Buckley–Leverett Theory for Two-Phase Immiscible Fluids Flow Model with Explicit Phase-Coupling Terms

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Abstract: The theory of two-phase immiscible flow in porous media is based on the extension of single phase models through the concept of relative permeabilities. It mimics Darcy's law for a fixed average saturation through the introduction of saturation-based permeabilities to model the momentum exchange between the phases. In this paper, we present a model of two-phase flow, based on the extension of Darcy's law including the effect of capillary pressure, but considering in addition the coupling between the phases modeled through flow cross-terms. In this work, we extend the Buckley–Leverett theory to the subsequent model, and provide numerical experiments shading the light on the effect of the coupling cross-terms in comparison to the classical Darcy's approach.

Keywords: Darcy's law; two-phase flows; phases coupling; fractional flow; Buckley–Leverett theory; capillary pressure

1. Introduction

Undoubtedly, modeling multiphase flows in porous media is of major importance in many fields of applications such as ground water storage, containment transport, geological CO_2 sequestration, and particularly in enhanced oil recovery applications of petroleum engineering. The classical mathematical models for multiphase flows are based on a straightforward generalization of Darcy's law for a single-phase flow, as proposed in Muskat [1]. A natural question arises: How important is the influence of a phase on the other phase? A rather intuitive expectation is that the effect will depend on the properties of the contact area between the two phases, and of course on other properties such as wettability, structure of the porous medium and the surface between the fluid and solid phases. Indeed, it is known that, in chemical and nuclear engineering, the scenario of strong friction between phases occurs when the contact area and the permeability are large; we refer for instance to [2–5]. On the opposite side, in some applications, it is shown that the coupling effects are small, and therefore negligible.

The idea of phases coupling in multiphase flows modeling has already been investigated by several authors; we refer for instance to Rose [6–8], De Gennes [9], De La Cruz and Spanos [10], Auriault and Scanchez-Palencia [11], Whitaker [12], Spanos et al. [13], Kalaydjian [2,3], Kalaydjian and Legait [14], and Spanos et al. [15]. In particular, Kalaydjian and Legait [16] studied the effect of the pore geometry and wettability for a counter-current two-phase flow in a capillary tube and in porous media to derive equations with coupling terms between phases. In Rose et al. [17], it was suggested that the Darcian-based equations poorly model the reservoir's transport processes events. They argued

that the reason is that viscous and/or other related coupling effects for dynamic multiphase-saturated media are not quantitatively accounted for in Darcy's law. In Dullien et al. [18], it is assumed that, if the pressure gradient of one of two phases fluids is equal to zero, the coefficient in the coupled equation can be defined, and they experimented with this idea with oil/water in a sand pack to compute the four transport coefficients with an enormous water saturation scale. They found coefficient values ranging from 10 to 35% of the value of the effective permeability to water and from 5 to 15% of the effective permeability to oil, respectively. Pasquier et al. [19] studied models of creeping flows that include an explicit coupling between both phases by adding a cross-term in the generalized Darcy's law describing the multiphase flow.

For more than a half century, Buckley and Leverett [20] (1942) has been the reference to forecast the behavior of flow in porous media for one-dimensional problems. Buckley and Leverett introduced the fractional flow calculus and computed the sharp saturation front position to model the displacement of fluids behavior through porous media. Welge's graphic method was proposed [21] in 1952 to describe the evolution of the saturation front. Sheldon and Cardwell [22] used an alternative method via the characteristic technique to solve the Buckley and Leverett equation. More recently, scientific literature as Mc-Whorter and Sunada [23] contributed to the developments of Buckley and Leverett theory. In the context of phase-coupled Darcy equations, Guérillot and Kalaydjian [24] proposed a numerical scheme based on the global pressure concept of Chavent et al. [25] through a mixed finite element code to solve the subsequent system of coupled Darcy equations. To the best of our knowledge, such numerical simulations, and therefore approximations, are mainly needed in applications because of the lack of analytical solutions of the mathematical system .

In this paper, we investigate the analysis and the numerical simulation of a coupled system of Darcy's equations through cross-flow terms. We consider several scenarios and study few particular situations related to the coupling terms and the permeabilities. Eventually, we present a method that allows for taking into account the cross-terms, and therefore the effect of one phase on the other, using any classical reservoir simulation software solving the classical non-coupled Darcy's multiphase approach. The idea consists of solving the classical Darcy's system with modified permeabilities (we give the precise expression of these permeabilities), and, using the latter solution, we reconstruct a solution of the coupled system. The paper is organized as follows: the next section is devoted to the introduction of the mathematical Darcy's equations governing the fluid domain and the one-dimensional, coupling approach. Section 3 is dedicated to the associated Buckley–Leverett theory and to the presentation of several numerical simulations. Section 4 is devoted to the discussion of several aspects of the coupled Darcy's equations and the study of few limiting and particular cases. In addition, in this section, we present the method mentioned earlier regarding the calculation of the solution of the coupled system from the solution of a classical Darcy's type system that can be solved using any existing software designed to solve the standard Darcy's approach.

2. Mathematical Formulation

We are interested in the description of incompressible fluids displacement in a porous medium reservoir. The mass conservation equation for the oil and the water phases, denoted with subscripts $\cdot_o$ and $\cdot_w$ respectively, reads as

$$\frac{\partial \rho_o S_o \phi}{\partial t} + \nabla \cdot (\rho_o v_o) = 0,$$
$$\frac{\partial \rho_w S_w \phi}{\partial t} + \nabla \cdot (\rho_w v_w) = 0, \tag{1}$$

where ϕ denotes the effective porosity of the reservoir, ρ_o, S_o and ρ_w, S_w denote the density and saturations of oil and water, respectively. Moreover, v_o, p_o and v_w, p_w denote the superficial velocity and the pressure of the oil and the water phases, respectively. The system above is complemented with the natural physical constraint $S_o + S_w = 1$. Observe that we intentionally neglect the residual saturations for clarity of the presentation, since they do not bring any additional difficulty. Indeed, taking such

saturation into account amounts to replace the saturation S_o by the reduced saturation $S_o^\star = \frac{S_o - S_o^{min}}{S_o^{max} - S_o^{min}}$ and, equivalently, for S_w. In the overwhelming majority of applications, particularly in reservoir engineering, the oil and water phases velocities are assumed to obey Darcy's law. That is, v_o and v_w satisfy

$$v_o = -\bar{\bar{k}}\frac{kr_o}{\mu_o}\left(\nabla p_o - \rho_o\,\mathbf{g}\right),$$

$$v_w = -\bar{\bar{k}}\frac{kr_w}{\mu_w}\left(\nabla p_w - \rho_w\,\mathbf{g}\right),$$

$$(2)$$

where g denotes the gravity acceleration vector, and μ_o and μ_w their respective viscosities. $\bar{\bar{k}}$ denotes the absolute permeability tensor, and kr_o and kr_w denote the relative permeabilities of oil and water phases, respectively. As mentioned in the Introduction, this approach clearly neglects the effect of the water phase on the oil phase and *vice versa*. In this paper, we follow the argument of Kalaydjian [3] by introducing an explicit coupling into system (2).

From now on, for the sake of clarity of the presentation, we shall assume that the flow is one-dimensional, along an horizontal direction, say $x-$direction; that is, we set $\mathbf{g} = 0$ and we consider only the $x-$component of the velocities v_o and v_w. By abuse notation, we still denote the velocities v_o and v_w, and consider the following extended Darcy's system

$$v_o = -k\frac{kr_o}{\mu_o}\frac{\partial p_o}{\partial x} - k\frac{kr_{o,w}}{\mu_o}\frac{\partial p_w}{\partial x},$$

$$v_w = -k\frac{kr_w}{\mu_w}\frac{\partial p_w}{\partial x} - k\frac{kr_{w,o}}{\mu_w}\frac{\partial p_o}{\partial x}.$$

$$(3)$$

In the sequel, all terms in system (3), more precisely kr_o, kr_w, $kr_{o,w}$, and $kr_{w,o}$, are assumed to be functions depending only on the water saturation. Let us mention that kr_w, $kr_{o,w}$ can be interpreted as transport coefficients that can be determined experimentally as in [18]. Obviously, system (3) differs from the traditional formulation (2) by the inclusion of the pressure cross-terms. In addition to this extension, we shall consider the important concept of capillary pressure. Capillary pressure is defined as the difference between the pressure of the non-wetting phase (the oil here) and the wetting phase (water here) and given by

$$p_{c,o,w} = p_o - p_w. \tag{4}$$

The capillary force in a porous media depends mainly on pore size, wettability, and interfacial tension, and therefore on how the fluid is distributed in the pores. In practical situations, fluid and rock properties can be considered as constant, and therefore the capillary pressure can be reasonably assumed to be a function of the water saturation S_w only. In addition, the displacement system is assumed to be incompressible, i.e., oil, water, and formation are all incompressible, which in turn induces that ρ_o and ρ_w are constant (set $= 1$ for simplicity). Let us emphasize that this incompressibility condition provides a good approximation in many applications, such as the one at hand, namely the displacement of oil by water both in laboratory experiments and in field scale thanks to the very small, and therefore negligible, compressibility of the involved fluids. Moreover, the porosity ϕ will be assumed constant. These assumptions lead to the conclusion that the total velocity $v_t := v_o + v_w$ is constant. Specifically, the total velocity is independent of the space variable x for all time. In addition, the total volumetric flow rate $Av_t = Av_o + Av_w = q_o + q_w =: q_t$ through the flow direction is also independent of time. Indeed, summing up both PDEs of system (1), we obtain

$$\phi\frac{\partial(S_o + S_w)}{\partial t} = -\frac{\partial(v_o + v_w)}{\partial x} = 0.$$

Since $S_o + S_w = 1$, we infer the desired property.

3. Buckley–Leverett Theory

3.1. Buckley–Leverett Equations

Following Willhite [26], we introduce the concept of fractional flow defined as the volumetric flux fraction of the phase that is flowing at position x and time t. That is, the water fractional flow is given

$$f_w = \frac{q_w}{q_t} = \frac{v_w}{v_t},$$

and equivalently for f_o. It is rather easy to see that $f_w + f_o = 1$. Therefore, in the sequel, we shall focus on finding an explicit expression of the water fractional flow in terms of the water saturation and the capillary pressure. For this purpose, let us first introduce the oil and water mobilities:

$$M_o = \frac{kr_o}{\mu_o}, \quad M_w = \frac{kr_w}{\mu_w}, \tag{5}$$

respectively. In addition, let us introduce the cross-mobilities $M_{o,w} = \frac{kr_{o,w}}{\mu_o}$ and $M_{w,o} = \frac{kr_{w,o}}{\mu_w}$ so that, thanks to the definition (4), system (3) can be recast as follows:

$$
\begin{aligned}
v_o &= kM_{o,w}\partial_x p_{c,o,w} - k(M_{o,w} + M_o)\partial_x p_o, \\
v_w &= kM_w\partial_x p_{c,o,w} - k(M_w + M_{w,o})\partial_x p_o.
\end{aligned}
\tag{6}
$$

Recalling that $v_t = v_o + v_w$, we can write

$$\partial_x p_o = \frac{k(M_{o,w} + M_w)\partial_x p_{c,o,w} - v_t}{k(M_{o,w} + M_o + M_{w,o} + M_w)}. \tag{7}$$

Substituting the expression of the pressure gradient (7) in system (6), we infer

$$
\begin{aligned}
v_o &= \frac{k(M_{o,w}M_{w,o} - M_oM_w)\partial_x p_{c,o,w} + (M_o + M_{o,w})v_t}{M_{o,w} + M_o + M_{w,o} + M_w}, \\
v_w &= \frac{k(M_oM_w - M_{o,w}M_{w,o})\partial_x p_{c,o,w} + (M_w + M_{w,o})v_t}{M_{o,w} + M_o + M_{w,o} + M_w}.
\end{aligned}
\tag{8}
$$

Therefore, the water fractional flow is given by

$$f_w = \frac{M_oM_w - M_{o,w}M_{w,o}}{M_{o,w} + M_o + M_{w,o} + M_w}\frac{k\,\partial_x p_{c,o,w}}{v_t} + \frac{M_w + M_{w,o}}{M_{o,w} + M_o + M_{w,o} + M_w}. \tag{9}$$

Observe that neglecting the capillary pressure reduces the water fractional flow to an analytic expression in terms of the water saturation since we assume all the permeabilities, $M_o, M_w, M_{o,w}$ and $M_{w,o}$ depend only on the water saturation. By setting $p_{c,o,w} = 0$, it follows from (9) that

$$f_w = \frac{M_w + M_{w,o}}{M_{o,w} + M_o + M_{w,o} + M_w}. \tag{10}$$

Therefore, neglecting the capillary pressure amounts to considering a modified classical Darcy system. Indeed, in this scenario, it holds that $\partial_x p_o = \partial_x p_w$ because $p_{c,o,w} = 0$ for all x, and therefore the previous expression corresponds to the water fractional flow associated with the following Darcy system:

$$v_o = -\frac{k(kr_o + kr_{o,w})}{\mu_o}\frac{\partial p_o}{\partial x},$$

$$v_w = -\frac{k(kr_w + kr_{w,o})}{\mu_w}\frac{\partial p_w}{\partial x},$$

Obviously, this system is not physical, in the *stricto sensu* of Darcy's law, since the permeabilities are modified compared to the original system (2). This suggests that the coupling between phases makes sense only when the capillary pressure is taken into consideration.

The Buckley–Leverett equation introduced by Buckley and Leverett in 1942 [20], also known as the frontal-advance equation, is the simplest equation for the description of one-dimensional, immiscible displacement in a linear reservoir. Indeed, it enables the recovery of the velocity of the water saturation motion from the derivative of the fractional flow with respect to the water saturation; specifically, it uses the fractional flow derivative to localize the movement of the sharp saturation front x through the linear reservoir. This equation (referred to as front-advance equation) reads as follows:

$$\left.\frac{dx}{dt}\right|_{S_w} = \frac{q_t}{A\phi}\left.\frac{\partial f_w}{\partial S_w}\right|_t \tag{11}$$

This equation states that, in a linear displacement process, a particular water saturation moves in the porous medium at a velocity that can be evaluated from the derivative of the fractional flow with respect to the water saturation. It is worth mentioning that this equation infers that, if the water fractional flow depends only on the water saturation, then this equation can be integrated to obtain analytical solution. Now, let us introduce the notation

$$\hat{f}_w = \frac{M_o M_w - M_{o,w} M_{w,o}}{M_{o,w} + M_o + M_{w,o} + M_w} \; ; \; \tilde{f}_w = \frac{M_w + M_{w,o}}{M_{o,w} + M_o + M_{w,o} + M_w}. \tag{12}$$

Thanks to (9) and the chain rule

$$\frac{\partial p_{c,o,w}}{\partial x} = \frac{dp_{c,o,w}}{dS_w}\frac{\partial S_w}{\partial x}, \tag{13}$$

we can write

$$\frac{\partial f_w}{\partial S_w} = \frac{k}{v_t}\frac{\partial S_w}{\partial x}\left\{\frac{\partial \hat{f}_w}{\partial S_w}\frac{dp_{c,o,w}}{dS_w} + f_w^1 \frac{d^2 p_{c,o,w}}{d^2 S_w}\right\} + \frac{\partial \tilde{f}_w}{\partial S_w}. \tag{14}$$

It is worth mentioning that, although the capillary pressure and all mobilities are assumed to depend only on the water saturation S_w, the expression (14) is not analytic in S_w because of the derivative of the water saturation with respect to space $\frac{\partial S_w}{\partial x}$. However, if the capillary pressure is neglected, the first term on the right-hand side of (14) vanishes, and $\tilde{f}_w$ coincides with f_w in (10). Therefore, there is no contribution of $\frac{\partial S_w}{\partial x}$ to the derivative of the water fractional flow, and its derivative is then analytic in S_w. Thus, the front advance Equation (11) can be integrated to obtain an analytic expression of the water front position to be used for oil recovery calculation purposes, EOR estimates etc.

In the sequel, we consider the following relative permeabilities, as proposed by Brooks and Corey [27]. In particular, for all S_w such that $S_{wc} \leq S_w \leq 1 - S_{or}$, we let

$$kr_w = k_{rw}^{max}\left(\frac{S_w - S_{wc}}{1 - S_{wc} - S_{or}}\right)^{n_w}, \quad kr_o = k_{ro}^{max}\left(\frac{1 - S_w - S_{or}}{1 - S_{wc} - S_{or}}\right)^{n_o}, \tag{15}$$

where S_{wc} denotes the connate and S_{or} is the residual oil saturation. k_{rw}^{max} and k_{ro}^{max} denote the maximal relative permeabilities for water and oil. n_w and n_o denote real numbers to fit observed laboratory data. The coupling terms in the system (3) are chosen as

$$kr_{o,w} = (S_w - S_{wc})^{n_{o,w}} (1 - S_w - S_{or})^{n_{w,o}}, \quad kr_{w,o} = (S_w - S_{wc})^{n_{o,w}} (1 - S_w - S_{or})^{n_{w,o}}, \quad (16)$$

where $n_{o,w}$ and $n_{w,o}$ depend on laboratory observations. The capillary pressure is assumed to depend only on the water saturation as follows (Van Genuchten capillary pressure [28]):

$$p_{c,o,w} = \tau(Se^{-1/m} - 1)^{1-m}; \; with \; S_e = \frac{S_w - S_{wc}}{1 - S_{wc} - S_{or}}$$

Observe that the capillary pressure (Figure 1) is a decreasing function of S_w, and therefore its derivative with respect to S_w is negative and is given for all $S_{wc} \leq S_w \leq 1 - S_{or}$ by

$$\frac{dp_{c,o,w}}{dS_w} = -\frac{1-m}{m}\tau \left(\frac{S_e^{-1-\frac{1}{m}}}{1 - S_{wc} - S_{or}} (S_e^{-1/m} - 1)^{-m} \right). \quad (17)$$
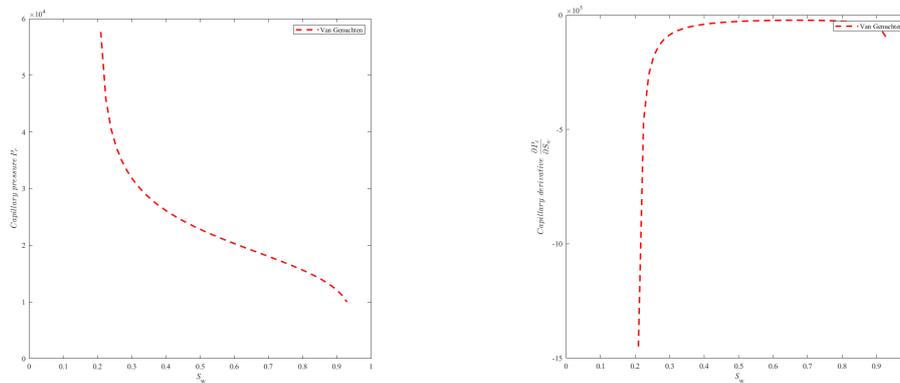
Figure 1. Van Genuchten capillary pressure and its derivative with respect to water saturation versus water saturation.

It is worth mentioning that the water fractional flow expression (9) involves the derivative with respect to space of the capillary pressure. Using the chain rule (13), one can infer that $\frac{\partial p_{c,o,w}}{\partial x} > 0$ thanks to the typical profile of the water saturation in terms of x as shown in Figure 2.
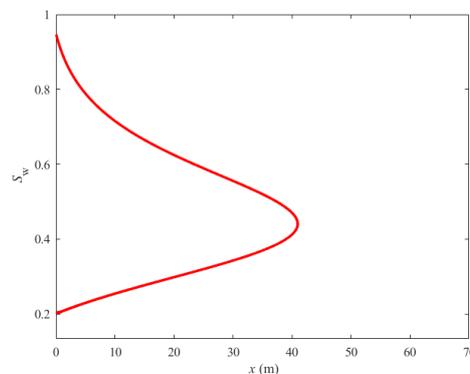
Figure 2. Typical water saturation profile versus space position x.

Since the capillary pressure contributes to the water saturation dynamics, as per Equation (14), one can expect that its maximum influence is reached at a maximum of its derivative with respect to the water saturation, which corresponds to the zero of the expression (17), in particular at

$$S_w = S_{wc} + \frac{1 - S_{wc} - S_{or}}{(1 + m)^m}$$

Therefore, one approach could be to neglect the capillary pressure and consider it only *close* to the front where it is expected to have significant contribution. In addition, thanks to the sign of the gradient of the capillary pressure, one can expect that this enhances the front penetration velocity and introduces some diffusion to the system which in turn removes the sharp singularity at the front of the Buckley–Leverett solution. Our main interest in this paper is the comparison between the coupled system given by (3) and the classical Darcy's approach described in (2), and therefore we both respective solutions presented below are obtained numerically using a Runge–Kutta approach.

To determine the front position, the equation to be solved reads

$$\left. \frac{dx}{dt} \right|_{S_w} = \frac{q_t}{A\phi} \left\{ \frac{k}{v_t} \frac{\partial S_w}{\partial x} \left\{ \frac{\partial f_w^1}{\partial S_w} \frac{\partial p_{c,o,w}}{\partial S_w} + f_w^1 \frac{\partial^2 p_{c,o,w}}{\partial^2 S_w} \right\} + \frac{\partial f_w^2}{\partial S_w} \right\} \right|_t \tag{18}$$

This equation can be solved using classical numerical methods by discretizing in time, space and water saturation. It is rather well-known that the difficulty resides in the fact that the typical water fractional flow curve has an inflection point, which provides two values of water saturation at the same time, since $\frac{df_w}{dS_w}$ will reach a maximum value ((see Figure 3) for typical examples with capillary pressure).
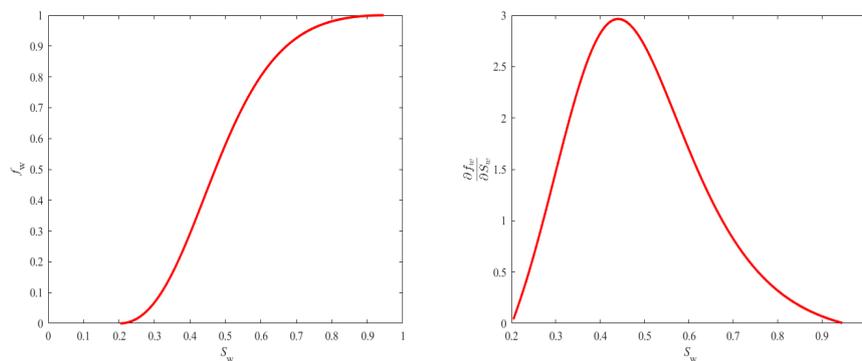
Figure 3. Typical fractional flow curve with respect to water saturation (**left**), and its derivative with respect to water saturation (**right**) versus water saturation.

One of the most commonly used methods to estimate the position of the schok water saturation front, solution of the Buckley–Leverett equation, is the simple and elegant Welge graphical method, [21]. The basic idea of the Welge's method is based on the observation that, starting from the injection point, traveling velocities of a saturation will increase, with an increase in saturation initially from its connate value, until the curve of the derivative of the water fractional flow with respect to the water saturation attains a maximum value. Therefore, the points where the curve of the fractional flow indicates a decreasing velocity with increasing saturation, a significant change is needed that is a schok is formed. From the graphical point of view, this method, using Welge's method to localize the front water saturation S_{wf}, amounts to determining the coordinates of to the point $(S_{wf}, f_w(S_{wf}))$ such that the line passing through this point and the connate water saturation point $(S_{wc}, 0)$ is tangent to the fractional flow curve. Let us mention that we shall use the Welge's method only for comparison purposes between different scenarios to see graphically that the front water saturation position changes when considering different values of the coupling cross-terms. However, for the the determination of the breakthrough time, we solve Buckley–Leverett numerically. Observe that, in the presence of

capillary pressure, there is no schok since the capillary pressure brings in some diffusion that removes the sharp singularity at the front of Buckley–Leverett equation.

In the sequel, we consider a one-dimensional, flow direction simulation with an horizontal reservoir which has a uniform cross-area A. We provide two scenarios

- Comparison: classical Darcy's system vs coupled system with $kr_{o,w} = kr_{w,o}$ (see (19))
- Effect of non-symmetric coupling: coupled system with $kr_{o,w} = \epsilon kr_{w,o}$ (we refer to ϵ as tolerance). For more examples, see (see (20))

In what follows, we shall use two sets of data. The first is theoretical and aims at showing the difference between both models. The second set of data is dedicated to a physically realistic case. In particular, we shall consider the following data:

(a) First set of data

Parameters	Value	Unit
Length of formation, L	1000.0	[m]
Cross-area of reservoir, A	10^4	[m^2]
Absolute permeability, k	2.96×10^{-13}	[m^2]
Oil phase viscosity, μ_o	5×10^{-3}	[Pa.s]
Water phase viscosity, μ_w	10^{-3}	[Pa.s]
Oil density, ρ_o	0.8×10^3	[kg/m^3]
Water density, ρ_w	10^3	[kg/m^3]
Initial water injection rate, q_t	400	[m^3/s]
Porosity, ϕ	0.30	[-]
Residual oil saturation, S_{or}	0.25	[-]
Connate water saturation, S_{wc}	0.20	[-]
Maximal relative permeability for oil, k_{ro}^{max}	0.8	[-]
Maximal relative permeability for water, k_{rw}^{max}	0.5	[-]
Power index of water relative permeability n_w	2.00	[-]
Power index of oil relative permeability, n_o	2.00	[-]
Power index of first coupling term $kr_{o,w}$, $n_{o,w}$	2.00	[-]
Power index of second coupling term $kr_{w,o}$, , $n_{w,o}$	2.00	[-]

(b) Second set of data

Parameters	Value	Unit
Length of formation, L	3280	[ft]
Cross-area of reservoir, A	$10,764 \times 10^4$	[ft^2]
Absolute permeability, k	300	[mD]
Oil phase viscosity, μ_o	5	[cP]
Water phase viscosity, μ_w	1	[cP]
Oil density, ρ_o	22.653	[kg/ft^3]
Water density, ρ_w	28.316	[kg/ft^3]
Initial water injection rate, q_t	2515.924	[bbl/s]
Porosity, ϕ	0.30	[-]
Residual oil saturation, S_{or}	0.25	[-]
Connate water saturation, S_{wc}	0.20	[-]
Maximal relative permeability for oil, k_{ro}^{max}	0.8	[-]
Maximal relative permeability for water, k_{rw}^{max}	0.5	[-]
Power index of water relative permeability n_w	2.00	[-]
Power index of oil relative permeability, n_o	2.00	[-]
Power index of first coupling term $kr_{o,w}$, $n_{o,w}$	2.00	[-]
Power index of second coupling term $kr_{w,o}$, , $n_{w,o}$	2.00	[-]

We shall consider the following two cases:

1. The case of $kr_{o,w} = kr_{w,o}$, we choose

$$kr_{o,w} = kr_{w,o} = (S_w - S_{wc})^2 (1 - S_w - S_{or})^2 \tag{19}$$

2. The case of $kr_{o,w} = \varepsilon kr_{w,o}$ we choose:

$$kr_{o,w} = (S_w - S_{wc})^2 (1 - S_w - S_{or})^2, \ kr_{w,o} = \frac{7}{5} (S_w - S_{wc})^2 (1 - S_w - S_{or})^2$$

$$kr_{o,w} = (S_w - S_{wc})^2 (1 - S_w - S_{or})^2, \ kr_{w,o} = 2 (S_w - S_{wc})^2 (1 - S_w - S_{or})^2 \tag{20}$$

3.2. Numerical Results

In this section, we present few numerical simulations aiming at comparing the introduced coupled system (3) and the classical Darcy's approach given by (2). The initial data, total constant velocity, and the system's parameters are chosen as in the previous tables. The first example describes a comparison between the solution of the classical Darcy's system (2) and the solution on the coupled Darcy's system (3) by varying the tolerance ε. For the first scenario of $kr_{w,o} = \varepsilon \, kr_{o,w}$, we choose the following values for the tolerance ε,

$$\varepsilon = \{2, 4, 6\}.$$

Figure 4 shows the classical graph of kr_o, kr_w, and $(kr_{w,o})_i$, i = 1,...,3, corresponding to the tolerances above.
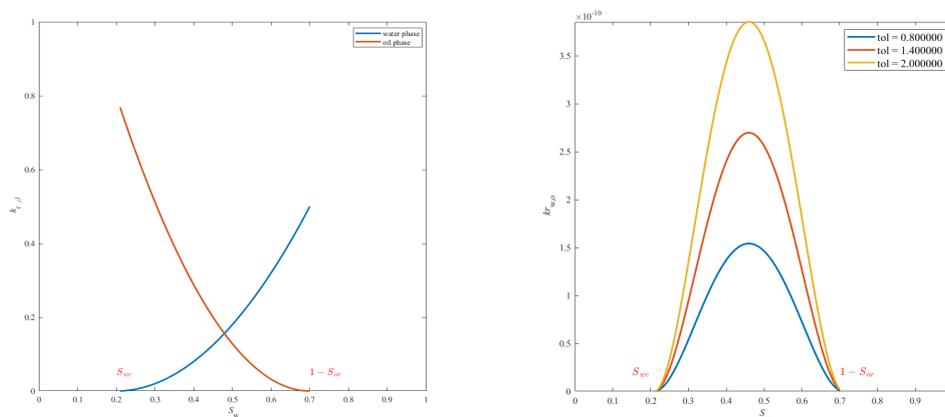
Figure 4. (**Left**) Relative permeability kr_o and kr_w in terms of water saturation. (**Right**) Coupling term permeability $kr_{w,o}$ versus water saturation.

Figure 5 shows the fractional flow graph associated with the classical Darcy's system together with the one associated with the coupled system for three values of tolerances as indicated above. For instance, if one chooses to use the Welge approach to locate the front, that is, if we trace a line passing through the saturation S_{wc}, corresponding to the beginning of the water injection, and tangent to the fractional flow, we observe different values of S_{wf} clearly affecting the water front position and therefore its advance.

The associated water saturation profiles are shown in Figure 6. We observe a more advanced water saturation front position compared to the one corresponding to system (2) for all values of tolerances. The front position is in advance of approximatively 1.77%, 3.55%, and 5.14% after 1500 days, respectively to the chosen tolerances, compared to the classical Darcy's one. Such differences clearly impact the volume of oil recovered from the reservoir.
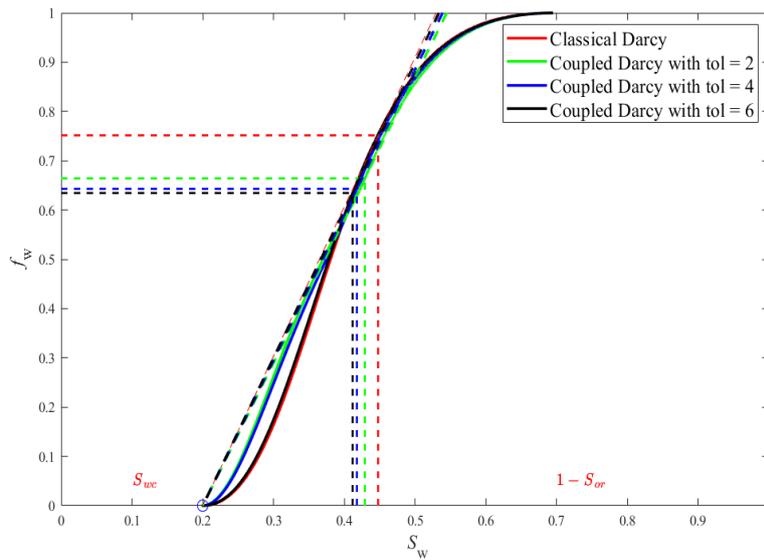
Figure 5. Fractional flow f_w in terms of water saturation (Time = 1500 days).
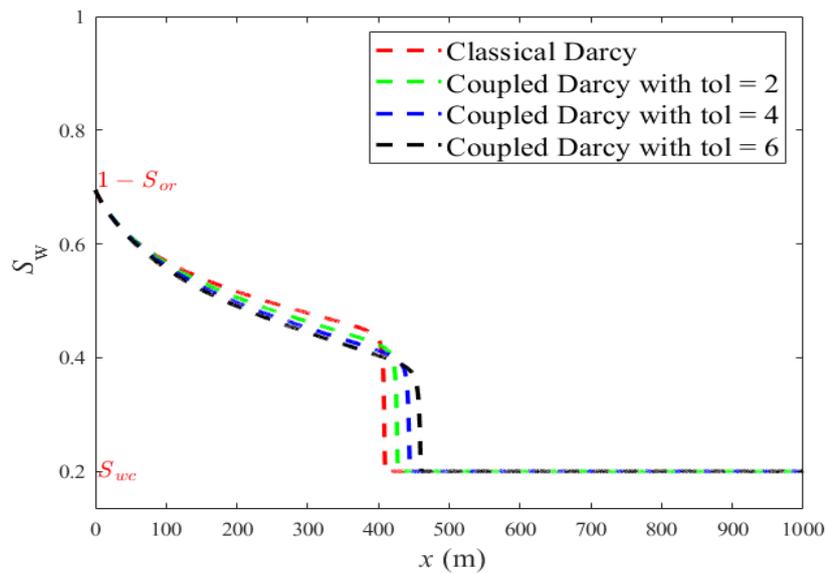
Figure 6. Water Saturation profile S_w in terms of the distance from the inlet (Time = 1500 days).

To emphasize the effect of coupling terms on the displacement of the two phase flow, a second scenario is considered. We assume that the permeabilities $kr_{o,w}$ and $kr_{w,o}$ are equal, and we consider the second set of data given in the table above. Figure 7 represents the graph of the water fractional flow for both systems (classical and coupled). The figure shows a difference in the water saturation front S_{wf} obtained by Welge's method, [21]. Therefore, one can see that, even in the case of $kr_{o,w} = kr_{w,o}$, a difference in the water saturation front position between both systems appears. Observe that, in this case, the cross-terms are not equal (see below for equal cross-terms). Such difference can be clearly seen in the graph of the respective water saturation profiles presented in Figure 8.
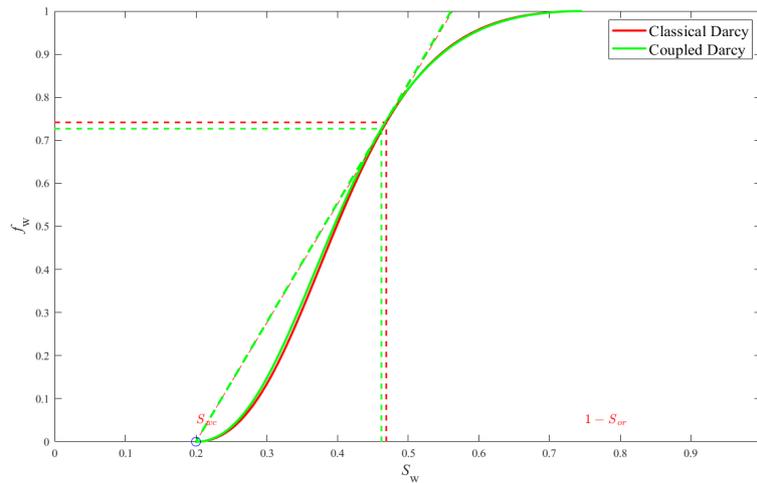
Figure 7. Fractional flow curve in terms of water saturation S_w (Time = 1500 days).
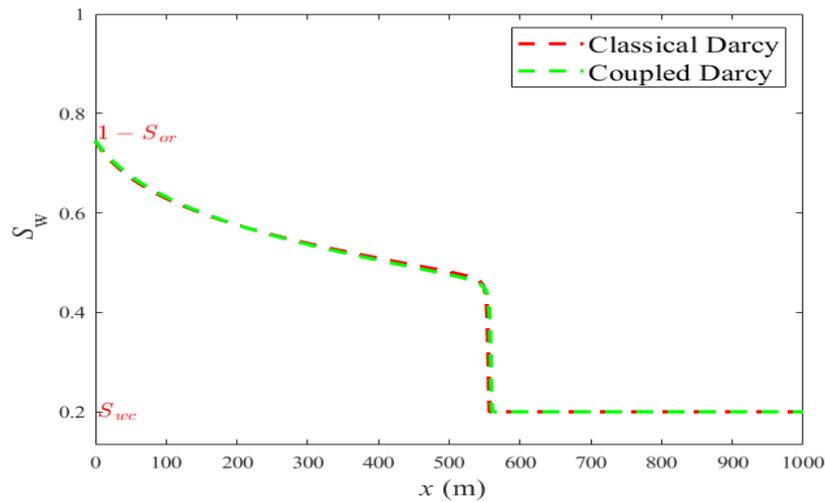
Figure 8. Water Saturation profile S_w in terms of distance from inlet (Time = 1500 days).

Eventually, let us consider the case of the specific tolerance ε given by

$$\varepsilon = \frac{\mu_w}{\mu_o},$$

which implies that the cross-terms coefficients are equal (cross-mobilities). In this scenario, Figure 9 shows that there is no difference in the water saturation front position which is rather intuitive and expected.

All in all, for a given displacement system with constant water injection rate, the solution of the Equation (11), corresponding to the coupled system (3) that is the water saturation front, depends clearly on the amplitude of the cross-terms and not only on the pressure gradients. Indeed, Figures 6 and 8 show a consequent difference, compared to the classical Darcy's system (2) in terms of water front position compared to Figure 9.
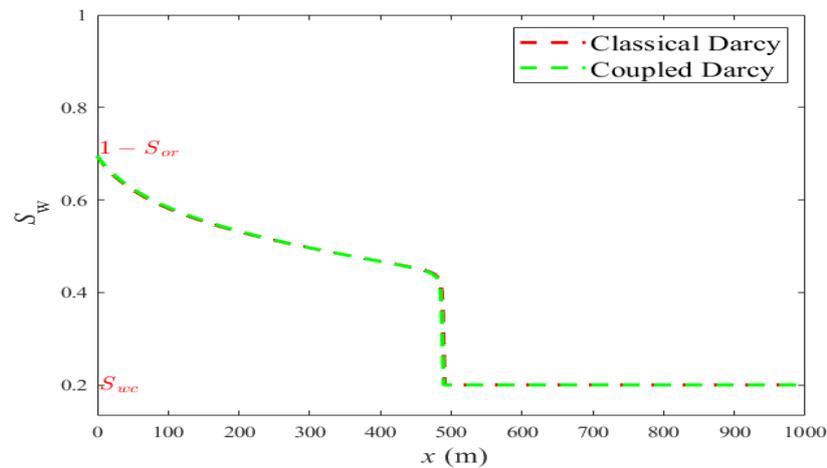
Figure 9. Water Saturation profile S_w in terms of distance from inlet (Time = 1500 days).

As an application of the previous results, we compute the breakthrough time for both scenarios using the following formula:

$$t_L = \frac{A\phi x_L}{q_t \left. \frac{\partial f_w}{\partial S_w}\right|_{S_{wf}}} \tag{21}$$

For the first scenario:

Classical Darcy	Tolerance $\varepsilon = 0.8$	Tolerance $\varepsilon = 1.4$	Tolerance $\varepsilon = 2$
$t_L = 2.7219 \times 10^3$ days	$t_L = 2.5843 \times 10^3$ days	$t_L = 2.5397 \times 10^3$ days	$t_L = 2.5024 \times 10^3$ days

For the second scenario:

Classical Darcy	Coupled Darcy with $kr_{w,o} = kro, w$
$t_L = 2.7219 \times 10^3$ days	$t_L = 2.7071 \times 10^3$ days

4. Discussion

4.1. On the Fractional Flow

As shown in the previous section, the cross-terms affect the water front position in the reservoir, and therefore play a role in the description of the flow dynamic when compared to the classical multiphase Darcy's approach (2). In this section, we make few comments on the fractional water flow expression associated with the coupled system (3), and given by (9). It is rather clear that (9) can be reformulated as follows:

$$f_w = \frac{\left[1 - \frac{M_{o,w}}{M_o}\frac{M_{w,o}}{M_w}\right]\frac{k M_o}{v_t}\partial_x p_{c,o,w} + \left[1 + \frac{M_{w,o}}{M_w}\right]}{\left(1 + \frac{M_o}{M_w}\right)\left[1 + \frac{M_{o,w}+M_{w,o}}{M_o+M_w}\right]}. \tag{22}$$

The classical Darcy's approach can be seen as a limiting, or particular case, of the coupled multiphase approach we developed in this paper. Indeed, setting $M_{o,w} = M_{w,o} = 0$ reduces system (3) to the classical Darcy's system (2) (with **g** = 0), and the fractional flow Formula (22)

reduces to the well-known water fractional flow expression associated with system (2). More precisely, setting $M_{o,w} = M_{w,o} = 0$ in (9), we get

$$f_w^{\text{Darcy}} = \frac{M_o M_w \frac{k \partial_x p_{c,o,w}}{v_t} + M_w}{M_o + M_w} = \frac{\frac{k M_o}{v_t} \partial_x p_{c,o,w} + 1}{1 + \frac{M_o}{M_w}}. \tag{23}$$

This is a well-known formula when capillary pressure is considered in the classical Darcy's approach (2). Clearly, when the capillary pressure is neglected, the formula reduces to

$$f_w^{\text{Darcy}} = \frac{1}{1 + \frac{M_o}{M_w}}. \tag{24}$$

Comparing the expressions (22) and (23), we see that the contribution of the coupling cross-terms in system (3) is described through the three terms between the square brackets. The expression (22) raises the natural question of the choices $M_{w,o} = M_w$, and $M_{o,w} = M_o$ leading to

$$f_w = \frac{M_w}{M_o + M_w} = \frac{1}{1 + \frac{M_o}{M_w}}, \tag{25}$$

corresponding to the classical Darcy's water fractional flow (24) when the capillary pressure is neglected or equal zero. This formula can be justified using a rescaling argument. Indeed, in the case of $M_{w,o} = M_w$, and $M_{o,w} = M_o$, system (3) turns out to be

$$v_o = -k M_o \partial_x p_o - k M_o \partial_x p_w,$$
$$v_w = -k M_w \partial_x p_w - k M_w \partial_x p_o.$$

Using the capillary pressure definition, we can write

$$v_o = -2k M_o \partial_x p_o + k M_o \partial_x p_{c,o,w} = -k M_o \partial_x \left(2p_o - p_{c,o,w} \right),$$
$$v_w = -2k M_w \partial_x p_w - k M_w \partial_x p_{c,o,w} = -k M_w \partial_x \left(2p_w + p_{c,o,w} \right).$$

Denoting $\tilde{p}_o := 2p_o - p_{c,o,w}$, and $\tilde{p}_w := 2p_w + p_{c,o,w}$, this system turns out to be the classical Darcy's system with oil and water phases velocities given by v_o and v_w, respectively, but with oil and water phases pressures given by $\tilde{p}_o$ and $\tilde{p}_w$, respectively. Therefore, the associated water fractional flow is given by the classical formula:

$$f_w^{\text{Darcy}} = \frac{1 + \frac{k M_o}{v_t} \partial_x \tilde{p}_{c,o,w}}{1 + \frac{M_o}{M_w}} = \frac{1 + \frac{k M_o}{v_t} \partial_x \underbrace{\left(2p_o - p_{c,o,w} - 2p_w - p_{c,o,w} \right)}_{=0}}{1 + \frac{M_o}{M_w}},$$

which justifies the Formula (25). In particular, this infers that one can perform post calculation based on the classical Darcy's system (2) (with **g** = 0) to include the effect of each phase on the other in the case of cross-terms chosen such that $M_{w,o} = M_w$, and $M_{o,w} = M_o$, which is in the case of

$$kr_{w,o} = kr_w \quad \text{and} \quad kr_{o,w} = kr_o,$$

under the assumption that the capillary pressure depends only on the water saturation. Indeed, let (v_o, p_o) and (v_w, p_w) be as in system (2) (with $\mathbf{g} = 0$), thus $(v_o, \tilde{p}_o = \frac{p_o + p_{c,o,w}}{2})$ and $(v_w, \tilde{p}_w = \frac{p_w + p_{c,o,w}}{2})$ satisfy the system

$$v_o = -kM_o\partial_x\tilde{p}_o - kM_o\partial_x\tilde{p}_w,$$

$$v_w = -kM_w\partial_x\tilde{p}_w - kM_w\partial_x\tilde{p}_o,$$

and we refer to Section 4.2 for a general algorithm scheme allowing to take into account any cross-terms that is independent of M_o and M_w (that is, kr_o and kr_w).

A situation that might be physically relevant is to consider the $M_{w,o} = 0$ and $M_{o,w} \neq 0$ leading to the following semi-coupled system:

$$v_o = -kM_o\partial_x p_o - kM_{o,w}\partial_x p_w,$$

$$v_w = -kM_w\partial_x p_w.$$

Thanks to (22), the associated water fractional flow expression reads

$$f_w = \frac{1}{\left[1 + \frac{M_{o,w}}{M_o + M_w}\right]} \frac{\frac{kM_o}{v_t}\partial_x p_{c,o,w} + 1}{1 + \frac{M_o}{M_w}} < \frac{\frac{kM_o}{v_t}\partial_x p_{c,o,w} + 1}{1 + \frac{M_o}{M_w}},$$

for all water saturation S_w since $1 + \frac{M_{o,w}}{M_o + M_w} > 1$. Since the right-hand side of the latter inequality corresponds to the water fractional flow associated with the classical Darcy's approach, one can argue that the derivative of f_w with respect to the saturation S_w will reach a smaller maximum than any classical Darcy's system fractional flow. In particular, this means that, if we consider the same value for S_{wc} for both systems, and use Welge's approach, then the slope of the tangent to f_w passing through S_{wc} is smaller than the slope of Darcy's classical system, suggesting a larger value for S_{wf} in the case of the semi-coupled system. This is coherent with the fact that, if $M_{o,w} \ll M_o + M_w$ (in the sense of $M_{o,w}$ is very small compared to $M_o + M_w$), then the factor $(1 + M_{o,w}/(M_o + M_w))^{-1}$ is very close to 1, and therefore the effect of the semi-coupling term is minimal compared to the classical Darcy's system.

In addition, observe that Formula (22) suggests, at least formally, that, if $M_{o,w} \ll M_o$ and $M_{w,o} \ll M_w$ at any space position x in the reservoir and time t, then the graph of the water fractional flow of the coupled system (3) will be very close to the graph of the water fractional flow corresponding to the classical Darcy's approach (2). Thus, one can reasonably expect that the slope of their respective tangents passing through S_{wc} will be very close, and therefore so is the value of S_{wf}. This observation makes sense from the physical point of view, since such small coupling cross-terms are expected to have a very small effect and impact on the dynamic of the system, and can be seen as a very small perturbation of the classical Darcy's system (2) (with $\mathbf{g} = 0$).

4.2. Solution of Coupled System by a Decoupling Approach

Most of the industrial codes use the classical Darcy approach (2) to model multiphase flows, and any deviation from this scenario will be immediately termed "non Darcy flow". In this section, we present a method allowing any classical existing code or software based on the classical Darcy's approach to take into account the coupling between phases. The idea consists of solving the classical Darcy's system (2) with modified permeabilities (or mobilities), and use the latter solution to obtain a second solution taking into account the effect of every phase on the other. Below, we present the idea in the case of two phases flow, for simplicity, but obviously it can be generalized to the case of more involved phases and higher dimensions. Following the notation of the previous section,

we aim to solve system (1) where the oil and water velocities are described by the following modified Darcy's system:

$$v_o = -k\overline{M}_o\, \partial_x p_o,$$
$$v_w = -k\overline{M}_w\, \partial_x p_w,$$

(26)

where we used the notation $\overline{M}_o = M_o + M_{w,o}$ and $\overline{M}_w = M_w + M_{o,w}$. This is nothing but Darcy's classical system (2) (with $\mathbf{g} = 0$) with modified permeabilities, that is, M_o and M_w are replaced by $\overline{M}_o$ and $\overline{M}_w$, respectively. Therefore, following the classical approach, a straightforward calculation shows that

$$\partial_x p_o = \frac{k\overline{M}_w \partial_x p_{c,o,w} - v_t}{k(\overline{M}_o + \overline{M}_w)}, \quad \text{and} \quad \partial_x p_w = -\frac{k\overline{M}_o \partial_x p_{c,o,w} + v_t}{k(\overline{M}_o + \overline{M}_w)},$$

(27)

inferring the following water and oil velocities' expressions

$$v_o = \frac{-k\overline{M}_w \partial_x p_{c,o,w} + v_t}{1 + \frac{\overline{M}_w}{\overline{M}_o}}, \quad \text{and} \quad v_w = \frac{k\overline{M}_o \partial_x p_{c,o,w} + v_t}{1 + \frac{\overline{M}_o}{\overline{M}_w}}.$$

Specifically, the water fractional flow expression reads

$$f_w = \frac{\frac{k\overline{M}_o}{v_t} \partial_x p_{c,o,w} + 1}{1 + \frac{\overline{M}_o}{\overline{M}_w}}.$$

(28)

Since all the mobilities and the capillary pressure are assumed to depend only on the water saturation, using this fractional expression the system (11) can be solved and a Buckley–Leverett solution obtained using, for instance, Welge's approach or any numerical scheme.

In the sequel, using a velocity rescaling argument, we show that the fractional flow (28) determines uniquely the fractional flow associated with the coupled system (3). Hence, in order to solve (3), any existing code or software can be used to determine the fractional flow (and its graph) associated with the modified Darcy's system (26), and therefore obtain the fractional flow (and its graph) associated (3), and eventually apply, Welge's method, for instance, can be used to determine the water front. Indeed, we have the following

Lemma 1. *If (v_o, p_o) and (v_w, p_w) satisfy system (26), then $(\tilde{v}_o, p_o)$ and $(\tilde{v}_w, p_w)$ given by*

$$\begin{cases} \tilde{v}_o = v_o + \dfrac{k\left(M_{o,w}\overline{M}_o + M_{w,o}\overline{M}_w\right)\partial_x p_{c,o,w} + (M_{o,w} - M_{w,o})v_t}{\overline{M}_o + \overline{M}_w}, \\[3mm] \tilde{v}_w = v_w - \dfrac{k\left(M_{o,w}\overline{M}_o + M_{w,o}\overline{M}_w\right)\partial_x p_{c,o,w} + (M_{o,w} - M_{w,o})v_t}{\overline{M}_o + \overline{M}_w}, \end{cases}$$

(29)

satisfy the following system

$$\tilde{v}_o = -kM_o\partial_x p_o - kM_{o,w}\partial_x p_w,$$
$$\tilde{v}_w = -kM_w\partial_x p_w - kM_{w,o}\partial_x p_o.$$

(30)

Proof. First, observe that $\tilde{v}_o + \tilde{v}_w = v_o + v_w = v_t$ which is constant as per the incompressibility property preserved by the flow. On the one side, thanks to (27), we can write

$$\frac{k\overline{M}_w \partial_x p_{c,o,w}}{\overline{M}_o + \overline{M}_w} = k\partial_x p_o + \frac{v_t}{\overline{M}_o + \overline{M}_w}, \quad \text{and} \quad \frac{k\overline{M}_o \partial_x p_{c,o,w}}{\overline{M}_o + \overline{M}_w} = -k\partial_x p_w - \frac{v_t}{\overline{M}_o + \overline{M}_w}.$$

On the opposite side, following $(29)_1$, we have

$$
\begin{aligned}
\tilde{v}_o &= v_o + M_{o,w}\frac{k\overline{M}_o\partial_x p_{c,o,w}}{\overline{M}_o + \overline{M}_w} + M_{w,o}\frac{k\overline{M}_w\partial_x p_{c,o,w}}{\overline{M}_o + \overline{M}_w} + \frac{M_{o,w} - M_{w,o}}{\overline{M}_o + \overline{M}_w}v_t \\
&= -k\overline{M}_o\,\partial_x p_o + M_{o,w}\left(-k\partial_x p_w - \frac{v_t}{\overline{M}_o + \overline{M}_w}\right) + M_{w,o}\left(k\partial_x p_o + \frac{v_t}{\overline{M}_o + \overline{M}_w}\right) + \frac{M_{o,w} - M_{w,o}}{\overline{M}_o + \overline{M}_w}v_t \\
&= -k\overline{M}_o\,\partial_x p_o - kM_{o,w}\partial_x p_w + kM_{w,o}\partial_x p_o \\
&= -k(M_o + M_{w,o})\partial_x p_o - kM_{o,w}\partial_x p_w + kM_{w,o}\partial_x p_o \\
&= -kM_o\partial_x p_o - kM_{o,w}\partial_x p_w,
\end{aligned}
$$

which is the desired result. The calculation for $\tilde{v}_o$ uses the change of variable given by $(29)_1$ and the same argument we used above, then we omit it.　□

In particular, the previous lemma allows us to write a relation between fractional flows of both system. In fact, using the Formula (28), we have

$$
\begin{aligned}
\tilde{f}_w &= f_w - \frac{\left(M_{o,w}\overline{M}_o + M_{w,o}\overline{M}_w\right)}{\overline{M}_o + \overline{M}_w}\frac{k\partial_x p_{c,o,w}}{v_t} - \frac{M_{o,w} - M_{w,o}}{\overline{M}_o + \overline{M}_w} \\
&= f_w - \frac{\left(M_{o,w}\overline{M}_o + M_{w,o}\overline{M}_w\right)}{\overline{M}_o + \overline{M}_w}\left(\frac{\overline{M}_o + \overline{M}_w}{\overline{M}_o\overline{M}_w}f_w - \frac{1}{\overline{M}_o}\right) - \frac{M_{o,w} - M_{w,o}}{\overline{M}_o + \overline{M}_w} \\
&= \frac{M_o M_w - M_{o,w}M_{wo}}{(M_o + M_{w,o})(M_w + M_{o,w})}f_w + \frac{M_{w,o}}{M_o + M_{w,o}}.
\end{aligned} \tag{31}
$$

Now, any existing code or software can easily solve system (1) with the oil and water velocities given by system (26), by only modifying the permeabilities input, which are assumed to depend analytically on the water saturation. Next, the value of the fractional flow of the coupled system (3) can be post calculated using the Formula (31). Eventually, any standard method, for instance analytical or Welge's method, can be used to find the water front corresponding to the coupled system (3).

Observe that, setting formally the cross-terms $M_{o,w} = M_{w,o}$ in (31), we obtain $\tilde{f}_w = f_w$ given by (28) (corresponding to (23) by setting $M_{w,o} = 0$ in (28) since system (26) turns out to be the classical Darcy's system now). Furthermore, observe that, if the cross-terms are chosen such that $M_o = M_{o,w}$ and $M_w = M_{w,o}$, then we get

$$
\tilde{f}_w = \frac{1}{1 + \frac{M_o}{M_w}},
$$

which is the same formula as (24) corresponding to the classical Darcy's system (2) (with $\mathbf{g} = 0$). Eventually, let us point out that one cannot expect a physical interpretation from the expression (31) since the system (26) is not physical, but rather an artificial mathematical system that can be solved using standard petroleum solutions and its artificial solution manipulated to obtain a solution of the physical system (3).

5. Conclusions

In this paper, a mathematical model describing the behavior of a multiphase flow system taking into account the effect of one phase on the other, through a mathematical coupling between the phases, was considered. The coupling is modeled through cross-terms involving the pressure gradients multiplied by respective pseudo-mobilities. In comparison to the classical Darcy's approach, we showed that a significant impact of such a coupling can occur in particular scenarios. This was shown by solving the associated Buckley–Leverett equations using a basic Runge–Kutta technique. In particular, the water front position is clearly shown to be significantly different from one model to the other. Obviously, several other aspects need to be investigated such as the injection rate, the effect of gravity, etc. Eventually, we investigated the mathematical relation between the coupled model

and the decoupled one—namely, a classical Darcy's type approach. Indeed, we were able to identify and introduce an artificial classical Darcy's type model, with modified permeabilities (and therefore non-physical permeabilities), that has the advantage to be solvable by any existing software designed to solve the Classical Darcy's system. Once the latter system was solved, and the velocities and the pressure obtained, we provided an explicit expression allowing a direct calculation of the solution of coupled system. More precisely, we proposed a method allowing to take into account the coupling between phases using any classical software able to solve classical Darcy's system.

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