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Nonlinear Gravitational Waves as Dark Energy in Warped Spacetimes

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Abstract: We find an azimuthal-angle dependent approximate wave like solution to second order on a warped five-dimensional manifold with a self-gravitating U(1) scalar gauge field (cosmic string) on the brane using the multiple-scale method. The spectrum of the several orders of approximation show maxima of the energy distribution dependent on the azimuthal-angle and the winding numbers of the subsequent orders of the scalar field. This breakup of the quantized flux quanta does not lead to instability of the asymptotic wavelike solution due to the suppression of the n-dependency in the energy momentum tensor components by the warp factor. This effect is triggered by the contribution of the five dimensional Weyl tensor on the brane. This contribution can be understood as dark energy and can trigger the self-acceleration of the universe without the need of a cosmological constant. There is a striking relation between the symmetry breaking of the Higgs field described by the winding number and the SO(2) breaking of the axially symmetric configuration into a discrete subgroup of rotations of about 180°. The discrete sequence of non-axially symmetric deviations, cancelled by the emission of gravitational waves in order to restore the SO(2) symmetry, triggers the pressure T_{zz} for discrete values of the azimuthal-angle. There could be a possible relation between the recently discovered angle-preferences of polarization axes of quasars on large scales and our theoretical predicted angle-dependency and this could be evidence for the existence of cosmic strings. Careful comparison of this spectrum of extremal values of the first and second order φ-dependency and the distribution of the alignment of the quasar polarizations is necessary. This can be accomplished when more observational data become available. It turns out that, for late time, the vacuum 5D spacetime is conformally invariant if the warp factor fulfils the equation of a vibrating "drum", describing standing normal modes of the brane.

Keywords: cosmic strings; warped brane world models; U(1) scalar-gauge field; multiple-scale analysis; nonlinear gravitational waves; quasar polarization

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1. Introduction

It is a great challenge for theoretical physicists and cosmologists to find an explanation for the dark energy needed for the observed acceleration of our universe without the need of a cosmological constant. This acceleration could be the result of the gravitational leakage into extra dimensions. Higher dimensional models originate from super string theory and predict the existence of sub-manifolds of the bulk spacetime, i.e., the branes. We would live on the brane where all the standard model fields reside, while only gravity can propagate into the bulk. The extra dimension might be very large compared to the ones predicted in string theory [1], i.e., of order of millimeters. Then the huge discrepancy between the electro-weak scale and the gravitational mass scale (hierarchy problem) will be suppressed by the curvature close to the brane. The infinite extra dimension makes a finite contribution to the 5D volume due to the warp factor. One obtains so-called effective 4D Kaluza-Klein (KK) modes of the perturbative 5D graviton on the brane. These KK-modes will be massive from the brane viewpoint [2]. One could even ensure a zero effective cosmological constant by fine-tuning the negative bulk cosmological constant (RS-fine-tuning) with respect to the brane cosmological constant. This would solve the problem of an incredibly fine-tuning of the effective cosmological constant we encounter in 4D general relativity. The recently found evidence of the acceleration of our universe could then be explained by self-acceleration without the need of a cosmological constant as dark energy source. The dark energy term in the effective Einstein equations on the brane will be provided by the projected 5D Weyl tensor [3].

There is, however, another possibility to test the existence of large extra dimensions. This can be done by relating the recently discovered alignment of quasar polarizations on very large scales [4] to warped cosmic strings [5–7]. Cosmic strings are U(1) scalar gauge vortex solutions in general relativity in the framework of GUT's. [8]. This U(1) scalar gauge field with a "Mexican hat" potential has lived up its reputation in the theory of superconductivity, where vortex lines occur as topological defects and in the standard model of particle physics. In cosmology it could trigger the inflationary period of expansion and could solve the horizon and flatness problem.

Topological defects, such as cosmic strings, monopoles and textures, can have cosmological implications. The U(1) vortex solution possesses mass, so it will couple to gravity. It came as a big surprise that there exists vortex-like solutions in general relativity [9,10]. It is conjectured that for any field theory which admits cosmic string solutions, a network of strings inevitably forms at some point during the early universe. They were thought to provide a possible origin for the density inhomogeneities from which galaxies develop.

The mass per unit length of a cosmic string can be of the order of 10^{18} kg, which is proportional to the square of the energy breaking scale η. The thickness is of order η^{-1} and the length is unbounded long. Cosmic strings can collide with each other and will intercommute to form loops. These loops will oscillate and loose energy via gravitational radiation and decay. There are already tight constraints on the gravitational wave signatures due to string loops via observations of the millisecond pulsar-timing data, the cosmic microwave background radiation (CMB) and analysis of data of the LIGO-Virgo gravitational-wave detector [11,12]. Evidence of these objects would give us information at very high energies in the early stages of the universe. However, the interest in cosmic strings faded away, mainly because of inconsistencies with the power spectrum of the CMB. Moreover, they will produce a very special pattern of lensing effect, that has not yet been found by observations.

A revival in interest in cosmic strings was seen when it was realized that in the framework of super string theory inspired cosmological models these objects will inevitably form [13]. In brane world models with large extra dimensions, long superstrings may be stable and appear at the same energy scale as the GUT cosmic string. Investigation of cosmic strings in warped brane world models shows consistency with observational bounds [14], while brane fluctuations can be triggered dynamically by the huge mass the cosmic string builds up in the bulk by the warp factor and can induce massive KK-modes felt on the brane [5]. These disturbances are no longer axially symmetric.

A powerful perturbative method to study nonlinear wavelike behavior of gravitational waves coupled to the scalar gauge field perturbations can be provided by the so-called multiple-scale method [15–17]. The method consists of assuming that changes in the dependent variables occur on two (or more) scales, for example the dimension of the background spacetime and the extra dimension. The advantage is that one can easily keep track of the different orders of approximations and can verify if one obtains a asymptotically bounded wavelike solution.

In Section 2 we outline the multiple-scale method on a warped brane world spacetime and calculate the metric perturbations to second order. In Section 3 we calculate the matter field equations to second order and indicate the possible relation with axially symmetric instabilities caused by radiation-reactions. In Section 4 we discuss the possible connection of the warp factor with conformal invariance. In the appendices we collected all the relevant equations in order to keep the main text readable.

2. The Multiple-Scale Approximation on a Warped Brane World Spacetime

We will investigate vortex-like solution on a warped five-dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) model in cylindrical coordinates [5–7]:

$$ds^2 = \mathcal{W}^2\Big[e^{2(\gamma-\psi)}(-dt^2+dr^2)+e^{2\psi}dz^2+r^2e^{-2\psi}d\varphi^2\Big]+dy^2, \tag{1}$$

with $\mathcal{W}$ a warpfactor dependent of r, t and the bulk dimension y. The self-gravitating scalar-gauge field, parameterized as

$$\Phi = \eta X(t,r)e^{in\varphi}, \qquad A_\mu = \frac{1}{\epsilon}\big[P(t,r)-n\big]\nabla_\mu\varphi, \tag{2}$$

resides on the brane. η is the vacuum expectation value of the scalar field, n the winding number and ϵ the coupling constant. As potential we take the well-known symmetry breaking potential $V(\Phi) = \frac{1}{8}\beta(\Phi^2-\eta^2)^2$. The winding number (number of jumps in phase of the scalar field when one goes around the flux tube) is related to the quantized flux in the Ginsberg Landau theory of superconductivity (Abrikosov vortices) and the discrete values of the topological charge in the sin-Gordon theory. The exact solution of $\mathcal{W}$ [5] follows from the 5D Einstein equation

$${}^5G_{\mu\nu} = -\Lambda_5{}^5g_{\mu\nu} + \kappa_5^2\delta(y)\Big(-\Lambda_4{}^{(4)}g_{\mu\nu}+{}^{(4)}T_{\mu\nu}\Big), \tag{3}$$

with $\kappa_5 = 8\pi{}^5G = 8\pi/{}^5M_{pl}^3$, Λ_4 the brane tension and $x^\mu = (t, x^i, y)$. The ${}^5M_{pl}$ is the fundamental 5D Planck mass. The scalar-gauge field equations become [9]

$$D^\mu D_\mu\Phi = 2\frac{dV}{d\Phi^*}, \qquad {}^4\nabla^\mu F_{\nu\mu} = \frac{1}{2}i\epsilon\Big(\Phi(D_\nu\Phi)^*-\Phi^*D_\nu\Phi\Big), \tag{4}$$

with $D_\mu\Phi \equiv {}^4\nabla_\mu\Phi + i\epsilon A_\mu\Phi$, ${}^4\nabla_\mu$ the covariant derivative with respect to ${}^4g_{\mu\nu}$, ϵ the gauge coupling constant and the star represents the complex conjugated. $F_{\mu\nu}$ is the Maxwell tensor. The modified Einstein equations become [1]

$${}^4G_{\mu\nu} = -\Lambda_{eff}{}^4g_{\mu\nu} + \kappa_4^2{}^4T_{\mu\nu} + \kappa_5^4\mathcal{S}_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{5}$$

with ${}^4G_{\mu\nu}$ the Einstein tensor calculated on the brane metric ${}^4g_{\mu\nu} = {}^5g_{\mu\nu} - n_\mu n_\nu$ and n_μ the unit vector normal to the brane. In Equation (5) the effective cosmological constant $\Lambda_{eff} = \frac{1}{2}(\Lambda_5 + \kappa_4^2\Lambda_4) = \frac{1}{2}(\Lambda_5 + \frac{1}{6}\kappa_5^4\Lambda_4^2)$ and Λ_4 is the vacuum energy in the brane (brane tension). We will take here $\Lambda_{eff} = 0$, so we are dealing with the RS-fine tuning condition [2]. The first correction term $\mathcal{S}_{\mu\nu}$ in Equation (5) is the quadratic term in the energy-momentum tensor arising from the extrinsic curvature terms in the projected Einstein tensor and the second correction term $\mathcal{E}_{\mu\nu}$ in Equation (5) is a part of the 5D Weyl tensor and carries information of the gravitational field outside the brane and is constrained by the motion of the matter on the brane, i.e., the Codazzi equation. Because gravity can propagate in the bulk, the cosmic string can build up a huge mass per unit length (or angle deficit) $G\mu >> 1$ by the warp factor and can induce massive KK-modes felt on the brane, while the manifestation in the brane will be warped down to GUT scale, consistent with observations. Disturbances in the spatial components of the stress-energy tensor cause cylindrical symmetric waves, amplified due to the presence of the bulk space and warp factor. They could survive the natural damping due to the expansion of the universe. It was found in context with the acceleration of our universe [3], that these disturbances could have a profound influence on the expansion of the universe. There could even be a self-acceleration without the need of an effective brane cosmological constant. Here we will consider the modified cosmic string features on the warped spacetime Equation (1).

Let us expand the metric field and the scalar-gauge fields in the multiple-scale scheme

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu}(\mathbf{x}) + \frac{1}{\omega}h_{\mu\nu}(\mathbf{x},\xi,\chi,..) + \frac{1}{\omega^2}k_{\mu\nu}(\mathbf{x},\xi,\chi,..) + ..., \\ A_\mu &= \bar{A}_\mu(\mathbf{x}) + \frac{1}{\omega}B_\mu(\mathbf{x},\xi,\chi,..) + \frac{1}{\omega^2}C_\mu(\mathbf{x},\xi,\chi,..) + ..., \\ \Phi &= \bar{\Phi}(\mathbf{x}) + \frac{1}{\omega}\Psi(\mathbf{x},\xi,\chi,..) + \frac{1}{\omega^2}\Xi(\mathbf{x},\xi,\chi,..) + ..., \end{aligned} \tag{6}$$

with $\bar{g}_{\mu\nu}$ the background metric and $\bar{\Phi}$, $\bar{A}_\mu$ the background scalar and gauge fields. Here ω represents a dimensionless parameter, which will be large (the "frequency", $\omega >> 1$). So $\frac{1}{\omega}$ is a small expansion parameter. Further, $\xi = \omega\Theta(\mathbf{x})$ with Θ a scalar (phase) functions on the manifold. The small parameter $\frac{1}{\omega}$ can be the ratio of the characteristic wavelength of the perturbation to the characteristic dimension of the background, or the smallness of a periodic deviation of an initially stationary rotational invariant system. On warped spacetimes it could also be the ratio of the extra dimension y to the background dimension. It turns out that this method comprises a powerful approximation scheme for handling highly nonlinear PDEs, in order to construct uniformly valid asymptotically bounded wavelike solutions. The method is very useful when one encounters non-uniformity in a regular perturbation expansion, i.e., secular terms. GRT is an example of a field theory where a linear approximation of wavelike solutions is not adequate in the case of high energy or strong curvature [15–17]. High-frequency gravitational waves interact with the background metric. This can be the case when the curvature becomes high due to the presence of a compact object. For a recent overview, see the textbook of Choquet-Bruhat [18].

For the scalar field we take different winding numbers, i.e., different magnetic flux quantization for the background field and higher order perturbations. We define $\bar{\Phi} = \eta\bar{X}(t,r)e^{in_1\varphi}, \Psi = Y(t,r,\xi)e^{in_2\varphi}$ and $\Xi = Z(t,r,\xi)e^{in_3\varphi}$. So we break up the original vortex with winding number n in our case in three strings with winding numbers n_1, n_2 and n_3. One can prove [19] that this breakup remains stable if the gauge to scalar mass is >1. In our case stability will be guaranteed by inverse powers of the warp factor.

Now we substitute the expansions into the effective Einstein equation and matter field equations. One can subsequently put equal zero the various powers of ω. One then obtains a system of partial differential equations for the fields $\bar{g}_{\mu\nu}, h_{\mu\nu}, k_{\mu\nu}$ and the scalar gauge fields $\bar{\Phi}, \Psi, \Xi, \bar{A}_\mu, B_\mu$ and C_μ. The perturbations can be φ-dependent. For the Einstein equation we obtain

$$\underline{\omega^{(-1)}}: \quad {}^4G^{(-1)}_{\mu\nu} = -\mathcal{E}^{(-1)}_{\mu\nu}, \tag{7}$$

$$\underline{\omega^{(0)}}: \quad {}^4\bar{G}_{\mu\nu} + {}^4G^{(0)}_{\mu\nu} = \kappa_4^2({}^4\bar{T}_{\mu\nu} + {}^4T^{(0)}_{\mu\nu}) + \kappa_5^4(\bar{\mathcal{S}}_{\mu\nu} + \mathcal{S}^{(0)}_{\mu\nu}) - \bar{\mathcal{E}}_{\mu\nu} - \mathcal{E}^{(0)}_{\mu\nu}, \tag{8}$$

$$\underline{\omega^{(1)}}: \quad {}^4G^{(1)}_{\mu\nu} = \kappa_4^2{}^4T^{(1)}_{\mu\nu} + \kappa_5^4\mathcal{S}^{(1)}_{\mu\nu} - \mathcal{E}^{(1)}_{\mu\nu}. \tag{9}$$

We will consider here the equations up to order $\omega^{(1)}$. We used the notation for the several terms in the expansion of a tensor, vector or scalar: $V_i = \omega V^{(-1)} + \bar{V}_i + V_i^{(0)} + \frac{1}{\omega}V_i^{(1)} + ...$ The contribution from the bulk space, $\mathcal{E}_{\mu\nu}$, must be calculated with the 5D Riemann tensor. From the $\omega^{(-1)}$ equations, we obtain constraint (or "gauge") conditions on $h_{\mu\nu}$ and B_μ, C_μ, i.e., $l^\alpha\left(\ddot{h}_{\alpha\nu} - \frac{1}{2}\bar{g}_{\alpha\nu}\ddot{h}\right) = 0$, $B_\mu = [B_0, B_0, 0, B, 0]$ and $l^\mu l_\mu = 0$. We consider here rapid variation in the direction of $l_\mu \equiv \frac{\partial\Theta}{\partial x^\mu}$ transversal to the sub-manifold Θ = constant. We have the differential rule

$$\frac{dg_{\mu\nu}}{dx^\sigma} = g_{\mu\nu,\sigma} + \omega l_\sigma \dot{g}_{\mu\nu} \qquad g_{\mu\nu,\sigma} \equiv \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \qquad \dot{g}_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial \xi}. \tag{10}$$

We will consider, as a simplified case, $l_\mu = [1, 1, 0, 0, 0]$, which fulfils the Eikonal equation $l_\mu l^\mu = 0$. This condition follows from the $\omega^{(-1)}$-scalar equations and is not an a priori implemented condition.

3. The Metric Perturbations up to Second Order

From the Einstein equation Equation (8) one can deduce a set of partial differential equations for the background fields $\bar{W}_1, \bar{\psi}, \bar{\gamma}$ and the first order perturbations $h_{\mu\nu}$. Here $\bar{W}_1$ represents the background warp factor. If one integrate the equation Equation (8) with respect to ξ and supposing that the perturbations are periodic in ξ, we then obtain the Einstein equations with back-reaction terms:

$$ {}^4\bar{G}_{\mu\nu} = \kappa_4^2 {}^4\bar{T}_{\mu\nu} + \kappa_5^4 \bar{\mathcal{S}}_{\mu\nu} - \bar{\mathcal{E}}_{\mu\nu} + \frac{1}{\tau}\int \Big(\kappa_4^2 T^{(0)}_{\mu\nu} + \kappa_5^4 S^{(0)}_{\mu\nu} - {}^4G^{(0)}_{\mu\nu} - \mathcal{E}^{(0)}_{\mu\nu}\Big) d\xi, \tag{11} $$

with τ de period of the high-frequency components. One can say that the term $-\int \mathcal{E}^{(0)}_{\mu\nu} d\xi$ in Equation (11) is the KK-mode contribution of the perturbative 5D graviton. It is an extra back-reaction term, which contain $\dot{h}_{55}$ amplified by the warp factor and with opposite sign with respect to the κ_4^2-term. So it can play the role of an effective cosmological constant. By substituting back these equations into the original equations, one gets propagation equations for the first order perturbations. In this way we obtain the set PDE's for the background fields and first order metric perturbation equations [6]. In Appendix A we collected for completeness these equations. We notice that in our simplified case of radiative coordinates $\Theta(x_\mu) = t + r$, the equations for the background metric separates from the perturbations. So this example is very suitable to investigate the perturbation equations. We omitted for the time being, the κ_5^4 contribution. It is manifest that to first order there is an interaction between the high-frequency perturbations from the bulk, the matter fields on the brane and the evolution of $\dot{h}_{ij}$, also found in the numerical solution [5]. These numerical solutions also show an interaction of the scalar-gauge field with the gravitational waves. The resulting fluctuations can act as a dark energy field. The numerical solution of the energy-momentum component $T_{\varphi\varphi}$ of Equation (A23) (without the bars) shows fluctuations and change of sign triggered by the several terms on the righthand side. This in contrast to the 4D case, where this change of sign is only initiated by the scalar to gauge mass ratio [10]. We observe again that the bulk contribution $\dot{h}_{55}$ is amplified by $\bar{W}_1^2$. It is a reflection of the massive KK modes felt on the brane. The most interesting equation is the differential equation for $\dot{h}_{14}$, i.e., the (t, φ) component, Equation (A5). It triggers the φ-dependent disturbances. The $sin[(n_2 - n_1)]\varphi$-term, amplified by warp factor $\bar{W}_1$, can have extremal values on $[0, \pi]$, if we choose, for example, $(n_2 - n_1) = 2$. We then have the term $\cos 2\varphi$, which has two extremal values on $[0, \pi]$ $mod(\frac{1}{2}\pi)$ (also found in [6]).

The next step is to investigate the higher order equations in ω , i.e., Equation (9), which will provide us first order equations of $\partial_t \dot{k}_{\mu\nu}$ and second order equations for $\partial_{tt} h_{\mu\nu}$. In this way, one can construct an approximate wave solution of the Einstein and scalar-gauge field equations and one can keep track of the different orders of perturbations.

With the help of an algebraic manipulation program, one obtains, for example, the equations for $\partial_t \dot{k}_{14}$ and $\partial_t \dot{k}_{55}$, were we took for the moment $k_{23} = k_{13}, k_{24} = k_{14}, k_{34} = 0$ and $k_{12} = \frac{1}{2}(k_{11} + k_{22})$. See Appendix B. We observe in Equation (A14) again a $cos[(n_2 - n_1)\varphi]$-term amplified by the warp factor. In the equation Equation (A15) for $\partial_t \dot{k}_{14}$, there appears besides the term $sin[(n_2 - n_1)\varphi]$, also a $sin[(n_3 - n_1)\varphi]$-term in connection with the second order perturbation $\dot{Z}$, amplified by the warp factor. In the equation for the first order counterpart equation, i.e., Equation (A5), there is only the $sin[(n_2 - n_1)\varphi]$-term. So if we take in Equation (A15) for $(n_3 - n_1) = 4$ (and $(n_2 - n_1) = 2$), then the maxima in φ of these two terms belonging to the perturbations of first and second order respectively, are out-of-phase. In the next section this will also become clear by considering the energy-current components of the energy momentum tensor. From Equations (A14) and (A7) we can obtain a second order PDE for h_{55} if we impose constraint conditions on $\dot{k}_{55}$ or integrate Equation (A14) with respect to ξ. This can also be done for the other components. This result is related to the Cauchy problem. In any field theory where there is a gauge freedom (as, for example, in GRT and Maxwell theory), one has to specify gauge conditions in order to determine the dynamical evolution for some initial set of Cauchy data. In Maxwell theory one usually chooses the Lorentz gauge. In GRT one has constraint equations because the system is over determined. These constraints are usually partial differential equations of second order. In our approximation scheme we have Equation (7) which leads to conditions on $\dot{h}_{\mu\nu}$. These first order equations Equations (A4)–(A8) can also be considered as constraint equations for the second order wave equations for $h_{\mu\nu}$, as example, in the case of h_{55}. So we can construct a dynamical evolution of the system of equations which fulfil the Cauchy data [20] .

In the next section we will see how our results can be related to the recently found spooky alignment of the rotation axes of quasars over large distances in two perpendicular directions.

4. The Matter Field Equations and the Energy-Momentum Tensor Expansion

The equations for the matter fields can be obtained in a similar way. From the first order equation of the scalar field we obtain

$$\bar{D}^\alpha \bar{D}_\alpha \bar{\Phi} - \frac{1}{2}\beta\bar{\Phi}(\bar{\Phi}\bar{\Phi}^* - \eta^2) = \frac{1}{\tau}\int \Big(h^{\mu\nu} l_\mu l_\nu \ddot{\Psi} + \bar{g}^{\mu\nu}\Gamma^{\alpha(0)}_{\mu\nu} l_\alpha \dot{\Psi}\Big) d\xi, \tag{12}$$

where we have integrated the equation with respect to ξ. On the right hand side we see again the high-frequency contribution to the field equation. In our case, this back reaction term turns out to be zero. So the first order equation is just the unperturbed equation for $\bar{X}$. See Equation (A9). The equation for $\bar{A}_\mu$ is also the same as in the unperturbed situation. See Equation (A10). If we substitute back the integrated equations into the original equations, we then obtain the first order perturbations (for $l^\alpha C_\alpha = 0; \Psi = Y(t, r, \xi)e^{in_2\varphi}$). See Equations (A11)–(A13).

For these matter field equations one needs the condition $l^\alpha \bar{A}_\alpha = 0$, otherwise the real and imaginary parts of $\dot{\Psi}$ interact as the propagation progresses. Again, there appears a φ-dependent term in the propagation equation for B_0, amplified by $\bar{W}_1$. So the approximate wave solution is no longer axially symmetric, also found by [15]. More insight in this φ-dependency can be obtained by studying the second-order matter field equations. One obtains for $\dot{Z}$ and $\dot{C}$ again first order differential equations. See Appendix B. We used the fact that the complex conjugate of the full complex second order scalar equation also must be satisfied. The appearance of the terms $\cos[(n_3 - n_2)\varphi], \cos[(n_3 - n_1)\varphi]$ and $\cos[(n_3 + n_2 - 2n_1)\varphi]$ will contribute to the next order modes of maxima in φ-dependent disturbances.

We can obtain, as in the case of the second order metric components, again second order PDE's for $\partial_{tt}B$ and $\partial_{tt}Y$ by suitable constraints on C and Z or integration with respect to ξ. After some rearrangement of Equation (A16), for example, we get the wave equation for Y

$$\begin{aligned}
\partial_{tt}Y = {} & \partial_{rr}Y + \frac{\partial_r Y}{r} - \frac{e^{2\bar{\gamma}}}{r^2}Y(n_2 - n_1 + \bar{P})^2 - \beta\bar{W}_1^2 e^{2\bar{\gamma}-2\bar{\psi}}Y\bar{X}^2 + 2\frac{\partial_r Y\partial_r\bar{W}_1 - \partial_t Y\partial_t\bar{W}_1}{\bar{W}_1} + 2e^{2\bar{\psi}}\frac{\dot{Y}}{\bar{W}_1^2 r^2}(\partial_r\bar{\psi} - \partial_t\bar{\psi} - \frac{1}{2r})h_{44} \\
& + 2e^{2\bar{\psi}-2\bar{\gamma}}\frac{\dot{Y}}{\bar{W}_1^2}(\partial_t\bar{\psi} - \partial_r\bar{\psi} + \partial_r\bar{\gamma} - \partial_t\bar{\gamma} + \frac{\partial_r\bar{W}_1 - \partial_t\bar{W}_1}{\bar{W}_1})h_{11} + 2e^{2\bar{\psi}-2\bar{\gamma}}\frac{\partial_r\dot{Y} - \partial_r\dot{Y}}{\bar{W}_1^2}h_{11} + \frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}(\partial_r Y - \partial_t Y)\dot{h}_{11} \\
& + \frac{\dot{Y}e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}(\partial_t h_{11} - \partial_r h_{11}) + \mathcal{A}\mathbf{cos}[(n_2 - n_1)\varphi] - \frac{1}{2}\beta Y\bar{W}_1^2\bar{X}^2 e^{2\bar{\gamma}-2\bar{\psi}}\mathbf{cos}[2(n_2 - n_1)\varphi] \\
& + 2\Big[\partial_r\dot{Z} - \partial_t\dot{Z} + \Big(\frac{\partial_r\bar{W}_1 - \partial_t\bar{W}_1}{\bar{W}_1} + \frac{1}{2r}\Big)\dot{Z}\Big]\mathbf{cos}[(n_3 - n_2)\varphi],
\end{aligned} \tag{13}$$

where $\mathcal{A}$ is an expression in backgrounds fields, h_{11} and h_{44}. We have again two periodic functions $\mathbf{cos}[2(n_2 - n_1)\varphi]$ and $\mathbf{cos}[(n_2 - n_1)\varphi]$ with frequency difference of a factor two and where one of the functions is amplified by $\bar{W}_1^2$. It will be necessary to study these equations numerically in order to compare the amplitudes of these two periodic functions with those of the azimuthal-angle dependent maxima of the quasar polarization alignment.

The second order equation for B_0 can be obtained from the sum of the t- and r-components of the second order gauge field equations.

We can calculate the three first terms of the energy momentum tensor ${}^4T_{\mu\nu}$. See Appendix C. In ${}^4T^{(1)}_{tt}$ there appears, for $(n_2 - n_1) = 2$ and $(n_3 - n_1) = 4$, the terms $cos(2\varphi), sin(2\varphi)$ and $cos(4\varphi)$, while in the first order term, Equation (A21), there is only the $cos(2\varphi)$. This is also true for the energy-current components ${}^4T^{(0)}_{t\varphi}$ and ${}^4T^{(1)}_{t\varphi}$. In the energy momentum tensor component ${}^4T^{(1)}_{tt}$ there is also a term proportional $(n_2 - n_1 + \bar{P})$. In the next order ${}^4T^{(2)}_{tt}$ there will be terms proportional with higher orders of $(n_2 - n_1 + \bar{P})$. These higher order terms will be suppressed by the warp factor, so the vortex will not become unstable as is the case when one breakup the vortex string in multiple flux [21].

The most interesting behavior arises in the angular component ${}^4T_{\varphi\varphi}$, i.e., Equations (A23)–(A25). As already noticed by Laguna-Castillo and Matzner [10], ${}^4\bar{T}_{\varphi\varphi}$ can alternate in sign dependent of the gauge to scalar mass. This can also happen dynamically [5]. In the next order ${}^4T^{(0)}_{\varphi\varphi}$ we have the $\dot{Y}$ contribution in front of $cos[2\varphi]$ and in the next order ${}^4T^{(1)}_{\varphi\varphi}$ the $\dot{Z}$ contribution in front of $cos[4\varphi]$ (for the chosen values of n_i as above). So the doubling of the frequency in obvious. From the expression for ${}^4T^{(0)}_{zz}$, Equation (A26), we see that the pressure in the z-direction is again dominated by the $cos[(n_2 - n_1)\varphi]$, because the second term is suppressed by $\bar{W}_1$. There is, however, a peculiar side effect: the term $(\partial_t \bar{X} - \partial_r \bar{X})$ can change sign dynamically. A numerical solution can give a decisive answer.

There is a relation between the phase freedom $e^{in\varphi}$ of our scalar field and the secular instability of an initially quasi-stationary axially symmetric configuration caused by radiative reaction. The small non-axially symmetric deformations turn out to be of the form $e^{im\varphi}$ with m an integer [22]. This broken symmetry, described by the inverse of the angular momentum J, is comparable with the symmetry breaking of the Higgs field considered in our model. An axially symmetric system is invariant under rotations in two dimensions, the SO(2) group. The breaking of this symmetry can be expressed in the equatorial eccentricity in the (r, φ)-plane. The particular orientation of the ellipsoid can be expressed through the azimuthal-angle φ. This discrete change into non-axially symmetry must be cancelled by emission of gravitational energy (and is amplified in our model by the 5D contribution), otherwise we are saddled with a helical time coordinate, $t \to t + J\varphi$ and must give up Lorentz invariance. This is clear from the fact that our metric will then possess a $g_{t\varphi}$ term. The angular momentum in (r, φ)-plane is determined by the currents of the momentum density, $\sim x^\rho T^{\nu\mu} - x^\nu T^{\rho\mu}$ and can be calculated in our case with the off-diagonal components of $T_{\mu\nu}$ of Appendix C. For example $J \sim \epsilon_{ij} \int d^2\mathbf{x}(x^i T^{0j} - x^j T^{0i})$.

5. The Warp Factor as Local Conformal Symmetry

Gravity theory invariant under $g_{\mu\nu}(\mathbf{x}) \to \Omega^2(\mathbf{x}) g_{\mu\nu}(\mathbf{x})$ is local conformal invariant and must be spontaneously broken because our world appears not to be scale invariant [23]. Let us rewrite our spacetime of Equation (1) with $\psi = \gamma = 0$

$$ds^2 = \Omega^2\Big[-dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 + \frac{1}{\Omega^2} dy^2\Big]. \tag{14}$$

where we renamed the warp factor as Ω. If we consider the (r,t)-dependent part of Ω and consider the flat (brane) case of the metric Equation (1),

$$\hat{ds}^2 = -dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 + \frac{1}{\Omega^2} dy^2. \tag{15}$$

we then obtain for the Ricci scalar

$${}^5\hat{R} = \frac{2}{\Omega}\Big(\partial_{rr}\Omega - \partial_{tt}\Omega + \frac{\partial_r \Omega}{r}\Big) + \frac{4}{\Omega^2}\Big(\partial_t \Omega^2 - \partial_r \Omega^2\Big). \tag{16}$$

The Ricci scalar transforms under $g_{\mu\nu}(\mathbf{x}) \to \Omega^2(\mathbf{x}) g_{\mu\nu}(\mathbf{x})$ as [24]

$${}^5\hat{R} \to \frac{1}{\Omega^2}\Big[{}^5\hat{R} - \frac{4\hat{g}^{\mu\nu}\hat{\nabla}_\mu \Omega \hat{\nabla}_\nu \Omega}{\Omega^2} - \frac{8\hat{g}^{\mu\nu}\hat{\nabla}_\mu \hat{\nabla}_\nu \Omega}{\Omega}\Big] = \frac{1}{\Omega^2}\Big[{}^5\hat{R} + \frac{4(\partial_r \Omega^2 - \partial_t \Omega^2)}{\Omega^2} + \frac{8(\partial_{tt}\Omega - \partial_{rr}\Omega - \frac{1}{r}\partial_r \Omega)}{\Omega}\Big]. \tag{17}$$

So for conformal invariancy of ${}^5\hat{R}$, the second term on the right hand side of Equation (17) must vanish. For ${}^5\hat{R} = 0$ we find in combination of Equation (16)

$$\partial_{tt}\Omega - \partial_{rr}\Omega - \frac{1}{r}\partial_r\Omega = 0, \tag{18}$$

with constraint equations $\partial_t\Omega^2 - \partial_r\Omega^2 = 0$. Equation (18) is just the equation of a vibrating circular drum. The general solution for the boundary conditions $\Omega(r,0) = f(r), \partial_t\Omega(r,0) = 0$ is

$$\Omega_n(r,t) = \Big(A\mathbf{cos}(c_n t) + B\mathbf{sin}(c_n t)\Big)\Big(J_0(c_n r) + Y_0(c_n r)\Big), \tag{19}$$

with J_0, Y_0 Bessel functions and C_n coefficients dependent of $f(r)$. These solutions represent for suitable boundary conditions, the standing normal modes of the brane in the vacuum case. In general, Ω can also depend on the azimuthal angle. This dependency is found, in our non-vacuum situation, in Sections 2 and 3 in the multiple-scale approximation. So one could conclude that the warp factor in the vacuum case fulfils a scalar wave equations representing fluctuations of the brane in the ground state. It represents the amount of local "stretching" of the 4D geometry. In the non-vacuum case, with the U(1) scalar gauge field in the brane, one can try to formulate again the conformal invariance. This is a peculiar issue in theoretical physics till now. Einstein equations and the scalar equation (Klein-Gordon equation) are not conformally invariant. One has to modify Einstein's equations to make it conformally invariant and make the energy momentum tensor traceless [24,25]. Our Ω-field can play a crucial role in this context if one introduces an unavoidable dilaton field.

6. Conclusions

It is found on a five dimensional warped brane world spacetime, using a multiple-scale approximation scheme, that, to second order, the metric and scalar gauge field show a spectrum of azimuthal-angle dependent wavelike modes with extremal values dependent of the winding numbers of the background, first and second order perturbations of the scalar field. The model can also explain the late-time acceleration of our universe without the need of a controversial cosmological constant. The disturbances do not fade away by natural damping, as is the case for the 4D model, but can survive due to the warp factor and the induced massive KK-modes felt on the brane. There is a relation between the symmetry breaking of the scalar-gauge field described by the winding number and the discrete non-axially symmetric deviations. Because a network of long non-intercommuting cosmic strings shows scale-invariant evolution, it would be desirable to find this scale invariance also in the quasar alignment. The results found here can be used to explain the recently found spooky alignment of the rotation axes of quasars over very large distances and can serve as an evidence for the existence of cosmic strings. Careful comparison of this spectrum of extremal values of the first and second order φ-dependency and the distribution of the alignment of the quasar polarizations is necessary. This can be accomplished when more observational data become available.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. The Background and First Order Perturbation Equations

In an earlier work [6] we obtained the equations for the background metric components $(\bar{W}_1, \bar{\psi}, \bar{\gamma})$, background matter fields $\bar{X}, \bar{Y}$ and first order approximation equations of $\dot{h}_{13}, \dot{h}_{14}, \dot{h}_{11}, \dot{h}_{44}, \dot{Y}, \dot{B}$ and $\dot{B}_0$. They are for completeness

$$\partial^2_{tt}\bar{W}_1 = -\partial^2_{rr}\bar{W}_1 + \frac{2}{\bar{W}_1}(\partial_t\bar{W}_1^2 + \partial_r\bar{W}_1^2) - \bar{W}_1(\partial_t\bar{\psi}^2 + \partial_r\bar{\psi}^2) + \frac{\bar{W}_1}{r}(\partial_r\bar{\gamma} - \partial_t\bar{\gamma}) + 2(\partial_r\bar{W}_1 - \partial_t\bar{W}_1)(\partial_t\bar{\psi} - \partial_r\bar{\psi} + \partial_r\bar{\gamma} - \partial_t\bar{\gamma})$$
$$+2\bar{W}_1\partial_r\bar{\psi}\partial_t\bar{\psi} - 4\frac{\partial_r\bar{W}_1\partial_t\bar{W}_1}{\bar{W}_1} + 2\partial_{rt}\bar{W}_1 - \frac{3}{4}\kappa_4^2\Big(e^{2\bar{\psi}}\frac{(\partial_t\bar{P} - \partial_r\bar{P})^2}{\bar{W}_1 r^2\epsilon^2} + \bar{W}_1(\partial_t\bar{X} - \partial_r\bar{X})^2\Big), \tag{A1}$$

$$\partial_{tt}\bar{\psi} = \partial_{rr}\bar{\psi} + \frac{\partial_r\bar{\psi}}{r} + \frac{2}{\bar{W}_1}(\partial_r\bar{W}_1\partial_r\bar{\psi} - \partial_t\bar{W}_1\partial_t\bar{\psi}) - \frac{\partial_r\bar{W}_1}{r\bar{W}_1} + \frac{3e^{2\bar{\psi}}}{4\bar{W}_1^2r^2\epsilon^2}\kappa_4^2\Big(\partial_t\bar{P}^2 - \partial_r\bar{P}^2 - \bar{W}_1^2\epsilon^2\bar{X}^2\bar{P}^2e^{2\bar{\gamma}-2\bar{\psi}}\Big), \tag{A2}$$

$$\begin{aligned}\partial_t\bar{\gamma} = \partial_r\bar{\gamma} + \frac{1}{\partial_t\bar{W}_1 - \partial_r\bar{W}_1 - \frac{\bar{W}_1}{2r}}\Big[\frac{1}{2}\bar{W}_1(\partial_t\bar{\psi} - \partial_r\bar{\psi})^2 + \frac{\partial_r\bar{W}_1}{r} - \partial_{tr}\bar{W}_1 - \frac{\partial_r\bar{W}_1^2 + 3\partial_t\bar{W}_1^2}{2\bar{W}_1}\\ +\partial_{rr}\bar{W}_1 + \frac{2\partial_r\bar{W}_1\partial_t\bar{W}_1}{\bar{W}_1} + (\partial_r\bar{W}_1 - \partial_t\bar{W}_1)(\partial_r\bar{\psi} - \partial_t\bar{\psi}) + \kappa_4^2\frac{\bar{W}_1}{16}\Big(7\partial_t\bar{X}^2 + 5\partial_r\bar{X}^2\\ -12\partial_r\bar{X}\partial_t\bar{X} + 5e^{2\bar{\gamma}}\frac{\bar{X}^2\bar{P}^2}{r^2} + 6e^{2\bar{\psi}}\frac{(\partial_r\bar{P} - \partial_t\bar{P})^2}{\bar{W}_1^2r^2\epsilon^2} + \bar{W}_1^2\beta e^{2\bar{\gamma}-2\bar{\psi}}(\bar{X}^2 - \eta^2)^2\Big)\Big].\end{aligned} \tag{A3}$$

$$\partial_t\dot{h}_{13} = \partial_r\dot{h}_{13} + \ddot{k}_{13} - \ddot{k}_{23} + 2\Big(\frac{\partial_t\bar{W}_1 - \partial_r\bar{W}_1}{\bar{W}_1} + \partial_t\bar{\psi} - \partial_r\bar{\psi}\Big)\dot{h}_{13}, \tag{A4}$$

$$\begin{aligned}\partial_t\dot{h}_{14} = \partial_r\dot{h}_{14} + \ddot{k}_{14} - \ddot{k}_{24} + 2\Big(\partial_r\bar{\psi} - \partial_t\bar{\psi} + \frac{\partial_t\bar{W}_1 - \partial_r\bar{W}_1}{\bar{W}_1} - \frac{1}{r}\Big)\dot{h}_{14}\\ +2\kappa_4^2e^{2\bar{\gamma}-2\bar{\psi}}\bar{W}_1^2\bar{X}\bar{P}\dot{Y}\mathbf{sin}[(n_2 - n_1)\varphi] + \partial_\varphi\Big[\bar{W}_1^2e^{2\bar{\gamma}-2\bar{\psi}}\dot{h}_{55} - \dot{h}_{11} - \frac{e^{2\bar{\gamma}}}{r^2}\dot{h}_{44}\Big],\end{aligned} \tag{A5}$$

$$\begin{aligned}\partial_t\dot{h}_{11} = \partial_r\dot{h}11 + \frac{e^{2\bar{\gamma}}}{r^2}\Big(\partial_r\bar{\psi} - \partial_t\bar{\psi} - \frac{1}{2r}\Big)\dot{h}_{44} + \frac{1}{2}(\ddot{k}_{22} + \ddot{k}_{11}) - \ddot{k}_{12} + \frac{2}{\bar{W}_1}\Big(\partial_t\bar{W}_1 - \partial_r\bar{W}_1 + \bar{W}_1(\partial_r\bar{\psi} - \partial_t\bar{\psi} + \partial_t\bar{\gamma} - \partial_r\bar{\gamma})\Big)\dot{h}_{11}\\ +e^{2\bar{\gamma}-2\bar{\psi}}\bar{W}_1^2\Big[\Big(\frac{1}{4r} + \frac{\partial_r\bar{W}_1 - \partial_t\bar{W}_1}{2\bar{W}_1}\Big)\dot{h}_{55} + \kappa_4^2(\partial_t\bar{X} - \partial_r\bar{X})\dot{Y}\Big]\mathbf{cos}[(n_2 - n_1)\varphi],\end{aligned} \tag{A6}$$

$$\partial_t\dot{h}_{55} - \partial_r\dot{h}_{55} = 0, \tag{A7}$$

$$\partial_t\dot{h}_{44} = \partial_r\dot{h}_{44} + \Big(2\partial_r\bar{\psi} - 2\partial_t\bar{\psi} - \frac{3}{2r} + \frac{\partial_t\bar{W}_1 - \partial_r\bar{W}_1}{\bar{W}_1}\Big)\dot{h}_{44} + \frac{\kappa_4^2}{\epsilon}(\partial_r\bar{P} - \partial_t\bar{P})\dot{B} + \frac{1}{2}\bar{W}_1^2r^2e^{-2\bar{\psi}}\Big(\partial_t\bar{\psi} - \partial_r\bar{\psi} + \frac{1}{2r}\Big)\dot{h}_{55}. \tag{A8}$$

The background matter fields $(\bar{X}, \bar{P})$ become [5]

$$\partial_{tt}\bar{X} = \partial_{rr}\bar{X} + \frac{\partial_r\bar{X}}{r} + 2\Big(\frac{\partial_r\bar{W}_1\partial_r\bar{X}}{\bar{W}_1} - \frac{\partial_t\bar{W}_1\partial_t\bar{X}}{\bar{W}_1}\Big) - \frac{e^{2\bar{\gamma}}\bar{X}\bar{P}^2}{r^2} - \frac{1}{2}\bar{W}_1^2e^{2\bar{\gamma}-2\bar{\psi}}\beta\bar{X}(\bar{X}^2 - \eta^2), \tag{A9}$$

$$\partial_{tt}\bar{P} = \partial_{rr}\bar{P} - \frac{\partial_r\bar{P}}{r} + 2(\partial_r\bar{P}\partial_r\bar{\psi} - \partial_t\bar{P}\partial_t\bar{\psi}) - \epsilon^2\bar{W}_1^2e^{2\bar{\gamma}-2\bar{\psi}}\bar{P}\bar{X}^2, \tag{A10}$$

and perturbation equations

$$\partial_t\dot{Y} = \partial_r\dot{Y} + \Big(\frac{\partial_r\bar{W}_1 - \partial_t\bar{W}_1}{\bar{W}_1} + \frac{1}{2r}\Big)\dot{Y} \tag{A11}$$

$$\partial_t\dot{B} = \partial_r\dot{B} + \Big(\partial_r\bar{\psi} - \partial_t\bar{\psi} - \frac{1}{2r}\Big)\dot{B} + \frac{e^{2\bar{\psi}}(\partial_t\bar{P} - \partial_r\bar{P})}{2r^2\bar{W}_1^2\epsilon}\dot{h}_{44}, \tag{A12}$$

$$\partial_t\dot{B}_0 = \partial_r\dot{B}_0 - \frac{e^{2\bar{\gamma}}}{r^2}\partial_\varphi\dot{B} + \frac{e^{2\bar{\psi}}(\partial_t\bar{P} - \partial_r\bar{P})}{\bar{W}_1^2r^2\epsilon}\dot{h}_{14} + \epsilon e^{2\bar{\gamma}-2\bar{\psi}}\dot{Y}\bar{X}\bar{W}_1^2\mathbf{sin}[\varphi(n_2 - n_1)]. \tag{A13}$$

The background equations don't contain perturbations terms, due to our (simplified) gauge conditions.

Appendix B. The Second Order Perturbation Equations

With the help of an algebraic manipulation program, we obtain from Equation (9), for example, the most interesting[1]:

$$\begin{aligned}\partial_t\dot{k}_{55} = \partial_r\dot{k}_{55} + \frac{1}{2}(\partial_{rr}h_{55} - \partial_{tt}h_{55}) + \frac{1}{2}(\partial_t h_{55} - \partial_r h_{55})\dot{h}_{55} + \frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}(\partial_t\dot{h}_{55} - \partial_r\dot{h}_{55})h_{11}\\ +\kappa_4^2\frac{e^{2\bar{\gamma}}}{r^2}\bar{X}Y\Big(2\bar{P}(n_1 - n_2 - \bar{P}) + e^{-2\bar{\psi}}r^2\bar{W}_1^2\beta(\eta^2 - \bar{X}^2)\Big)\mathbf{cos}[(n_2 - n_1)\varphi]\\ +2\frac{e^{2\bar{\psi}}}{\bar{W}_1^2r^2}\Big[\kappa_4^2\frac{e^{2\bar{\gamma}}}{r^2}\bar{X}^2\bar{P}^2 + \partial_{rr}\bar{\psi} - \partial_{tt}\bar{\psi} + \frac{e^{2\bar{\psi}}}{\bar{W}_1^2r^2}\Big(2\partial_r\bar{\psi} - 2\partial_t\bar{\psi} + \frac{\partial_t\bar{W}_1 - \partial_r\bar{W}_1}{\bar{W}_1} - \frac{3}{2r}\Big)\dot{h}_{44}\end{aligned}$$

[1] The interested reader can obtain the computer-algebra program from the author.

$$
\begin{aligned}
&+\frac{2(\partial_t\bar{W}_1\partial_t\bar{\psi}-\partial_r\bar{W}_1\partial_r\bar{\psi})}{\bar{W}_1}+\frac{\partial_r\bar{\psi}}{r}-\frac{\partial_r\bar{W}_1}{\bar{W}_1 r}\Big]h_{44}+2\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}\Big[\kappa_4^2e^{2\bar{\gamma}}\Big(\frac{\bar{X}^2\bar{P}^2}{2r^2}+\frac{1}{8}e^{-2\bar{\psi}}\beta(\bar{X}^2-\eta^2)^2\bar{W}_1^2\Big)\\
&+\frac{\partial_t\bar{W}_1^2-\partial_r\bar{W}_1^2}{\bar{W}_1^2}+\frac{\partial_{tt}\bar{W}_1-\partial_{rr}\bar{W}_1}{\bar{W}_1}-2\frac{\partial_r\bar{W}_1}{\bar{W}_1 r}\Big]h_{11}+\frac{e^{2\bar{\psi}}}{\bar{W}_1^2r^2}\Big[2(\partial_r\bar{\psi}-\partial_t\bar{\psi}+\frac{\partial_r\bar{W}_1-\partial_t\bar{W}_1}{\bar{W}_1})\partial_\varphi h_{14}\\
&+2\frac{e^{2\bar{\psi}}}{\bar{W}_1^2r^2}(\partial_r h_{44}-\partial_t h_{44})\dot{h}_{44}+\partial_{r\varphi}h_{14}-\partial_{t\varphi}h_{14}+\partial_{\varphi\varphi}h_{11}+\frac{e^{2\bar{\gamma}}}{r^2}\partial_{\varphi\varphi}h_{44}\Big]-2\kappa_4^2\frac{e^{2\bar{\gamma}}}{r^2}\epsilon\bar{X}^2\bar{P}B,
\end{aligned}
\tag{A14}
$$

and

$$
\begin{aligned}
\partial_t\dot{k}_{14}=&\,\partial_r\dot{k}_{14}+2\Big(\partial_r\bar{\phi}-\partial_t\bar{\phi}+\frac{\partial_t\bar{W}_1-\partial_r\bar{W}_1}{\bar{W}_1}-\frac{1}{r}\Big)\dot{k}_{14}+\frac{1}{2}\dot{h}_{55}(\partial_r h_{14}-\partial_t h_{14})+h_{14}(\partial_r\dot{h}_{55}-\partial_t\dot{h}_{55})\\
&+2\kappa_4^2e^{2\bar{\gamma}-2\bar{\psi}}\Big[\bar{W}_1^2\Big(\partial_t\bar{X}Y(n_1-n_2-\bar{P})+\bar{X}(\bar{P}\partial_tY+\epsilon\dot{Y}B)\Big)\mathbf{sin}[(n_2-n_1)\varphi]\\
&+\bar{W}_1^2\bar{X}\bar{P}\dot{Z}\mathbf{sin}[(n_3-n_1)\varphi]+(\partial_t\bar{X}-\partial_r\bar{X})\dot{Y}e^{2\bar{\psi}-2\bar{\gamma}}h_{14}\mathbf{cos}[(n_2-n_1)\varphi]\Big]\\
&+\frac{1}{2}\dot{h}_{55}(\partial_\varphi h_{11}-\bar{W}_1^2e^{2\bar{\gamma}-2\bar{\psi}}\partial_\varphi h_{55})-\frac{e^{2\bar{\psi}+2\bar{\gamma}}}{\bar{W}_1^2r^4}h_{44}\partial_\varphi\dot{h}_{44}-\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}(h_{11}\partial_\varphi\dot{h}_{11}+\dot{h}_{11}\partial_\varphi h_{11})\\
&+\partial_\varphi\Big[e^{2\bar{\gamma}-2\bar{\psi}}(\bar{W}_1^2\partial_t\bar{\psi}-\bar{W}_1\partial_t\bar{W}_1)h_{55}+2h_{11}(\partial_t\bar{\gamma}+2\frac{\partial_t\bar{W}_1}{\bar{W}_1}-\partial_t\bar{\psi})-\partial_t h_{11}+\frac{1}{2}(\dot{k}_{22}-\dot{k}_{11}\\
&-2\frac{e^{2\bar{\gamma}}}{r^2}\dot{k}_{44})+e^{2\bar{\gamma}-2\bar{\psi}}\bar{W}_1^2\dot{k}_{55}+2\frac{e^{2\bar{\gamma}}}{r^2}(\frac{\partial_t\bar{W}_1}{\bar{W}_1}-2\partial_t\bar{\psi})h_{44}-\frac{e^{2\bar{\gamma}}}{r^2}(\partial_t h_{44}-e^{-2\bar{\psi}}r^2\bar{W}_1^2\partial_t h_{55})\Big]\\
&+\frac{e^{2\bar{\gamma}}}{r^2}(h_{44}\partial_\varphi\dot{h}_{55}-\frac{1}{2}\dot{h}_{44}\partial_\varphi h_{55})+2\kappa_4^2\bar{W}_1^2e^{2\bar{\gamma}-2\bar{\psi}}\Big(\bar{X}^2\bar{P}\epsilon B_0-\frac{1}{8}\beta h_{14}(\bar{X}^2-\eta^2)^2\Big)+\mathcal{H},
\end{aligned}
\tag{A15}
$$

with $\mathcal{H}$ a function of the background fields and the fields h_{ij}. The other components of the metric perturbations are obtained in the same way. The second order equations for $\dot{Z}$ and $\dot{C}$ are

$$
\begin{aligned}
\partial_t\dot{Z}=&\,\partial_r\dot{Z}+\Big(\frac{\partial_r\bar{W}_1-\partial_t\bar{W}_1}{\bar{W}_1}+\frac{1}{2r}\Big)\dot{Z}-\frac{1}{2}\beta\bar{W}_1^2\bar{X}^2Ye^{2\bar{\gamma}-2\bar{\psi}}\mathbf{cos}[(n_3+n_2-2n_1)\varphi]\\
&+\Big[\partial_{rr}Y-\partial_{tt}Y+\frac{\partial_rY}{r}+2\frac{\partial_rY\partial_r\bar{W}_1-\partial_tY\partial_t\bar{W}_1}{\bar{W}_1}+\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}\Big((\partial_rY-\partial_tY)\dot{h}_{11}+\dot{Y}(\partial_t h_{11}\\
&-\partial_r h_{11})\Big)-\frac{e^{2\bar{\gamma}}}{r^2}Y(n_2-n_1+\bar{P})^2-\beta e^{2\bar{\gamma}-2\bar{\psi}}Y\bar{X}^2\bar{W}_1^2+2\frac{e^{2\bar{\psi}}}{\bar{W}_1^2r^2}h_{44}\dot{Y}(\partial_r\bar{\psi}-\partial_t\bar{\psi}-\frac{1}{2r})\\
&+2\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}h_{11}\dot{Y}\Big(\partial_r\bar{\gamma}-\partial_t\bar{\gamma}-\partial_r\bar{\psi}+\partial_t\bar{\psi}+\frac{\partial_r\bar{W}_1-\partial_t\bar{W}_1}{\bar{W}_1}+\frac{\partial_r\dot{Y}-\partial_t\dot{Y}}{\dot{Y}}\Big)\Big]\mathbf{cos}[(n_3-n_2)\varphi]\\
&+\Big[\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}\Big(\frac{1}{2}(\partial_t\bar{X}-\partial_r\bar{X})(\dot{k}_{11}-\dot{k}_{22})+(\partial_t\bar{X}\partial_t h_{11}-\partial_r\bar{X}\partial_r h_{11})+h_{11}\Big(\partial_{rr}\bar{X}-\partial_{tt}\bar{X}\\
&+2\frac{\partial_r\bar{X}\partial_r\bar{W}_1-\partial_t\bar{X}\partial_t\bar{W}_1}{\bar{W}_1}+2(\partial_t\bar{X}\partial_t\bar{\psi}-\partial_r\bar{X}\partial_t\bar{\psi}+\partial_r\bar{X}\partial_r\bar{\gamma}-\partial_t\bar{X}\partial_t\bar{\gamma})\Big)\Big)\\
&+\frac{2e^{2\bar{\psi}}h_{44}}{\bar{W}_1^2r^2}\Big(\partial_r\bar{X}\partial_r\bar{\psi}-\partial_t\bar{X}\partial_t\bar{\psi}-\frac{\partial_r\bar{X}}{2r}+\frac{e^{-2\bar{\gamma}}\bar{X}\bar{P}^2}{2r^2}\Big)-\frac{2\epsilon e^{2\bar{\gamma}}}{r^2}\bar{X}\bar{P}B\Big]\mathbf{cos}[(n_3-n_1)\varphi],
\end{aligned}
\tag{A16}
$$

$$
\begin{aligned}
\partial_t\dot{C}=&\,\partial_r\dot{C}+(\partial_r\bar{\psi}-\partial_t\bar{\psi}-\frac{1}{2r})\dot{C}+\frac{1}{2}e^{2\bar{\gamma}-2\bar{\psi}}\epsilon^2\bar{X}^2\bar{W}_1^2B+\frac{1}{2}(\partial_{rr}B-\partial_{tt}B+\partial_{t\varphi}B_0-\partial_{r\varphi}B_0)\\
&+\frac{1}{2}e^{2\bar{\gamma}-2\bar{\psi}}\epsilon\bar{X}Y\bar{W}_1^2(n_2-n_1+2\bar{P})\mathbf{cos}[(n_2-n_1)\varphi]+\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}\Big[\partial_t\dot{B}-\partial_r\dot{B}+\frac{1}{2\epsilon}(\partial_{tt}\bar{P}-\partial_{rr}\bar{P}+\frac{\partial_r\bar{P}}{r})+\dot{B}(\frac{\partial_r\bar{W}_1-\partial_t\bar{W}_1}{\bar{W}_1}+\frac{1}{2r})\\
&+\frac{\partial_r\bar{P}\partial_r\bar{W}_1-\partial_t\bar{P}\partial_t\bar{W}_1}{\epsilon\bar{W}_1}\Big]h_{11}+\frac{e^{2\bar{\psi}}}{\bar{W}_1^2r^2\epsilon}\Big[2(\partial_t\bar{P}\partial_t\bar{\psi}-\partial_r\bar{P}\partial_r\bar{\psi})+2\dot{B}\epsilon(\partial_t\bar{\psi}-\partial_r\bar{\psi}+\frac{1}{2r})+\frac{\partial_r\bar{P}}{r}+\frac{e^{2\bar{\psi}}}{2\bar{W}_1^2r^2}(\partial_r\bar{P}-\partial_t\bar{P})\dot{h}_{44}\Big]h_{44}\\
&+\frac{e^{2\bar{\psi}}}{2\bar{W}_1^2r^2\epsilon}\Big((\partial_t\bar{P}-\partial_r\bar{P})\dot{k}_{44}+\partial_t\bar{P}\partial_t h_{44}-\partial_r\bar{P}\partial_r h_{44}+\epsilon\dot{B}(\partial_t h_{44}-\partial_r h_{44})\Big).
\end{aligned}
\tag{A17}
$$

There are the second order partial derivative terms $\partial_{tt}Y-\partial_{rr}Y$ and $\partial_{tt}B-\partial_{rr}B$ in Equations (A16) and (A17) respectively. They can be isolated in order to get a wave equation for the first order perturbations.

Appendix C. The Energy Momentum Tensor Components

For the several orders of the energy-momentum tensor components we find

$$
{}^4\bar{T}_{t\varphi}=0,\qquad {}^4T^{(0)}_{t\varphi}=\bar{X}\bar{P}\dot{Y}\mathbf{sin}[(n_2-n_1)\varphi]
\tag{A18}
$$

$$
\begin{aligned}
{}^4T^{(1)}_{t\varphi}=&\Big[\partial_t\bar{X}Y(n_1-n_2-\bar{P})+\bar{X}(\bar{P}\partial_tY+\epsilon B\dot{Y})\Big]\mathbf{sin}[(n_2-n_1)\varphi]+\bar{X}\bar{P}\dot{Z}\mathbf{sin}[(n_3-n_1)\varphi]+\bar{X}^2\bar{P}\epsilon B_0-\frac{1}{8}\beta(\bar{X}^2-\eta^2)^2h_{14}\\
&+\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2}\dot{Y}h_{14}(\partial_t\bar{X}-\partial_r\bar{X})\mathbf{cos}[(n_2-n_1)\varphi]-\frac{e^{2\bar{\psi}-2\bar{\gamma}}}{2\bar{W}_1^2}h_{14}\Big[\frac{e^{2\bar{\psi}}}{r^2\bar{W}_1^2\epsilon^2}(\partial_t\bar{P}-\partial_r\bar{P})^2+\partial_r\bar{X}^2-\partial_t\bar{X}^2+e^{2\bar{\gamma}}\frac{\bar{X}^2\bar{P}^2}{r^2}\Big].
\end{aligned}
\tag{A19}
$$

$$^4\bar{T}_{tt} = \frac{e^{2\bar{\psi}}}{2\bar{W}_1^2 r^2 \epsilon^2}(\partial_r \bar{P}^2 + \partial_t \bar{P}^2) + \frac{1}{2}(\partial_t \bar{X}^2 + \partial_r \bar{X}^2) + \frac{1}{2r^2} e^{2\bar{\gamma}} \bar{X}^2 \bar{P}^2 + \frac{1}{8} e^{2\bar{\gamma}-2\bar{\psi}} \bar{W}_1^2 \beta(\bar{X}^2 - \eta^2)^2 \tag{A20}$$

$$^4T_{tt}^{(0)} = \dot{Y}^2 + \dot{Y}(\partial_t \bar{X} + \partial_r \bar{X})\mathbf{cos}[(n_2 - n_1)\varphi] + \frac{e^{2\bar{\psi}}}{\bar{W}_1^2 r^2 \epsilon}\Big(\epsilon \dot{B}^2 + \dot{B}(\partial_r \bar{P} + \partial_t \bar{P})\Big) \tag{A21}$$

$$\begin{aligned} ^4T_{tt}^{(1)} &= \dot{Z}(\partial_t \bar{X} + \partial_r \bar{X})\mathbf{cos}[(n_3 - n_1)\varphi] + 2\dot{Y}\dot{Z}\mathbf{cos}[(n_3 - n_2)\varphi] + 2\epsilon \bar{X}\dot{Y}B_0 \mathbf{sin}[(n_2 - n_1)\varphi] \\ &+ \bar{X}Y\Big(\frac{\beta}{2} e^{2\bar{\gamma}-2\bar{\psi}} \bar{W}_1^2 (\bar{X}^2 - \eta^2) + \frac{\partial_r \bar{X} \partial_r Y + \partial_t \bar{X} \partial_t Y}{\bar{X}Y} + \frac{e^{2\bar{\gamma}}}{r^2} Y(n_2 - n_1 + \bar{P})\Big)\mathbf{cos}[(n_2 - n_1)\varphi] \\ &- \frac{e^{4\bar{\psi}}}{\bar{W}_1^4 r^4 \epsilon^2}\Big(\frac{1}{2}(\partial_t \bar{P}^2 + \partial_r \bar{P}^2) + \epsilon \dot{B}(\partial_t \bar{P} + \partial_r \bar{P}) + \epsilon^2 \dot{B}^2 + \frac{1}{2} e^{2\bar{\gamma}-2\bar{\psi}} \bar{X}^2 \bar{P}^2 \bar{W}_1^2 \epsilon^2\Big) h_{44} \\ &- \Big(\frac{1}{8}\beta(\bar{X}^2 - \eta^2)^2 + \frac{e^{2\bar{\psi}}}{2\bar{W}_1^2 r^2} \bar{X}^2 \bar{P}^2\Big) h_{11} + \dot{Y}(\partial_t Y + \partial_r Y) + \frac{e^{2\bar{\psi}}}{\bar{W}_1^2 r^2 \epsilon} \dot{C}(\partial_t \bar{P} + \partial_r \bar{P} + 2\epsilon \dot{B}) + \frac{\epsilon e^{2\bar{\gamma}}}{r^2} \bar{X}^2 \bar{P}^2 B \end{aligned} \tag{A22}$$

$$^4\bar{T}_{\varphi\varphi} = \frac{e^{2\bar{\psi}-2\bar{\gamma}}}{2\bar{W}_1^2 \epsilon^2}(\partial_r \bar{P}^2 - \partial_t \bar{P}^2) + \frac{1}{2} e^{-2\bar{\gamma}} r^2 (\partial_t \bar{X}^2 - \partial_r \bar{X}^2) + \frac{1}{2}\bar{X}^2 \bar{P}^2 - \frac{1}{8} e^{-2\bar{\psi}} \bar{W}_1^2 r^2 \beta(\bar{X}^2 - \eta^2)^2 \tag{A23}$$

$$^4T_{\varphi\varphi}^{(0)} = e^{-2\bar{\gamma}} r^2 \dot{Y}(\partial_t \bar{X} - \partial_r \bar{X})\mathbf{cos}[(n_2 - n_1)\varphi] + e^{2\bar{\psi}-2\bar{\gamma}} \frac{\dot{B}}{\bar{W}_1^2 \epsilon}(\partial_r \bar{P} - \partial_t \bar{P}) \tag{A24}$$

$$\begin{aligned} ^4T_{\varphi\varphi}^{(1)} &= \Big[\frac{e^{-2\bar{\psi}}}{2} \bar{W}_1^2 r^2 Y \bar{X} \beta(\eta^2 - \bar{X}^2) + e^{-2\bar{\gamma}} r^2 (\partial_t \bar{X} \partial_t Y - \partial_r \bar{X} \partial_r Y) + \bar{X}\bar{P}Y(n_2 - n_1 + \bar{P}) \\ &+ \frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2} \dot{Y}(\partial_t \bar{X} - \partial_r \bar{X})(h_{44} + r^2 e^{-2\bar{\gamma}} h_{11})\Big]\mathbf{cos}[(n_2 - n_1)\varphi] + \Big(r^2 e^{-2\bar{\gamma}} \dot{Z}(\partial_t \bar{X} - \partial_r \bar{X})\Big)\mathbf{cos}[(n_3 - n_1)\varphi] \\ &- \frac{1}{8}\beta h_{44}(\bar{X}^2 - \eta^2)^2 + r^2 e^{-2\bar{\gamma}} \dot{Y}(\partial_t Y - \partial_r Y) + \frac{e^{2\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2 \epsilon} \dot{C}(\partial_r \bar{P} - \partial_t \bar{P}) \\ &+ \frac{\bar{X}\bar{P}\epsilon \dot{B}}{r^2} + \frac{e^{2\bar{\psi}-4\bar{\gamma}}}{2\bar{W}_1^2}(r^2 h_{11} + e^{2\bar{\gamma}} h_{44}) + \frac{e^{4\bar{\psi}-4\bar{\gamma}}}{2\bar{W}_1^4 \epsilon^2} h_{11}\Big(\partial_r \bar{P}^2 - \partial_t \bar{P}^2 + \epsilon \dot{B}(\partial_r \bar{P} - \partial_t \bar{P})\Big). \end{aligned} \tag{A25}$$

$$^4T_{zz}^{(0)} = e^{4\bar{\psi}-2\bar{\gamma}} \dot{Y}(\partial_t \bar{X} - \partial_r \bar{X})\mathbf{cos}[(n_2 - n_1)\varphi] + \frac{e^{6\bar{\psi}-2\bar{\gamma}}}{\bar{W}_1^2 r^2 \epsilon} \dot{B}(\partial_t \bar{P} - \partial_r \bar{P}). \tag{A26}$$

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