quad \delta \neq 0; \\
(2) a \neq 0, \quad \beta = \gamma = 0, \quad \alpha + \delta \neq 0; \\
(3) a \neq 0, \quad \delta = \gamma = 0. 
\end{align*} \]
Lemma 26 ([7]). The Kobayashi–Nomizu connection $\nabla^k$ of $G_6$ is given by

$$
\begin{align*}
\nabla_{\tilde{e}_1} \tilde{e}_1 &= 0, \quad \nabla_{\tilde{e}_1} \tilde{e}_2 = 0, \quad \nabla_{\tilde{e}_1} \tilde{e}_3 = \delta \tilde{e}_3, \\
\nabla_{\tilde{e}_2} \tilde{e}_1 &= -\alpha \tilde{e}_2, \quad \nabla_{\tilde{e}_2} \tilde{e}_2 = \alpha \tilde{e}_1, \quad \nabla_{\tilde{e}_2} \tilde{e}_3 = 0, \\
\nabla_{\tilde{e}_3} \tilde{e}_1 &= -\gamma \tilde{e}_3, \quad \nabla_{\tilde{e}_3} \tilde{e}_2 = 0, \quad \nabla_{\tilde{e}_3} \tilde{e}_3 = 0.
\end{align*}
$$

(124)

Then,

$$
\begin{align*}
\rho^k(\tilde{e}_1, \tilde{e}_1) &= -(a^2 + \beta \gamma), \quad \rho^k(\tilde{e}_1, \tilde{e}_2) = 0, \quad \rho^k(\tilde{e}_1, \tilde{e}_3) = 0, \\
\rho^k(\tilde{e}_2, \tilde{e}_1) &= -a^2, \quad \rho^k(\tilde{e}_2, \tilde{e}_3) = -\frac{a \gamma}{2}, \quad \rho^k(\tilde{e}_3, \tilde{e}_3) = 0.
\end{align*}
$$

(125)

By (5), we have

$$
\begin{align*}
(\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_2, \tilde{e}_1) &= (\nabla_{\tilde{e}_2}^k \rho^k)(\tilde{e}_1, \tilde{e}_1) = (\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_2, \tilde{e}_2) = 0, \\
(\nabla_{\tilde{e}_2}^k \rho^k)(\tilde{e}_1, \tilde{e}_2) &= a \beta \gamma, \quad (\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_2, \tilde{e}_3) = (\nabla_{\tilde{e}_3}^k \rho^k)(\tilde{e}_1, \tilde{e}_3) = 0, \\
(\nabla_{\tilde{e}_3}^k \rho^k)(\tilde{e}_1, \tilde{e}_3) &= (\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_3, \tilde{e}_1) = 0, \quad (\nabla_{\tilde{e}_3}^k \rho^k)(\tilde{e}_2, \tilde{e}_2) = -\frac{a \gamma^2}{2}, \\
(\nabla_{\tilde{e}_2}^k \rho^k)(\tilde{e}_3, \tilde{e}_3) &= -(a^2 \gamma), \quad (\nabla_{\tilde{e}_2}^k \rho^k)(\tilde{e}_3, \tilde{e}_1) = (\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_3, \tilde{e}_2) = 0, \\
(\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_3, \tilde{e}_2) &= (\nabla_{\tilde{e}_1}^k \rho^k)(\tilde{e}_2, \tilde{e}_3) = (\nabla_{\tilde{e}_2}^k \rho^k)(\tilde{e}_2, \tilde{e}_2) = 0.
\end{align*}
$$

(126)

Then, if $\tilde{\rho}^k$ is a Codazzi tensor on $(G_6, \nabla^k)$, by (6) and (7), we have the following two equations:

$$
\begin{align*}
\alpha \beta \gamma &= 0, \\
\alpha^2 \gamma &= 0.
\end{align*}
$$

(127)

By solving (127), we get

Theorem 26. $\tilde{\rho}^k$ is a Codazzi tensor on $(G_6, \nabla^k)$ if and only if

$$
\begin{align*}
(1) & \quad \alpha = \beta = 0, \quad \delta \neq 0; \\
(2) & \quad \alpha \neq 0, \quad \beta \delta = \gamma = 0, \quad \alpha + \delta \neq 0.
\end{align*}
$$

4.7. Codazzi Tensors of $G_7$

Lemma 27 ([7]). The canonical connection $\nabla^c$ of $G_7$ is given by

$$
\begin{align*}
\nabla^c_{\tilde{e}_1} \tilde{e}_1 &= a \tilde{e}_2, \quad \nabla^c_{\tilde{e}_1} \tilde{e}_2 = -a \tilde{e}_1, \quad \nabla^c_{\tilde{e}_1} \tilde{e}_3 = 0, \\
\nabla^c_{\tilde{e}_2} \tilde{e}_1 &= \beta \tilde{e}_2, \quad \nabla^c_{\tilde{e}_2} \tilde{e}_2 = -\beta \tilde{e}_1, \quad \nabla^c_{\tilde{e}_2} \tilde{e}_3 = 0, \\
\nabla^c_{\tilde{e}_3} \tilde{e}_1 &= (\gamma - \beta) \tilde{e}_2, \quad \nabla^c_{\tilde{e}_3} \tilde{e}_2 = (\beta - \gamma) \tilde{e}_1, \quad \nabla^c_{\tilde{e}_3} \tilde{e}_3 = 0.
\end{align*}
$$

(128)

Then,

$$
\begin{align*}
\tilde{\rho}^c(\tilde{e}_1, \tilde{e}_1) &= -(a^2 + \frac{\beta^2 \gamma}{2}), \quad \tilde{\rho}^c(\tilde{e}_1, \tilde{e}_2) = 0, \quad \tilde{\rho}^c(\tilde{e}_1, \tilde{e}_3) = -\frac{1}{2}(a \gamma + \delta \gamma), \\
\tilde{\rho}^c(\tilde{e}_2, \tilde{e}_2) &= -(a^2 + \frac{\beta^2 \gamma}{2}), \quad \tilde{\rho}^c(\tilde{e}_2, \tilde{e}_3) = \frac{1}{2}(a^2 + \frac{\delta \gamma}{2}), \quad \tilde{\rho}^c(\tilde{e}_3, \tilde{e}_3) = 0.
\end{align*}
$$

(129)

By (5), we have
\((\nabla^c_{\tilde{e}_1} \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_1) = (\nabla^c_{\tilde{e}_2} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = (\nabla^c_{\tilde{e}_1} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_2) = 0,\)
\((\nabla^c_{\tilde{e}_2} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_2) = 0, \ (\nabla^c_{\tilde{e}_1} \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_3) = -\frac{\alpha}{2} (a\gamma + \frac{\beta\gamma}{2}), \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_3) = -\frac{\beta}{2} (a^2 + \frac{\beta\gamma}{2}),\)
\((\nabla^c_{\tilde{e}_1} \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_1) = -\frac{\alpha}{2} (a^2 + \frac{\beta\gamma}{2}), \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_1) = 0, \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_2) = -\frac{\alpha}{2} (a\gamma + \frac{\beta\gamma}{2}),\)
\((\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_2) = (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = 0, \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_1, \tilde{e}_3) = (\beta - \frac{\gamma}{2})(a^2 + \frac{\beta\gamma}{4}),\)
\((\nabla^c_{\tilde{e}_2} \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_1) = -\frac{\beta}{2} (a^2 + \frac{\beta\gamma}{2}), \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_1) = 0, \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_2) = -\frac{\beta}{2} (a\gamma + \frac{\beta\gamma}{2}),\)
\((\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_2) = -2a^2, \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_3, \tilde{e}_3) = 0, \ (\nabla^c_{\tilde{e}_3} \tilde{\rho}^c)(\tilde{e}_2, \tilde{e}_3) = \frac{1}{2} (\beta - \frac{\gamma}{2})(a\gamma + \frac{\beta\gamma}{2}). \) \(\tag{130}\)

Then, if \(\tilde{\rho}^c\) is a Codazzi tensor on \((G^*_7, \nabla^c)\), by (6) and (7), we have the following seven equations:
\[
\begin{align*}
\beta(2a^2 + \beta\gamma) - a(2a\gamma + \delta) &= 0 \\
\alpha(2a^2 + \beta\gamma) &= 0 \\
\alpha\gamma(2a + \delta) &= 0 \\
(\gamma - 2\beta)(2a^2 + \beta\gamma) &= 0 \\
\beta(2a^2 + \beta\gamma) &= 0 \\
\beta\gamma(2a + \delta) &= 0 \\
(\gamma - 2\beta)(2a\gamma + \delta\gamma) &= 0.
\end{align*}
\(\tag{131}\)

The above equations induce the following theorem:

**Theorem 27.** \(\tilde{\rho}^c\) is a Codazzi tensor on \((G^*_7, \nabla^c)\) if and only if \(a = \gamma = 0, \ \delta \neq 0.\)

**Lemma 28 ([7]).** The Kobayashi–Nomizu connection \(\nabla^k\) of \(G^*_7\) is given by
\[
\begin{align*}
\nabla^k_{\tilde{e}_1} \tilde{e}_1 &= a\tilde{e}_2, \quad \nabla^k_{\tilde{e}_1} \tilde{e}_2 = -a\tilde{e}_1, \quad \nabla^k_{\tilde{e}_1} \tilde{e}_3 = \beta\tilde{e}_3, \\
\nabla^k_{\tilde{e}_2} \tilde{e}_1 &= \beta\tilde{e}_2, \quad \nabla^k_{\tilde{e}_2} \tilde{e}_2 = -\beta\tilde{e}_1, \quad \nabla^k_{\tilde{e}_2} \tilde{e}_3 = \delta\tilde{e}_3, \\
\nabla^k_{\tilde{e}_3} \tilde{e}_1 &= -a\tilde{e}_1 - \beta\tilde{e}_2, \quad \nabla^k_{\tilde{e}_3} \tilde{e}_2 = -\gamma\tilde{e}_1 - \delta\tilde{e}_2, \quad \nabla^k_{\tilde{e}_3} \tilde{e}_3 = 0.
\end{align*}
\(\tag{132}\)

Then,
\[
\tilde{\rho}^k(\tilde{e}_1, \tilde{e}_1) = -a^2, \quad \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_2) = \frac{\beta}{2} (\delta - a), \quad \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_3) = \beta(\alpha + \delta),
\]
\[
\tilde{\rho}^k(\tilde{e}_2, \tilde{e}_2) = -(a^2 + \beta^2 + \beta\gamma), \quad \tilde{\rho}^k(\tilde{e}_2, \tilde{e}_3) = \frac{\alpha\delta + \beta\gamma + 2\alpha^2}{2}, \quad \tilde{\rho}^k(\tilde{e}_3, \tilde{e}_3) = 0.
\(\tag{133}\)

By (5), we have
If \( \tilde{\rho}^k \) is a Codazzi tensor on \((G_7, \nabla^k)\), by (6) and (7), we have the following nine equations:

\[
\begin{align*}
\beta(\alpha \gamma + \beta \delta) & = 0 \\
\beta(\alpha \delta - \alpha^2 - \beta^2 - \beta \gamma) & = 0 \\
\beta(\alpha + \delta)^2 & = 0 \\
4 \alpha^3 - 4 \beta^2 \delta - 2 \alpha \delta^2 - (\alpha \beta \gamma + \alpha^2 \delta) & = 0 \\
\beta(2 \alpha^2 + \beta^2 + \beta \gamma + \delta^2) + 2 \alpha^2 \gamma & = 0 \\
2 \alpha \beta (\alpha + \delta) + \beta(\alpha \delta + \beta \gamma + 2 \delta^2) & = 0 \\
2 \alpha^2 \gamma + 2 \beta^3 + 3 \alpha \beta \gamma - 3 \alpha \beta \delta - 5 \beta \delta^2 & = 0 \\
2(\alpha \beta \gamma - 3 \delta^2 + 2 \alpha^2 \delta) - \beta \delta \gamma - \alpha \delta^2 & = 0 \\
2(\alpha \beta \gamma + \delta^3) + 3 \beta \delta \gamma + \alpha \delta^2 & = 0.
\end{align*}
\] (135)

By solving (135), we get \( \alpha = \delta = 0 \), and there is a contradiction. So,

**Theorem 28.** \( \tilde{\rho}^k \) is not a Codazzi tensor on \((G_7, \nabla^k)\).

5. **Quasi-Statistical Structure Associated with Canonical Connections and Kobayashi–Nomizu Connections on Three-Dimensional Lorentzian Lie Groups**

The torsion tensor of \((G_i, g, \nabla^i)\) is defined by

\[ T^i(X,Y) = \nabla^i_X Y - \nabla^i_Y X - [X,Y] \] (136)

The torsion tensor of \((G_i, g, \nabla^k)\) is defined by

\[ T^k(X,Y) = \nabla^k_X Y - \nabla^k_Y X - [X,Y] \] (137)

Then, for \( G_1 \), we have

\[ T^c(\tilde{e}_1, \tilde{e}_2) = \beta \tilde{e}_3, \quad T^c(\tilde{e}_1, \tilde{e}_3) = \alpha \tilde{e}_1 + \frac{\beta}{2} \tilde{e}_2, \quad T^c(\tilde{e}_2, \tilde{e}_3) = -\frac{\beta}{2} \tilde{e}_1 - \alpha \tilde{e}_2 - \alpha \tilde{e}_3. \] (138)
\[ \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = \frac{a\beta^2}{4}, \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{a^2\beta}{2}, \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \]
\[ \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = -a(a^2 + \frac{\beta^2}{2}), \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_2) = -\frac{\beta}{2}(a^2 + \frac{\beta^2}{2}), \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_3) = \frac{a^2\beta}{2}, \]
\[ \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = \frac{\beta}{4}(a^2 + \beta^2), \quad \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = \frac{\alpha}{2}(a^2 + \beta^2), \quad \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = -a\left(\frac{\beta^2}{8} + \frac{a^2}{2}\right). \quad (139) \]

Then, if \((G_1, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure, by (57) and (58), we have the following three equations:

\[ \begin{cases} 
\alpha\beta = 0 \\
\alpha(a^2 + \beta^2) = 0 \\
\beta(a^2 + \beta^2) = 0.
\end{cases} \quad (140) \]

By solving (140), we get \(\alpha = 0\), and there is a contradiction. So,

**Theorem 29.** \((G_1, \nabla^c, \tilde{\rho}^c)\) is not a quasi-statistical structure.

Similarly,

\[ T^k(\tilde{e}_1, \tilde{e}_2) = \beta\tilde{e}_3, \quad T^k(\tilde{e}_1, \tilde{e}_3) = T^k(\tilde{e}_2, \tilde{e}_3) = 0, \quad (141) \]
\[ \tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = 0, \quad \tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{a^2\beta}{2}, \quad \tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \quad (142) \]
\[ \tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = 0, \quad \tilde{\rho}^k(T^k(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = 0, \]

where \(1 \leq j \leq 3\).

Then, if \((G_1, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure, by (57) and (58), we have the following three equations:

\[ \begin{cases} 
\alpha\beta = 0 \\
\alpha = 0 \\
\alpha(a^2 - \beta^2) = 0.
\end{cases} \quad (143) \]

By solving (143), we get \(\alpha = 0\), and this condition does not hold. Therefore,

**Theorem 30.** \((G_1, \nabla^k, \tilde{\rho}^k)\) is not a quasi-statistical structure.

For \(G_2\), we have

\[ T^c(\tilde{e}_1, \tilde{e}_2) = \beta\tilde{e}_3, \quad T^c(\tilde{e}_1, \tilde{e}_3) = (\beta - \frac{\alpha}{2})\tilde{e}_2 + \gamma\tilde{e}_3, \quad T^c(\tilde{e}_2, \tilde{e}_3) = -\frac{\alpha}{2}\tilde{e}_1. \quad (144) \]
\[ \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = 0, \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \beta\gamma(\frac{\beta}{2} - \frac{\gamma}{4}), \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \]
\[ \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = 0, \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_2) = \frac{\alpha}{4} - \frac{\beta}{2}(a\beta + \gamma^2), \]
\[ \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_3) = (\beta - \frac{\alpha}{2})(\frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4}), \quad \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = \frac{\alpha}{2}(\gamma^2 + \frac{\alpha\beta}{2}), \]
\[ \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_3) = \frac{\alpha}{2}(a^2 + \beta^2), \quad \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_3) = 0. \quad (145) \]
Then, if \((G_2, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure, by (57) and (58), we have the following four equations:

\[
\begin{cases}
\beta \gamma (2\beta - \alpha) = 0 \\
\gamma (2\beta - \alpha) = 0 \\
(2\beta - \alpha)(\gamma^2 + \alpha \beta) = 0 \\
\alpha (\gamma^2 + \alpha \beta) + 2\beta \gamma^2 = 0.
\end{cases}
\] (146)

By solving (146), we obtain

**Theorem 31.** \((G_2, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure if and only if \(\gamma \neq 0, \quad \alpha = \beta = 0.\)

Similarly,

\[
T^k(\bar{\epsilon}_1, \bar{\epsilon}_2) = \beta \bar{\epsilon}_3, \quad T^k(\bar{\epsilon}_1, \bar{\epsilon}_3) = T^k(\bar{\epsilon}_2, \bar{\epsilon}_3) = 0,
\] (147)

\[
\tilde{\rho}^k(T^k(\bar{\epsilon}_1, \bar{\epsilon}_2), \bar{\epsilon}_1) = 0, \quad \tilde{\rho}^k(T^k(\bar{\epsilon}_1, \bar{\epsilon}_2), \bar{\epsilon}_2) = \frac{\alpha \beta \gamma}{2}, \quad \tilde{\rho}^k(T^k(\bar{\epsilon}_1, \bar{\epsilon}_2), \bar{\epsilon}_3) = 0,
\] (148)

\[
\tilde{\rho}^k(T^k(\bar{\epsilon}_1, \bar{\epsilon}_3), \bar{\epsilon}_1) = 0, \quad \tilde{\rho}^k(T^k(\bar{\epsilon}_2, \bar{\epsilon}_3), \bar{\epsilon}_1) = 0,
\]

where \(1 \leq j \leq 3.\)

Then, if \((G_2, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure, by (57) and (58), we have the following three equations:

\[
\begin{cases}
\beta \gamma (2\beta - \alpha) = 0 \\
\gamma (2\beta - \alpha) = 0 \\
\alpha \beta \gamma = 0.
\end{cases}
\] (149)

By solving (149), we get

**Theorem 32.** \((G_2, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure if and only if \(\gamma \neq 0, \quad \alpha = \beta = 0.\)

For \(G_3,\) we have

\[
T^c(\bar{\epsilon}_1, \bar{\epsilon}_2) = \gamma \bar{\epsilon}_3, \quad T^c(\bar{\epsilon}_1, \bar{\epsilon}_3) = (\beta - m_3)\bar{\epsilon}_2, \quad T^c(\bar{\epsilon}_2, \bar{\epsilon}_3) = (m_3 - \alpha)\bar{\epsilon}_1,
\] (150)

\[
\tilde{\rho}^c(T^c(\bar{\epsilon}_1, \bar{\epsilon}_2), \bar{\epsilon}_1) = 0, \quad \tilde{\rho}^c(T^c(\bar{\epsilon}_1, \bar{\epsilon}_2), \bar{\epsilon}_2) = \tilde{\rho}^c(T^c(\bar{\epsilon}_1, \bar{\epsilon}_2), \bar{\epsilon}_3) = 0, \quad \tilde{\rho}^c(T^c(\bar{\epsilon}_1, \bar{\epsilon}_3), \bar{\epsilon}_1) = 0,
\] (151)

\[
\tilde{\rho}^c(T^c(\bar{\epsilon}_2, \bar{\epsilon}_3), \bar{\epsilon}_1) = \tilde{\rho}^c(T^c(\bar{\epsilon}_2, \bar{\epsilon}_3), \bar{\epsilon}_2) = \gamma m_3(\beta - m_3), \quad \tilde{\rho}^c(T^c(\bar{\epsilon}_2, \bar{\epsilon}_3), \bar{\epsilon}_1) = \gamma m_3(\alpha - m_3),
\]

Then, if \((G_3, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure, by (57) and (58), we have the following two equations:

\[
\begin{cases}
m_3 \gamma (m_3 - \beta) = 0 \\
m_3 \gamma (\alpha - m_3) = 0.
\end{cases}
\] (152)

By solving (152), we obtain

**Theorem 33.** \((G_3, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure if and only if

\[
(1) \quad \gamma = 0, \\
(2) \quad \gamma \neq 0, \quad \alpha + \beta - \gamma = 0.
\]
Similarly,
\[ T^k(\tilde{e}_1, \tilde{e}_2) = \gamma \tilde{e}_3, \quad T^k(\tilde{e}_1, \tilde{e}_3) = T^k(\tilde{e}_2, \tilde{e}_3) = 0, \]  
(153)

\[ \hat{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = \hat{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = \hat{\rho}^k(T^k(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = 0, \]  
(154)

where \( 1 \leq j \leq 3 \).

Obviously, the following theorem holds:

**Theorem 34.** \((G_4, \nabla^k, \hat{\rho}^k)\) is a quasi-statistical structure.

For \( G_4 \), we have
\[ T^c(\tilde{e}_1, \tilde{e}_2) = (\beta - 2\eta)\tilde{e}_3, \quad T^c(\tilde{e}_1, \tilde{e}_3) = (\beta - n_3)\tilde{e}_2 - \tilde{e}_3, \quad T^c(\tilde{e}_2, \tilde{e}_3) = (n_3 - \alpha)\tilde{e}_1, \]  
(155)

\[ \hat{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = 0, \quad \hat{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_2) = \frac{(\beta - 2\eta)(n_3 - \beta)}{2}, \quad \hat{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = 0, \]  
\[ \hat{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = 0, \quad \hat{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_2) = (\beta - n_3)(2\eta - \beta)n_3 - \frac{1}{2}, \]  
\( \hat{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = -\frac{(n_3 - \beta)^2}{2}, \quad \hat{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = (n_3 - \alpha)(2\eta - \beta)n_3 - 1], \]  
\[ \hat{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_2) = \hat{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_3) = 0. \]  
(156)

Then, if \((G_4, \nabla^c, \hat{\rho}^c)\) is a quasi-statistical structure, by (57) and (58), we have the following five equations:
\[
\begin{cases}
(\beta - 2\eta)(n_3 - \beta) = 0 \\
n_3 - \beta = 0 \\
(\beta - n_3)(2\eta - \beta)n_3 - 1] = 0 \\
(2n_3 - \beta)(n_3 - \beta) = 0 \\
\beta - n_3 + 2(n_3 - \alpha)(2\eta - \beta)n_3 - 1] = 0.
\end{cases}
\]  
(157)

By solving (157), we get

**Theorem 35.** \((G_4, \nabla^c, \hat{\rho}^c)\) is a quasi-statistical structure if and only if
\[
(1) \alpha = \beta = 2\eta, \\
(2) \alpha = 0, \quad \beta = \eta.
\]

Similarly,
\[ T^k(\tilde{e}_1, \tilde{e}_2) = (\beta - 2\eta)\tilde{e}_3, \quad T^k(\tilde{e}_1, \tilde{e}_3) = T^k(\tilde{e}_2, \tilde{e}_3) = 0, \]  
(158)

\[ \hat{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = 0, \quad \hat{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{\alpha(\beta - 2\eta)}{2}, \quad \hat{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \]  
\[ \hat{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = 0, \quad \hat{\rho}^k(T^k(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = 0, \]  
(159)

where \( 1 \leq j \leq 3 \).
Then, if \((G_4, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure, by (57) and (38), we have the following three equations:

\[
\begin{align*}
(2\eta - \beta)(\alpha - 2\beta) &= 0 \\
\alpha - 2\beta &= 0 \\
\alpha\beta &= 0.
\end{align*}
\] (160)

By solving (160), we get

**Theorem 36.** \((G_4, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure if and only if \(\alpha = \beta = 0\).

For \(G_5\), we have

\[
\begin{align*}
\nabla^c(\tilde{e}_1, \tilde{e}_2) &= -\alpha\tilde{e}_1 - \frac{\beta + \gamma}{2}\tilde{e}_2, \quad \nabla^c(\tilde{e}_2, \tilde{e}_3) = -\frac{\beta + \gamma}{2}\tilde{e}_1 - \delta\tilde{e}_2, \\
\tilde{\rho}^c(\nabla^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) &= \tilde{\rho}^c(\nabla^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^c(\nabla^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\end{align*}
\] (161)

where \(1 \leq j \leq 3\).

So,

**Theorem 37.** \((G_5, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure.

Similarly,

\[
\begin{align*}
\nabla^k(\tilde{e}_1, \tilde{e}_2) &= T^k(\tilde{e}_1, \tilde{e}_3) = T^k(\tilde{e}_2, \tilde{e}_3) = 0, \\
\tilde{\rho}^k(\nabla^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) &= \tilde{\rho}^k(\nabla^k(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^k(\nabla^k(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\end{align*}
\] (164)

where \(1 \leq j \leq 3\).

Obviously,

**Theorem 38.** \((G_5, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure.

For \(G_6\), we have

\[
\begin{align*}
\nabla^c(\tilde{e}_1, \tilde{e}_2) &= -\beta\tilde{e}_3, \quad \nabla^c(\tilde{e}_1, \tilde{e}_3) = -\beta + \gamma \tilde{e}_2 - \delta\tilde{e}_3, \quad \nabla^c(\tilde{e}_2, \tilde{e}_3) = \frac{\beta - \gamma}{2}\tilde{e}_1, \\
\tilde{\rho}^c(\nabla^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) &= 0, \quad \tilde{\rho}^c(\nabla^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \beta + \gamma \frac{\beta - \gamma}{2}\tilde{e}_1, \quad \tilde{\rho}^c(\nabla^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\end{align*}
\] (165)

Then, if \((G_6, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure, by (57) and (58), we have the following five equations:

\[
\begin{align*}
\beta[2\alpha\gamma - \delta(\beta - \gamma)] &= 0 \\
\alpha[2\alpha\gamma - \delta(\beta - \gamma)] &= 0 \\
(\beta + \gamma)[2\alpha^2 - \beta(\beta - \gamma)] + \delta[2\alpha\gamma - \delta(\beta - \gamma)] &= 0 \\
\gamma[2\alpha\gamma - \delta(\beta - \gamma)] &= 0 \\
\alpha[2\alpha\gamma - \delta(\beta - \gamma)] + (\beta - \gamma)[2\alpha^2 - \beta(\beta - \gamma)] &= 0.
\end{align*}
\] (167)
By solving (167), we obtain

**Theorem 39.** \((G_6, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure if and only if

\[
\begin{align*}
(1) & \alpha = \beta = \gamma = 0, \quad \delta \neq 0, \\
(2) & \alpha \neq 0, \quad \delta = \gamma = 0, \quad 2\alpha^2 = \beta^2, \\
(3) & \alpha \neq 0, \quad \beta = \gamma = 0.
\end{align*}
\]

Similarly,

\[
\begin{align*}
T^k(\tilde{e}_1, \tilde{e}_2) &= -\beta \tilde{e}_3, \quad T^k(\tilde{e}_1, \tilde{e}_3) = T^k(\tilde{e}_2, \tilde{e}_3) = 0, \quad (168) \\
\tilde{\rho}^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_j) &= \tilde{\rho}^k(\tilde{e}_1, \tilde{e}_3), \tilde{e}_j) = \tilde{\rho}^k(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0, \quad (169)
\end{align*}
\]

where \(1 \leq j \leq 3\).

Then, if \((G_6, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure, by (57) and (58), we have the following two equations:

\[
\begin{align*}
\begin{cases}
\alpha \beta \gamma &= 0 \\
\alpha^2 \gamma &= 0.
\end{cases} \quad (170)
\end{align*}
\]

By solving (170), we obtain

**Theorem 40.** \((G_6, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure if and only if

\[
\begin{align*}
(1) & \alpha = \beta = 0, \quad \delta \neq 0, \\
(2) & \alpha \neq 0, \quad \beta \delta = \gamma = 0.
\end{align*}
\]

For \(G_7\), we have

\[
\begin{align*}
T^c(\tilde{e}_1, \tilde{e}_2) &= \beta \tilde{e}_3, \quad T^c(\tilde{e}_1, \tilde{e}_3) = -a\tilde{e}_1 - \frac{\gamma}{2} \tilde{e}_2 - \beta \tilde{e}_3, \quad T^c(\tilde{e}_2, \tilde{e}_3) = -(\beta + \frac{\gamma}{2}) \tilde{e}_1 - \delta \tilde{e}_2 - \delta \tilde{e}_3, \quad (171) \\
\tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) &= -\frac{\beta}{2}(a\gamma + \delta \gamma), \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{\beta}{2}(a^2 + \frac{\beta \gamma}{2}), \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \quad (172) \\
\tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) &= a^3 + a\beta \gamma + \frac{\beta \delta \gamma}{4}, \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_2) = (\frac{a^2}{2} + \frac{\beta \gamma}{4})(\gamma - \beta), \\
\tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) &= a^3 + a\beta \gamma + \frac{\beta \delta \gamma}{4}, \quad \tilde{\rho}^c(T^c(\tilde{e}_1, \tilde{e}_3), \tilde{e}_2) = (\frac{a^2}{2} + \frac{\beta \gamma}{4})(\gamma - \beta), \\
\tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) &= \frac{\gamma(a + \delta)}{4} - \frac{\beta \gamma^2}{8}, \quad \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) = (\beta + \frac{\gamma}{2})(a^2 + \frac{\beta \gamma}{2}) + \frac{\delta \gamma}{2}(a + \delta), \\
\tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_1) &= \frac{\delta}{2}(a^2 + \frac{\beta \gamma}{2}), \quad \tilde{\rho}^c(T^c(\tilde{e}_2, \tilde{e}_3), \tilde{e}_3) = \frac{\alpha \beta \gamma - a^2 \delta}{2} + \frac{\alpha \gamma^2}{4} + \frac{\delta \gamma^2}{8}.
\end{align*}
\]

Then, if \((G_7, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure, by (57) and (58), we have the following nine equations:
\[
\begin{aligned}
\beta \gamma (2a + \delta) &= 0 \\
\beta (2a^2 + \beta \gamma) &= 0 \\
\alpha \gamma (2a + \delta) &= 0 \\
2a^3 + 3a \beta \gamma + \beta \delta \gamma &= 0 \\
(\gamma - \beta)(4a^2 + \beta \gamma) - a \gamma (2a + \delta) &= 0 \\
(\gamma - 2\beta)(2a^2 + \beta \gamma) + 2(\alpha \delta \gamma + \alpha^2 \gamma) - \beta \gamma^2 &= 0 \\
\delta \gamma (2a + \delta) + (\gamma - \beta)(2a^2 + \beta \gamma) &= 0 \\
\delta (2a^2 + \beta \gamma) - \beta \gamma (2a + \delta) &= 0 \\
4(\alpha \beta \gamma - a^2 \delta) + 2a \gamma^2 + \delta \gamma^2 - (2\beta - \gamma) (2a \gamma + \delta \gamma) &= 0.
\end{aligned}
\]

By solving (173), we obtain

**Theorem 41.** \((G_7, \nabla^c, \tilde{\rho}^c)\) is a quasi-statistical structure if and only if \(\alpha = \gamma = 0\).

Similarly,
\[
T^k(\tilde{e}_1, \tilde{e}_2) = \beta \tilde{e}_3, \quad T^k(\tilde{e}_1, \tilde{e}_3) = T^k(\tilde{e}_2, \tilde{e}_3) = 0, \tag{174}
\]
\[
\tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_1) = \beta^2 (\alpha + \delta), \quad \tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_2) = \frac{\beta}{2} (\alpha \delta + \beta \gamma + 2\delta^2), \quad \tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_2), \tilde{e}_3) = 0, \tag{175}
\]
\[
\tilde{\rho}^k(T^k(\tilde{e}_1, \tilde{e}_3), \tilde{e}_1) = 0, \quad \tilde{\rho}^k(T^k(\tilde{e}_2, \tilde{e}_3), \tilde{e}_j) = 0,
\]

where \(1 \leq j \leq 3\).

Then, if \((G_7, \nabla^k, \tilde{\rho}^k)\) is a quasi-statistical structure, by (57) and (58), we have the following nine equations:
\[
\begin{aligned}
\beta (\alpha \gamma + \beta \delta + 2\beta \delta) &= 0 \\
\beta (2a^2 - 2a^2 - 2\beta^2 + 3a \delta - \beta \gamma) &= 0 \\
\beta (\alpha + \delta) &= 0 \\
4a^3 - 2a^2 \delta^2 - 4\beta \delta - (a \beta \gamma + a^2 \delta) &= 0 \\
\beta (2a^2 + 2\beta^2 + \beta \gamma + a^2 \delta) + 2a^2 \gamma &= 0 \\
2a^3 (\alpha + \delta) + \beta (\beta \gamma + a \delta + 2\delta^2) &= 0 \\
2(\alpha^2 \gamma + \beta^3) + 3a^2 \beta + \beta^2 \gamma - 3a \beta \delta - 5\beta \delta^2 &= 0 \\
2(\alpha \beta \gamma - \delta^3 + 3\beta^2 \delta + 2a^2 \delta) + \beta \delta \gamma - a \delta^2 &= 0 \\
2(\alpha \beta \gamma + \delta^3) + 3\delta \delta \gamma + a \delta^2 &= 0.
\end{aligned}
\]

By solving (176), we obtain \(\alpha = \delta = 0\), and there is a contradiction. So,

**Theorem 42.** \((G_7, \nabla^k, \tilde{\rho}^k)\) is not a quasi-statistical structure.

6. Conclusions

It is clearly shown in the above two tables that there are Codazzi tensors associated with affine connections on three-dimensional Lorentzian Lie groups and there is a quasi-statistical structure associated with affine connections on three-dimensional Lorentzian Lie groups.

Table 1 shows the conditions that Ricci tensors associated with Bott connections, canonical connections and Kobayashi–Nomizu connections are Codazzi tensors associated with Bott connections, canonical connections and Kobayashi–Nomizu connections on \(\{G_i\}_{i=1,...,7}\). For Bott connections, there is a contradiction on \(G_1, G_2, G_7\), there is a permanent establishment on \(G_3, G_5\) and there are corresponding conditions on \(G_4, G_6\). For
canonical connections, there is a contradiction on \( G_1 \), there is a permanent establishment on \( G_3, G_5 \) and there are corresponding conditions on \( G_2, G_4, G_6, G_7 \). For Kobayashi–Nomizu connections, there is a contradiction on \( G_1, G_7 \), there is a permanent establishment on \( G_3, G_5 \) and there are corresponding conditions on \( G_2, G_4, G_6 \).

As is shown in Table 2, we can obtain \( \{ G_i \}_{i=1,...,7} \) with the quasi-statistical structure associated with Bott connections, canonical connections and Kobayashi–Nomizu connections. For Bott connections, there is a contradiction on \( G_1, G_7 \), there is a permanent establishment on \( G_3 \) and there are corresponding conditions on \( G_2, G_3, G_4, G_6 \). For canonical connections, there is a contradiction on \( G_1 \), there is a permanent establishment on \( G_3, G_5 \) and there are corresponding conditions on \( G_2, G_4, G_6, G_7 \). For Kobayashi–Nomizu connections, there is a contradiction on \( G_1, G_7 \), there is a permanent establishment on \( G_3, G_5 \) and there are corresponding conditions on \( G_2, G_4, G_6 \).

Table 1. Codazzi tensors associated with affine connections on three-dimensional Lorentzian Lie groups.

<table>
<thead>
<tr>
<th>Codazzi Tensors</th>
<th>Bott Connections</th>
<th>Canonical Connections</th>
<th>Kobayashi–Nomizu Connections</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>No solution</td>
<td>No solution</td>
<td>No solution</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>No solution</td>
<td>( \gamma \neq 0, \alpha = 2\beta )</td>
<td>( \gamma \neq 0, \alpha = \beta = 0 )</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>( \alpha = \beta = 0 )</td>
<td>( \frac{\alpha}{2} + \eta - \beta = 0 )</td>
<td>( \alpha = \beta = 0 )</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>(1) ( \alpha = \beta = 0, \delta \neq 0 ) ( \Rightarrow ) (2) ( \alpha \neq 0, \beta \delta = \gamma = 0 )</td>
<td>(1) ( \alpha = \beta = \gamma = 0, \delta \neq 0 ) ( \Rightarrow ) (2) ( \alpha \neq 0, \beta = \gamma = 0, \alpha + \delta \neq 0 ) ( \Rightarrow ) (3) ( \alpha \neq 0, \delta = \gamma = 0 )</td>
<td>(1) ( \alpha = \beta = 0, \delta \neq 0 ) ( \Rightarrow ) (2) ( \alpha \neq 0, \beta \delta = \gamma = 0, \alpha + \delta \neq 0 )</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>No solution</td>
<td>( \alpha = \gamma = 0, \delta \neq 0 )</td>
<td>No solution</td>
</tr>
</tbody>
</table>

Table 2. The quasi-statistical structure associated with affine connections on three-dimensional Lorentzian Lie groups.

<table>
<thead>
<tr>
<th>Quasi-Statistical Structure</th>
<th>Bott Connections</th>
<th>Canonical Connections</th>
<th>Kobayashi–Nomizu Connections</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>( \alpha = \beta = 0, \gamma \neq 0 )</td>
<td>( \gamma \neq 0, \alpha = \beta = 0 )</td>
<td>No solution</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( \alpha = \beta = 0, \gamma \neq 0 )</td>
<td>( \gamma \neq 0, \alpha = \beta = 0 )</td>
<td>( \gamma \neq 0, \alpha = \beta = 0 )</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>(1) ( \gamma = 0 ) ( \Rightarrow ) (2) ( \gamma \neq 0, \alpha + \beta - \gamma = 0 )</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>( \alpha = \beta = 0 )</td>
<td>(1) ( \alpha = \beta = 2\eta ) ( \Rightarrow ) (2) ( \alpha = 0, \beta = \eta )</td>
<td>( \alpha = \beta = 0 )</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
<td>Permanent establishment</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>(1) ( \alpha = \beta = 0, \delta \neq 0 ) ( \Rightarrow ) (2) ( \alpha \neq 0, \beta \delta = \gamma = 0 )</td>
<td>(1) ( \alpha = \beta = 0, \delta \neq 0 ) ( \Rightarrow ) (2) ( \alpha \neq 0, \beta \delta = \gamma = 0, \alpha + \delta \neq 0 ) ( \Rightarrow ) (3) ( \alpha \neq 0, \beta = \gamma = 0 )</td>
<td>(1) ( \alpha = \beta = 0, \delta \neq 0 ) ( \Rightarrow ) (2) ( \alpha \neq 0, \beta \delta = \gamma = 0, \alpha + \delta \neq 0 )</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>No solution</td>
<td>( \alpha = \gamma = 0, \delta \neq 0 )</td>
<td>No solution</td>
</tr>
</tbody>
</table>

The two tables have something in common in that there are three kinds of situations for three different affine connections on seven connected or simply connected three-dimensional Lorentzian Lie groups. The main difference is Bott connections. The results of no solution and constancy in Table 1 are more than those in Table 2. Therefore, we know that for three kinds of connections, torsion tensor has a stronger effect on Bott connections than the other two.
Author Contributions: Conceptualization, T.W. and Y.W.; methodology, Y.W.; software, T.W.; validation, T.W. and Y.W.; formal analysis, T.W.; investigation, T.W.; resources, Y.W.; data curation, T.W.; writing—original draft preparation, T.W.; writing—review and editing, T.W.; visualization, Y.W.; supervision, Y.W.; project administration, Y.W.; funding acquisition, Y.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China: No.11771070.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The second author was supported in part by NSFC No. 11771070. The authors are deeply grateful to the referees for their valuable comments and helpful suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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