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Reliable Portfolio Selection Problem in Fuzzy Environment: An m_λ Measure Based Approach

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Academic Editor: Oscar Castillo
Received: 16 February 2017; Accepted: 13 April 2017; Published: 18 April 2017

Abstract: This paper investigates a fuzzy portfolio selection problem with guaranteed reliability, in which the fuzzy variables are used to capture the uncertain returns of different securities. To effectively handle the fuzziness in a mathematical way, a new expected value operator and variance of fuzzy variables are defined based on the m_λ measure that is a linear combination of the possibility measure and necessity measure to balance the pessimism and optimism in the decision-making process. To formulate the reliable portfolio selection problem, we particularly adopt the expected total return and standard variance of the total return to evaluate the reliability of the investment strategies, producing three risk-guaranteed reliable portfolio selection models. To solve the proposed models, an effective genetic algorithm is designed to generate the approximate optimal solution to the considered problem. Finally, the numerical examples are given to show the performance of the proposed models and algorithm.

Keywords: portfolio selection problem; m_λ measure; expected value operator; genetic algorithm

1. Introduction

The portfolio selection problem is a well-known problem in the field of economics, which aims to allocate the capital to a pre-given set of securities and meanwhile obtain the maximum return. Theoretically, this problem can be characterized through a standard linear programming model, in which the decision variable corresponds to the investment ratio of the involved capital to each considered security, and the total return is typically a linear form with respect to the decision variables. If all of the parameters in the portfolio selection problem are pre-specified, the corresponding model can be easily solved by the simplex method or some existing classical algorithms.

In the real-world applications, there exist two types of uncertainties in the decision-making process. One is randomness; the other is fuzziness. In general, if enough sample data are available, we can use the statistics methods to estimate the probability distribution of the involved uncertain parameters, and the probability theory can be used as an effective tool to deal with them. On the other hand, when there are not enough sample data or even no sample data, a common method is to treat the uncertain parameters as fuzzy variables by using professional judgments or expert experiences. With these concerns, two classes of methods can be adopted in the literature to investigate the portfolio selection problem, i.e., random optimization and fuzzy optimization, in order to maximize the total return and decrease the risks in the uncertain environment. In the following discussion, we aim to review the existing works in the literature along these two lines.

For the earlier works, Markowitz [1,2] first proposed the mean-variance models in stochastic environments, in which the variance is usually used to quantify the existing risks in the uncertain return. In detail, to measure and control the risks of the investment, a threshold is firstly pre-given for the portfolio, and an investment strategy is called a feasible plan if the variance of its random return is not over this threshold. Afterwards, for the stochastic portfolio selection problem, a variety of existing research works focus on improving or extending this type of model to more complex decision environments. For instance, Shen [3] investigated a mean-variance-based portfolio selection problem in a complete market with unbounded random coefficients, which was solved by the stochastic linear-quadratic control theory and the Lagrangian method. Lv et al. [4] explored a continuous-time mean-variance portfolio selection problem with random market parameters and a random time horizon in an incomplete market, which was formulated as a linear constrained stochastic linear quadratic optimal control problem. He and Qu [5] considered a multi-period portfolio selection problem with market random uncertainty of asset prices. They formulated the problem as a two-stage stochastic mixed-integer program with recourse and designed a simplification and hybrid solution method to solve the problem of interest. Najafi and Mushakhiann [6] considered three indexes to model the portfolio selection problem, including the expected value, semivariance and conditional value-at-risk. A hybrid algorithm with a genetic algorithm and a particle swarm optimization algorithm was designed to solve the proposed model. Low et al. [7] estimated the expected returns by sampling from a multivariate probability model that explicitly incorporated distributional asymmetries to enhance the performance of mean-variance portfolio selection. Shi et al. [8] proposed three multi-period behavioral portfolio selection models under cumulative prospect theory. Shen et al. [9] discussed a mean-variance portfolio selection problem under a constant elasticity of variance model on the basis of backward stochastic Riccati equation. Kim et al. [10] resolved the high-cardinality of mean-variance portfolios through applying the semi-definite relaxation method to a cardinality-constrained optimal tangent portfolio selection model. Zhang and Chen [11] investigated a mean-variance portfolio selection problem with regime switching under the constraint of short-selling being prohibited. Chiu and Wong [12] further enriched the literature of the mean-variance portfolio selection by considering correlation risk among risky asset returns. Fulga [13] proposed a quantile-based risk measure, which is defined using the modified loss distribution according to the decision maker's risk and loss aversion. For other research, interested readers can refer to Alexander et al. [14], Villena and Reus [15], Maillet et al. [16], Huang [17], etc.

In the condition of incomplete information, fuzzy set theory can be used as an efficient tool to deal with this situation. Along this line, Zhang and Zhang [18] investigated a multi-period fuzzy portfolio selection problem to maximize the terminal wealth imposed by risk control, in which the returns of assets are characterized by possibilistic mean values, and a possibilistic absolute deviation is defined as the risk control of the portfolio. Li and Xu [19] studied the multi-objective portfolio selection model with fuzzy random returns for investors through three criteria, i.e., return, risk and liquidity, and a compromise approach-based genetic algorithm was designed to solve the proposed model. Gupta et al. [20] proposed a multi-objective credibilistic model with fuzzy chance constraints for the portfolio selection problem, which was solved by a fuzzy simulation-based genetic algorithm. Mehlawat [21] dealt with fuzzy multi-objective multi-period portfolio selection problems. A fuzzy credibilistic programming approach with multi-choice goal programming was proposed to obtain investment strategies. Huang and Di [22] discussed a new uncertain portfolio selection model in which background risk was considered, and the returns of the securities and the background assets were given by experts' evaluations instead of historical data. Huang and Zhao [23] investigated a mean-chance model for portfolio selection based on an uncertain measure and developed an effective genetic algorithm to solve the proposed nonlinear programming problem. Bermudez [24] extended genetic algorithms from their traditional domain of optimization to the fuzzy ranking strategy for selecting efficient portfolios of restricted cardinality. Rebiasz [25] presented a new method for the selection of efficient portfolios, where parameters in the calculation of effectiveness were expressed by interactive fuzzy numbers and the probability distribution. For other research about the fuzzy optimization technique on this topic, we can refer to Zhang et al. [26], Liu et al. [27], Bhattacharyya et al. [28], Saborido [29], etc.

As can be seen in the literature, in the case of incomplete historical data, a variety of fuzzy approaches has been proposed to deal with portfolio selection under uncertain environments. The evaluation indexes are usually associated with the possibility measure, credibility measure, expected values, variance, semivariance, etc. In this paper, we aim to propose new definitions for the expected value operator and variance to characterize the feature of fuzzy variables, which are defined on the basis of the m_λ measure proposed by Yang and Iwamura [30]. Actually, the m_λ measure is a linear combination of the possibility measure and necessity measure, which provides an effective method to make a trade-off between the optimistic and pessimistic decisions. Based on the newly-proposed expected value operator and variance, we handle the portfolio selection problem with different indexes from the literature. To the best our knowledge, no related research can be found to investigate the portfolio selection problem in the fuzzy environment, which motivates us to study this problem to get suitable strategies with fuzzy parameters.

The rest of this research is organized as follows. Section 2 gives a detailed description for the considered problem and formulates the reliable mathematical model with fuzzy parameters. In Section 3, some equivalent models have been proposed based on the rigorous mathematical analysis. In Section 4, an effective genetic algorithm is proposed to search for the approximate optimal solution of the proposed model. Finally, some numerical experiments are implemented to show the application and performance of the proposed methods.

2. Problem Statement and Mathematical Models

The portfolio selection problem deals with how to allocate the pre-given capital to a set of securities with the maximized returns. When all of the parameters are deterministic variables in this process, this classical problem can be essentially formulated as a linear programming model with the selection ratio constraints. In order to characterize this problem, we first introduce some notations in the formulation process.

S: total number of involved securities;

s: index of involved securities, $s \in \{1, 2, \ldots, S\}$;

r_s: return of the s-th security;

x_s: the investment ratio for security s, which is a decision variable.

If all of the parameters in this process are constant, we can formulate this problem as the following linear programming model.

$$\begin{cases} \max r_1x_1 + r_2x_2 + \ldots + r_Sx_S \\ s.t. \\ x_1 + x_2 + \cdots + x_S = 1 \\ x_s \geq 0, s \in \{1, 2, \ldots, S\} \end{cases} \tag{1}$$

In this model, the objective function is to maximize the total return among different feasible investment strategies. The constraint ensures that the sum of the investment proportions should be unity, and each proportion is larger than or equal to zero. Obviously, if all of the parameters are pre-given constants, it is typically a linear programming model and can be solved by the simplex method. Note that the portfolio selection plan is usually made before the return can be fulfilled. Thus, the actual return of each security is practically uncertain. In the following discussion, we particularly treat the return of each security as a fuzzy variable to suitably describe the practical uncertainty. Theoretically, if we consider the uncertainty in the portfolio selection problem, this problem should be formulated as a robust or reliable optimization programming with fuzzy parameters. For the completeness of this study, we shall introduce the basic knowledge in fuzzy set theory in the following discussion.

2.1. m_λ Measure and Expected Value Operator

Fuzzy set theory was first proposed by Zadeh in 1965 and further developed by many researchers, such as Nahmias [31], Liu and Liu [32], etc. In this theory, the possibility measure and necessity measure are two effective tools to characterize the chance of a fuzzy event. For instance, let ξ be a fuzzy variable with membership function $\mu_\xi(x)$ and B a subset of real numbers. Then, a fuzzy event can be expressed as $\{\xi \in B\}$. Its possibility and necessity, respectively, can be calculated as follows:

$$\mathrm{Pos}\{\xi \in B\} = \sup_{x \in B} \mu_\xi(x), \ \mathrm{Nec}\{\xi \in B\} = 1 - \sup_{x \notin B} \mu_\xi(x).$$

In general, we have the following relationship between these two measures, i.e., $\mathrm{Pos}\{\cdot\} \geq \mathrm{Nec}\{\cdot\}$ for any fuzzy event $\{\cdot\}$. Additionally, even if the possibility of a fuzzy event achieves one, it cannot necessarily guarantee the occurrence of this event; on the other hand, if the necessity of a fuzzy event is zero, it is possible that this event can occur. In this sense, in the process of optimizing the chance of fuzzy events, the possibility measure is more suitable for the optimistic decision makers, while the necessity measure is more favored by pessimistic decision makers. In order to make a trade-off between the optimism and pessimism in the decision-making process, Yang and Iwamura [30] proposed a linear combination of this two fuzzy measures by introducing a weighted parameter λ, called the m_λ measure, which has successfully applied to a variety of fields in handling the fuzziness of the decision-making process, for instance carbon capture, utilization and storage (Dai et al. [33]), water quality management (Li et al. [34]), etc. In detail, the m_λ measure is defined as follows:

$$m_\lambda\{\cdot\} = \lambda \mathrm{Pos}\{\cdot\} + (1-\lambda)\mathrm{Nec}\{\cdot\}.$$

Theoretically, if parameter λ is close to one, this measure is more suitable for risk-loving decision makers; on the contrary, it is suitable for risk-averse decision makers. In particular, if λ is set as 0.5, this measure degenerates to the credibility measure proposed by Liu and Liu [32]. For this measure, Yang and Iwamura [30] proved that for any fuzzy event A, we have $m_\lambda\{A\} + m_{1-\lambda}\{A^c\} = 1$, which implies that m_λ and $m_{1-\lambda}$ are two dual measures in scaling the chance of fuzzy events. With this property, we hereinafter define the expected value of a fuzzy variable, which has the form of a scalar integral.

Definition 1. *Let ξ be a fuzzy variable with the membership function $\mu_\xi(x)$. Then, the λ-expected value of this fuzzy variable is defined by:*

$$E_\lambda[\xi] = \int_0^{+\infty} m_\lambda\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 m_{1-\lambda}\{\xi \leq r\}\mathrm{d}r \tag{2}$$

provided that at least one integral is finite.

Typically, this definition has a common form of the expected value operator for random variables where the probability measure is self-dual. To illustrate the calculation of expected value operator, we here give several illustrations for clarity.

Example 1. *Let $\xi = (a, b, c)$ be a triangular fuzzy variable with the following membership function:*

$$\mu_\xi(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ \dfrac{x-c}{b-c}, & \text{if } b \leq x \leq c \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then, we have:

$$E_\lambda[\xi] = \frac{(1-\lambda)a + b + \lambda c}{2}.$$

Proof. In the following, we only consider the case of $0 < a < b < c$, and the other situations can be proven similarly. In this case, it is sufficient to calculate the first integral in the definition. In fact, we have:

$$m_\lambda\{\xi \geq r\} = \begin{cases} 1, & \text{if } 0 < r \leq a \\ 1 - (1-\lambda) \cdot \dfrac{r-a}{b-a}, & \text{if } a < r \leq b \\ \lambda \cdot \dfrac{r-c}{b-c}, & \text{if } b < r \leq c \\ 0, & \text{if } r > c \end{cases}$$

Thus,

$$\begin{aligned} E_\lambda[\xi] &= \int_0^a \mathrm{d}r + \int_a^b \left[1 - (1-\lambda) \cdot \frac{r-a}{b-a}\right] \mathrm{d}r + \int_b^c \lambda \cdot \frac{r-c}{b-c} \mathrm{d}r \\ &= a + (b-a) - (1-\lambda) \int_a^b \frac{r-a}{b-a} \mathrm{d}r + \lambda \int_b^c \frac{r-c}{b-c} \mathrm{d}r \\ &= b - \frac{(1-\lambda)(b-a)}{2} + \int_b^c \frac{\lambda(c-b)}{2} \mathrm{d}r \\ &= \frac{(1-\lambda)a + b + \lambda c}{2}. \end{aligned}$$

□

Remark 1. *If we take $\lambda = 0.5$, the expected value of fuzzy variable can be simplified as:*

$$E_\lambda[\xi] = \frac{a+2b+c}{4},$$

which just coincides with the situation proposed by Liu and Liu [32].

Example 2. *Let $\xi = (a,b,c,d)$ be a trapezoidal fuzzy variable with the following membership function:*

$$\mu_\xi(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } b \leq x \leq c \\ \dfrac{x-c}{b-c}, & \text{if } c \leq x \leq d \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Then, we have:

$$E[\xi;\lambda] = \frac{(1-\lambda)a + (1-\lambda)b + \lambda c + \lambda d}{2}.$$

Example 3. *Let $\xi = [a,b]$ be an interval fuzzy variable with the following membership function:*

$$\mu_\xi(x) = \begin{cases} 1, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Then, we have:

$$E_\lambda[\xi] = (1-\lambda)a + \lambda b.$$

In this definition, we have the following relationship with respect to the different parameter λ.

Theorem 1. *Let ξ be a fuzzy variable and $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 \leq \lambda_2$; we have $E_{\lambda_1}[\xi] \leq E_{\lambda_2}[\xi]$.*

Proof of Theorem 1. Since $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 \le \lambda_2$, we have $m_{\lambda_1}\{\cdot\} \le m_{\lambda_2}\{\cdot\}$ for any fuzzy event $\{\cdot\}$ (see Yang and Iwamura [30]). Then, we have:

$$\begin{aligned} E_{\lambda_1}[\xi] &= \int_0^{+\infty} m_{\lambda_1}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 m_{1-\lambda_1}\{\xi \le r\}\mathrm{d}r \\ &\le \int_0^{+\infty} m_{\lambda_2}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 m_{1-\lambda_2}\{\xi \le r\}\mathrm{d}r = E_{\lambda_2}[\xi]. \end{aligned}$$

The proof is thus completed. □

To capture the variant of the expected values with respect to parameter λ, we show the curve of the λ-expected value of a triangular fuzzy variable in Figure 1. Clearly, the expected value has a linear increasing relationship with respect to λ. If $\lambda = 1$, the expected value turns out to be $\frac{b+c}{2}$; if $\lambda = 0$, the expected value takes the value $\frac{a+b}{2}$. Actually, if ξ denotes the fuzzy return, this expected value function is reasonable for different types of decision makers, since optimistic decision makers can usually overrate the expected return, and pessimistic decision makers prefer to underestimate the expected return.

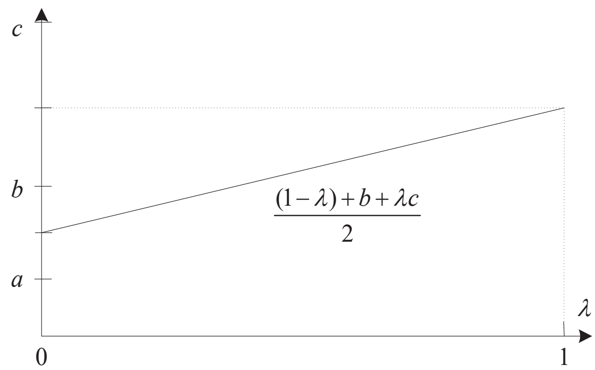

Figure 1. Expected value curves with respect to parameter λ.

Next, we aim to investigate the linearity of the proposed λ-expected value operator. To this end, we firstly introduce some closely related concepts and theorems in the following discussion.

Definition 2. *(Liu [35]) Suppose that ξ is a fuzzy variable and $\alpha \in (0,1]$. Then:*

$$\xi_{\sup}(\alpha) = \sup\{r | \mathrm{Pos}\{\xi \ge r\} \ge \alpha\}$$

is called the α-optimistic value to ξ; and:

$$\xi_{\inf}(\alpha) = \inf\{r | \mathrm{Pos}\{\xi \le r\} \ge \alpha\}$$

is called the α-pessimistic value to ξ.

Theorem 2. *(Liu [35]) Suppose that ξ and η are two fuzzy variables. Then, for any $\alpha \in (0,1]$, we have:*

(a) $(\xi+\eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) + \eta_{\sup}(\alpha)$;
(b) $(\xi+\eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha) + \eta_{\inf}(\alpha)$;
(c) *if $\gamma \ge 0$, then $(\gamma\xi)_{\sup}(\alpha) = \gamma\xi_{\sup}(\alpha)$ and $(\gamma\xi)_{\inf}(\alpha) = \gamma\xi_{\inf}(\alpha)$;*
(d) *if $\gamma < 0$, then $(\gamma\xi)_{\sup}(\alpha) = \gamma\xi_{\inf}(\alpha)$ and $(\gamma\xi)_{\inf}(\alpha) = \gamma\xi_{\sup}(\alpha)$.*

A fuzzy variable ξ with membership function $\mu_\xi(x)$ is called a normalized fuzzy variable if it has the following characteristics: (1) there exists a real number x^* with $\mu_\xi(x^*) = 1$; (2) $\mu_\xi(x)$ is nondecreasing for $x \le x^*$ and nonincreasing for $x \ge x^*$. Typically, the interval fuzzy variables, triangular fuzzy variables, trapezoidal fuzzy variable and fuzzy variables with unimodal membership functions are all normalized fuzzy variables.

Theorem 3. *Let ξ be a normalized fuzzy variable with finite expected value. Then, we have:*

$$E_\lambda[\xi] = \int_0^1 \left(\lambda\xi_{\sup}(\alpha) + (1-\lambda)\xi_{\inf}(\alpha)\right) \mathrm{d}\alpha,$$

in which $\xi_{\sup}(\alpha)$ and $\xi_{\inf}(\alpha)$, respectively, are α-optimistic and α-pessimistic values to fuzzy variable ξ.

Proof of Theorem 3. Since ξ is a normalized fuzzy variable, there exists a real number x^* with $\mu_\xi(x^*) = 1$, and $\mu_\xi(x)$ is nondecreasing for all $x < x^*$ and nonincreasing for all $x \ge x^*$. In the following, we only consider the case of $x^* > 0$, and the other situation can be proven similarly. In this case, we have:

$$\begin{aligned}
E_\lambda[\xi] &= \int_0^{+\infty} m_\lambda\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 m_{1-\lambda}\{\xi \le r\}\mathrm{d}r \\
&= \int_0^{x^*} m_\lambda\{\xi \ge r\}\mathrm{d}r + \int_{x^*}^{+\infty} m_\lambda\{\xi \ge r\}\mathrm{d}r - \left[\int_{-\infty}^{x^*} m_{1-\lambda}\{\xi \le r\}\mathrm{d}r - \int_0^{x^*} m_{1-\lambda}\{\xi \le r\}\mathrm{d}r\right] \\
&= \lambda x^* + \lambda \int_{x^*}^{+\infty} \mathrm{Pos}\{\xi \ge r\}\mathrm{d}r - \left[(1-\lambda)\int_{-\infty}^{x^*} \mathrm{Pos}\{\xi \le r\}\mathrm{d}r - (1-\lambda)x^*\right] \\
&= \lambda\left[x^* + \int_{x^*}^{+\infty} \mathrm{Pos}\{\xi \ge r\}\mathrm{d}r\right] + (1-\lambda)\left[x^* - \int_{-\infty}^{x^*} \mathrm{Pos}\{\xi \le r\}\mathrm{d}r\right] \\
&= \lambda \int_0^1 \xi_{\sup}(\alpha)\mathrm{d}\alpha + (1-\lambda)\int_0^1 \xi_{\inf}(\alpha)\mathrm{d}\alpha \\
&= \int_0^1 (\lambda\xi_{\sup}(\alpha) + (1-\lambda)\xi_{\inf}(\alpha))\mathrm{d}\alpha.
\end{aligned}$$

The proof is completed. □

The linearity of the expected value operator of random variables is an important property in the real-world applications. Likewise, for the λ-expected value operator proposed in this study, we also have a similar linearity for the fuzzy variables.

Theorem 4. *Let ξ and η be normalized fuzzy variables with finite expected values. If a and b are two real numbers, we then have:*

$$E_\lambda[a\xi + b\eta] = aE_\lambda[\xi] + bE_\lambda[\eta].$$

This theorem can be easily proven by using Theorems 2 and 3.

In addition, based on the expected value operator, we can define the concept of variance to measure the diversity degree of the uncertain information, given below.

Definition 3. *Let ξ be a fuzzy variable with λ-expected value e. Then, λ-variance of ξ is defined as:*

$$V_\lambda[\xi] = E_\lambda[(\xi - e)^2]. \tag{6}$$

Example 4. *Let $\xi = [a, b]$ be an interval fuzzy variable. Then, the λ-variance of ξ is:*

$$V[\xi; \lambda] = \lambda \cdot \max\{\lambda^2(b-a)^2, (1-\lambda)^2(b-a)^2\}.$$

Proof. According to Example 3, we have $E_\lambda[\xi] = (1-\lambda)a + \lambda b$. According to Zadeh's extension principle, both $\xi - E[\xi;\lambda]$ and $(\xi - E[\xi;\lambda])^2$ are two interval fuzzy variables with the following distributions:

$$\xi - E_\lambda[\xi] = [\lambda(a-b), (1-\lambda)(b-a)],$$
$$(\xi - E_\lambda[\xi])^2 = [0, \max\{\lambda^2(b-a)^2, (1-\lambda)^2(b-a)^2\}].$$

Then, the λ-variance of ξ is calculated as follows.

$$V_\lambda[\xi] = \lambda \cdot \max\{\lambda^2(b-a)^2, (1-\lambda)^2(b-a)^2\}.$$

□

Remark 2. *Typically, if parameter λ approaches one, the λ-variance of ξ is close to $(b-a)^2$. On the other hand, if parameter λ is close to zero, the λ-variance is also close to zero. If ξ represents the fuzzy return, it shows that the optimistic decision makers face large risk when they set a large λ, and the pessimistic decision makers have a small risk when they take a small parameter λ. Actually, this situation just coincides with the real conditions. In detail, if λ is equal to zero; the expected return should be a, which is the least realization in the fuzzy return and typically can be realized. Then, the return risk will be reduced to zero. However, if λ is taken as one, the expected return should be b, which is the upper bound of realizations in the fuzzy return, and can be realized with the largest risk.*

Theorem 5. *Let ξ be a fuzzy variable with λ-expected value e and a, b two real numbers. We then have:*

$$V_\lambda[a\xi + b] = a^2 V_\lambda[\xi].$$

Proof of Theorem 5. It follows that:

$$V_\lambda[a\xi + b] = E_\lambda[(a\xi + b - ae - b)^2] = a^2 V_\lambda[\xi].$$

The proof is thus finished. □

2.2. New Reliable Models

In this problem, if we use the least expected return to evaluate the quality of the portfolio selection strategy, we can formulate the following least expected return model for the fuzzy portfolio selection problem.

$$\begin{cases} \max E_\lambda[r_1x_1 + r_2x_2 + \ldots + r_Sx_S] \\ s.t. \\ x_1 + x_2 + \ldots + x_S = 1 \\ x_s \geq 0, s \in \{1, 2, \cdots, S\} \end{cases} \tag{7}$$

In this model, the objective function aims to minimize the expected return in the decision-making process. However, we note that the optimal strategy for this model is not always favored in all situations, because the least expected return strategy, occasionally creating large, systemic and poorly understood risks, might be subject to high risks under some extreme scenarios and would be specifically undesirable to risk-averse travelers. By recognizing this critical requirement, we here particularly propose three models to reduce the the diversity of the fuzzy total return. In addition, to avoid the excessive decentralization of the final investigation, the threshold for each sector needs to be given. That is, let h_s be the threshold of investigation ratio x_s, and then, we should have $x_s = 0$ or $x_s \geq h_s$, $s = 1, 2, \cdots, S$ in the process of the formulation.

Firstly, we add the variance of the total return into the objective function, and then, an expectation-variance reliable model is formulated, given by:

$$\begin{cases} \max E_\lambda\left[\sum_{s=1}^{S} r_s x_s\right] - \beta\sqrt{V_\lambda\left[\sum_{s=1}^{S} r_s x_s\right]} \\ s.t. \\ x_1 + x_2 + \ldots + x_S = 1 \\ x_s = 0 \text{ or } x_s \geq h_s, s \in \{1, 2, \ldots, S\} \end{cases} \tag{8}$$

In the literature, this type of index has been successfully applied to the transportation field (e.g., Xing and Zhou [36], Sen et al. [37]) as a guidance for finding a reliable path. In this model, the parameter β is a reliability coefficient to reflect the significance of total return variability. This reliability coefficient can also vary for different decision makers. In general, if the decision maker is risk-averse, he/she can set a relative large reliability coefficient; otherwise, a smaller coefficient can be taken for the optimistic decision makers. Typically, in the process of maximizing the objective, a small variance of the fuzzy return is desirable in evaluating the investment strategies.

Secondly, to reduce the risk, the variance of total return is assumed to be less than a threshold $\bar{V}$, which is regarded as a constraint in the formulation. This constraint can be referred to as a side constraint, which has been widely used to handle the routing optimization problems effectively (e.g., Wang et al. [38], Wang et al. [39]). By maximizing the expected return, we hereinafter reformulate the risk-guaranteed reliable model as a variance-constrained reliable model, given by:

$$\begin{cases} \max E_\lambda\left[\sum_{s=1}^{S} r_s x_s\right] \\ s.t. \\ x_1 + x_2 + \cdots + x_S = 1 \\ V_\lambda\left[\sum_{s=1}^{S} r_s x_s\right] \leq \bar{V} \\ x_s = 0 \text{ or } x_s \geq h_s, s \in \{1, 2, \ldots, S\} \end{cases} \tag{9}$$

In Model (9), the parameter $\bar{V}$ is an upper limit of the variance, which is determined by decision makers. Obviously, a risk-seeker often select a relatively large threshold $\bar{V}$, while a smaller $\bar{V}$ will lead to a risk-averse strategy.

Thirdly, in the condition of minimizing the variance of the total return, we regard the expected return as a resource constraint, and thus, the expectation-constrained reliable model is formulated by:

$$\begin{cases} \min V_\lambda\left[\sum_{s=1}^{S} r_s x_s\right] \\ s.t. \\ x_1 + x_2 + \cdots + x_S = 1 \\ E_\lambda\left[\sum_{s=1}^{S} r_s x_s\right] \geq \bar{E} \\ x_s = 0 \text{ or } x_s \geq h_s, s \in \{1, 2, \ldots, S\} \end{cases} \tag{10}$$

Model (10) aims to minimize the risk in the condition that the expected return is not less than a given threshold $\bar{E}$. In other words, the parameter $\bar{E}$ is regarded as the lower bound of the expected return. In this model, the risk-seeker prefers a larger $\bar{E}$, while the risk-averser often selects a smaller one.

3. Property Analysis

Next, we shall analyze the computational properties of some special cases. If all of the returns are interval fuzzy variables, we have the following results.

Theorem 6. *Let $r_s = [a_s, b_s]$ be interval fuzzy variables, $s = 1, 2, \ldots, S$. Then, the variance can be calculated according to the following equation.*

$$V\left[\sum_{s=1}^{S} r_s x_s\right] = \lambda \cdot \max\left\{\lambda^2\left(\sum_{s=1}^{S} b_s x_s - \sum_{s=1}^{S} a_s x_s\right)^2, (1-\lambda)^2\left(\sum_{s=1}^{S} b_s x_s - \sum_{s=1}^{S} a_s x_s\right)^2\right\}$$

Proof of Theorem 6. Since the returns r_s, $s = 1, 2, \ldots, S$ are interval fuzzy variables, then the total fuzzy return is also an interval fuzzy variable for each feasible solution, given below.

$$\sum_{s=1}^{S} r_s x_s = \left[\sum_{s=1}^{S} a_s x_s, \sum_{s=1}^{S} b_s x_s\right].$$

Then, by using Examples 1 and 4, we can easily prove this theorem. □

Typically, if all of the parameters r_s, $s = 1, 2, \ldots, S$ are interval fuzzy variables, we can use analytical methods to calculate the objective function for each feasible solution X.

Theorem 7. *(i) Let $r_s = (a_s, b_s, c_s)$ be a triangular fuzzy variable, $s = 1, 2, \ldots, S$. Then, the expected total return can be calculated according to the following equation.*

$$E\left[\sum_{s=1}^{S} r_s x_s\right] = \frac{(1-\lambda)\sum_{s=1}^{S} a_s x_s + \sum_{s=1}^{S} b_s x_s + \lambda \sum_{s=1}^{S} c_s x_s}{2}.$$

(ii) Let $r_s = (a_s, b_s, c_s, d_s)$ be a trapezoidal fuzzy variable, $s = 1, 2, \ldots, S$. Then, the expected total return can be calculated according to the following equation.

$$E\left[\sum_{s=1}^{S} r_s x_s\right] = \frac{(1-\lambda)\sum_{s=1}^{S} a_s x_s + (1-\lambda)\sum_{s=1}^{S} b_s x_s + \lambda \sum_{s=1}^{S} c_s x_s + \lambda \sum_{s=1}^{S} c_s x_s}{2}.$$

By using Examples 1 and 2, the proof of this theorem is obvious.

In general, for a common fuzzy variable with a complex membership function, it is difficult to calculate its expected value and variance by analytic methods. In this situation, we have to use a simulation algorithm to achieve the approximate values of these two indexes. In the following, we shall design the detailed simulation algorithm to simulate the expected value and variance of a fuzzy variable.

Consider a function $f(\boldsymbol{x}, \boldsymbol{\xi})$. We next design the simulation procedure of the expected value $E[f(\boldsymbol{x}, \boldsymbol{\xi}); \lambda]$. Typically, if we set $f(\boldsymbol{x}, \boldsymbol{\xi}) = \xi$, this procedure corresponds to the simulation process of the expected value of ξ; moreover, if we set $f(\boldsymbol{x}, \boldsymbol{\xi}) = (\xi - E[\xi; \lambda])^2$, this procedure corresponds to the simulation process of variance of ξ. For this purpose, we firstly need to compute the chance measure with form $m_\lambda\{f(\boldsymbol{x}, \boldsymbol{\xi}) \le r\}$, and chance measure $m_\lambda\{f(\boldsymbol{x}, \boldsymbol{\xi}) \ge r\}$ can be simulated similarly. Yang and Iwamura [30] give the simulation procedure as follows.

- The procedure of simulating $m_\lambda\{f(\boldsymbol{x}, \boldsymbol{\xi}) \ge r\}$:

  Step 1. Randomly generate crisp vectors $\boldsymbol{u}_k$ from the ε-level set of fuzzy vector $\boldsymbol{\xi}$, respectively, where ε is a sufficiently small number and $k = 1, 2, \ldots, N$;

  Step 2. Let $\nu_k = \mu_{\boldsymbol{\xi}}(\boldsymbol{u}_k)$ for $k = 1, 2, \ldots, N$ where $\mu_{\boldsymbol{\xi}}$ is the membership function of $\boldsymbol{\xi}$;

  Step 3. Return the following value:

$$L = \lambda \cdot \max_{1 \le k \le N}\{\nu_k | f(\boldsymbol{x}, \boldsymbol{u}_k) \le r\} + (1-\lambda) \cdot \min_{1 \le k \le N}\{1 - \nu_k | f(\boldsymbol{x}, \boldsymbol{u}_k) > r\}$$

- The procedure of simulating $E[f(\boldsymbol{x},\boldsymbol{\xi});\lambda]$:

  Step 1. Set $e=0$;

  Step 2. Randomly generate crisp vectors $\boldsymbol{u}_k$ from the ε-level set of fuzzy vector $\boldsymbol{\xi}$, respectively, where ε is a sufficiently small number and $k=1,2,\ldots,N$;

  Step 3. Set $a=\bigwedge_{k=1}^{N} f(\boldsymbol{x},\boldsymbol{u}_k)$ and $b=\bigvee_{k=1}^{N} f(\boldsymbol{x},\boldsymbol{u}_k)$;

  Step 4. Randomly generate r in interval $[a,b]$;

  Step 5. If $r\geq 0$, let $e\leftarrow e+m_\lambda\{f(\boldsymbol{x},\boldsymbol{\xi})\geq r\}$; otherwise, let $e\leftarrow e-m_\lambda\{f(\boldsymbol{x},\boldsymbol{\xi})\leq r\}$;

  Step 6. Repeat Step 4 to Step 5 for a total of N times;

  Step 7. $E[f(\boldsymbol{x},\boldsymbol{\xi});\lambda]=a\vee 0+b\wedge 0+e\cdot(b-a)/N$.

Next, we aim to give a numerical example to show the accuracy of the proposed simulation algorithm. Specifically, let us consider the sum of ten triangular fuzzy variables $\xi_i=(5+i,7+i,10+i)$, $i=1,2,\ldots,10$. If we set $\lambda=0.8$, by using the linear property of the expected value operator, we typically have:

$$E[\xi_1+\xi_2+\ldots+\xi_{10};0.8]=E[\xi_1;0.8]+E[\xi_2;0.8]+\ldots+E[\xi_{10};0.8]=135.$$

On the other hand, when we use the simulation algorithm, we then have $E=134.90$ after implementing a total of 1000 cycles, which has a relative error of 0.07%. More specifically, we simulate a total of ten cases for parameter λ (i.e., $\lambda=0.0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0$); the comparison between the exact values and simulated values is illustrated in the following Table 1. The results demonstrate the effectiveness of the proposed simulation algorithm.

Table 1. Comparison of the expected value (Exp.) and simulated value (Sim.).

λ	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
Exp.	115.00	117.5	120.00	122.50	125.00	127.50	130.00	132.50	135.00	137.50	140.00
Sim.	115.17	117.63	120.10	122.57	125.04	127.51	129.97	132.44	134.91	137.38	139.85

4. Solution Method

As discussed above, the proposed models are typically non-linear programming models due to the complexity of the objective function or constraints. In this case, it is impossible to adopt the optimization solver to find the near-optimal solution to the considered problem. In the following, we shall design a genetic algorithm-based approach to solve the proposed model. The genetic algorithm is an effective algorithm to solve the optimization problems. Up to now, this type of algorithm has been successfully applied to a variety of real-world fields, such as transportation, economy, finance, etc. Next, we aim to design the technical details for the genetic algorithm for the problem of interest in this paper.

4.1. Solution Representation

In this paper, the decision variables are the investment ratios associated with different securities. In total, there are S securities that can be invested. Then, we can use x_s to denote the decision variable. Then, in the genetic algorithm, we can use an S-dimensional array to represent the decision variables, listed below:

$$X=(x_1,x_2,\ldots,x_S),$$

where we need guarantee $x_s\in[0,1]$, $s=1,2,\ldots,S$. Take Model (9) for example: three types of constraints should be satisfied for the decision variables. Then, in the process of initializing the population, we need to generate a total of pop_size feasible solutions in the initial population. This process can be fulfilled through the following operation. Firstly, we need to generate a solution that satisfies the first constraint. In general, we can finish this part according to the following procedure.

Step 1. Randomly generate a sequence of nonnegative real numbers $y_1, y_2, \ldots, y_S$;
Step 2. Let $x_s = y_s / \sum_{s=1}^{S} y_s$, $s = 1, 2, \ldots, S$.

Based on the above procedure, we can obtain a solution satisfying the first constraint (i.e., the sum of investigation ratios is a unit). With this form, we need to further correct this solution by the following procedure.

Step 1. Let $s = 1$;
Step 2. If $x_s = 0$ or $x_s \geq h_s$, go to Step 4; otherwise, go to Step 3;
Step 3. Randomly find an index s' with $x_{s'} > 0$; let $x_{s'} \leftarrow x_{s'} + x_s$, $x_s \leftarrow 0$;
Step 4. If $s < S$, let $s++$, go to Step 3; otherwise, stop.

After the above operations, we finally produce a solution that satisfies the first and third constraints. If it also satisfies the variance constraint, it is necessarily a feasible solution for the proposed model. When a total of Pop_size chromosomes is generated, the initial population is produced.

4.2. Selection Operation

The selection operation is used to select the chromosomes for the crossover and mutation operations, which is the basis of the genetic algorithm. In this paper, we shall adopt a common selection operation frequently used in the literature (e.g., Yang and Iwamura [30]) to perform this operation. However, for the completeness of this paper, we still give a detailed description in this part.

To implement the selection operation, we first arrange all of the chromosomes in the population from good to bad according to their objective functions, denoted by $X_1, X_2, \ldots, X_S$. This operation is performed based on the fitness of each chromosome. To show the superiority of each chromosome, we can define different evaluation functions for each chromosome. In the following, we introduce two commonly-used methods.

Objective function-based evaluation: In this method, the objective function will be used to evaluate each chromosome, given below.

$$Eval(X_i) = f(X_i), i = 1, 2, \ldots, pop\_size.$$

Rank-based evaluation: In this method, the fitness function can be defined according to the ranked order from good to bad, i.e.,

$$Eval(X_i) = \alpha(1-\alpha)^{i-1}, i = 1, 2, \ldots, pop\_size.$$

In the following, the roulette wheel will be used to select chromosomes for crossover operation. In detail, we firstly make an increasing sequence $\{q_i\}_{i=0}^{pop\_size}$ according to the following formula.

$$q_0 = 0, q_i = \sum_{k=1}^{i} Eval(X_k), i = 1, 2, \ldots, pop\_size.$$

We repeat the following procedures for pop_size times to select the new population for the crossover operation: randomly generate a number l in interval $(0, q_{pop\_size}]$; if there exists a k such that $q_{k-1} < l \leq q_k$, then X_k will be selected for the new population. After a total of pop_size cycles, a new population, which may have overlapped chromosomes, finally comes into being.

4.3. Crossover Operation

The crossover operation is a key operation in the procedure of the genetic algorithm, which aims to produce the new chromosomes in the population. Through this process, we can expectedly find the better solutions as soon as possible in the following solution process. The crossover operation is carried out on the basis of selected chromosomes. In the process of biological evolution, not all of the chromosomes can finally produce the offspring due to the crossover probability. Thus, following this rule, we finally select a total of $pop\_size \cdot P_c$ chromosomes expectedly to implement the crossover operations, in which P_c is the pre-given crossover probability. In this operation, we select chromosomes according to the following procedures: for each individual in the population, we randomly generate a number w in $[0, 1]$; if $w \leq P_c$, this individual will be selected to take part in the crossover operation.

Next, we denote the selected chromosomes by $X'_1, X'_2, \ldots, X'_H$ ("H" represents the number of selected chromosomes). In the following operation, any two individuals can be grouped as a pair of parents for the crossover operation. Without loss of generality, assume that X'_1 and X'_2 are grouped as a pair of parents. We carry out the crossover operation according to the following formula.

$$X''_1 = \lambda X'_1 + (1 - \lambda)X'_2,\ X''_2 = (1 - \lambda)X'_1 + \lambda X'_2,$$

where λ is a pre-specified parameter in the crossover process. Obviously, the offspring X''_1 and X''_2 satisfy the first constraint in our Model (9), but they do not necessarily satisfy the second and third constraints. To satisfy the third constraint, the procedure proposed in initializing the population should be implemented to correct offspring. Once the X''_1 and X''_2 also satisfy the variance constraints, they will be used to replace their parents in the population. At the end of this operation, at most a total of $2\lfloor H/2 \rfloor$ chromosomes in the population can be expectedly updated.

4.4. Mutation Operation

In the process of biological evolution, the mutation always occurs to increase the diversity of individuals. To simulate this process, the genetic algorithm also includes the mutation operation in the population. Theoretically, different approaches can be designed to mutate a chromosome. In this study, we can perform this operation as follows.

Since only a part of individuals is involved in this operation, we first consider a selection probability P_m in the process of mutating chromosomes. Thus, an expected number $P_m \cdot pop\_size$ of individuals will be selected. The following is the detailed method. For each chromosome in the population, we generate a random number w in $[0, 1]$; if $w \leq P_m$, this chromosome will be selected to take part in this operation. Let X_i be a selected chromosome. We only change the values of two elements in this array. For instance, assume that X_i has the following form:

$$X_i = \{x_1, \ldots, x_s, \ldots, x'_s, \ldots, x_S\},$$

where one of x_s and x'_s has positive values. We can change the value of these two elements, e.g., letting $x_s \leftarrow x_s + x'_s$ and $x'_s \leftarrow 0$ if $x'_s > 0$. Through this operation, a new chromosome typically satisfies the first constraint. We can also correct this chromosome so as to satisfy the third constraint by the procedure in Section 4.1. If this chromosome also satisfies the second constraint, it can be used to replace the original one in the population. At the termination, a new population with the updated chromosomes can be produced.

- Procedure of the genetic algorithm:

  With the technical details designed above, the framework of the genetic algorithm can be summarized in the following.

  Step 1. Determine the parameters for the algorithm, including population size pop_size, fitness value parameter α, crossover probability P_c, mutation probability P_m, number of generation M, etc.;

Step 2. Initialize the population, in which a total of pop_size feasible individuals should be produced;

Step 3. Implement the selection operation based on the objective functions of different chromosomes;

Step 4. Implement the crossover operation with crossover probability P_c;

Step 5. Implement the mutation operation with mutation probability P_m;

Step 6. Repeat Step 3 to Step 5 for M times;

Step 7. Output the best individual found in this procedure as the near-optimal solution to the proposed model.

5. Numerical Examples

In this section, we aim to implement a series of numerical experiments to test the performance of the proposed approaches for the variance-constrained Model (9). All of the experiments are implemented in a personal computer with a 1.60-GHz CPU and 4.00 GB of memory.

Assume that there are 20 securities in our decision-making process. We use the triangular fuzzy variables to denote the security returns in the model, given in Table 2. In addition, to generate a favorite investment strategy, we set the threshold for each security as 0.1 in the experiments. Thus, at most ten securities can be finally selected for investment.

Table 2. Fuzzy returns for each security in the decision-making process.

Security Index	Fuzzy Return	Security Index	Fuzzy Return
1	(35, 45, 50)	11	(45, 53, 70)
2	(54, 80, 90)	12	(44, 60, 70)
3	(44, 60, 65)	13	(33, 45, 50)
4	(35, 40, 50)	14	(43, 54, 70)
5	(55, 70, 75)	15	(45, 66, 70)
6	(63, 70, 80)	16	(44, 50, 60)
7	(45, 54, 60)	17	(35, 50, 55)
8	(72, 80, 85)	18	(28, 40, 50)
9	(66, 73, 90)	19	(33, 40, 45)
10	(35, 40, 50)	20	(45, 50, 55)

(1) Steadiness of the proposed algorithm

With the above-mentioned decision data, we implement this experiment by the genetic algorithm in C++ software, in which the relevant model parameters are set as $\lambda = 0.8$, $\bar{V} = 40$. The computational results are listed in Table 3. Specifically, we randomly choose the critical parameters in the algorithm, including the crossover probability, mutation probability and population size, to test the steadiness of the algorithmic implementation. The algorithm terminated after 500 cycles, i.e., $M = 500$, and a total of ten tests is finally executed. To further show the performance of the computational results, we compute the relatively errors among different results with respect to the best objective function (i.e., 80.23) according to the following equation,

$$Relative\ Error = \frac{Best\ Objective\ Value - Current\ Objective\ Value}{Best\ Objective\ Value} \times 100\%.$$

Typically, among all of these experiments, the relative errors are not greater than 4.00%, which implies the steadiness of the proposed algorithm with respect to these critical parameters. For the parameter setting $P_c = 0.6$, $P_m = 0.7$ and $Pop\_size = 40$, we can find the near-optimal solution with objective value 80.23, and the corresponding optimal solution is $x_3 = 0.11$, $x_6 = 0.13$, $x_8 = 0.76$.

Table 3. The computational results with different parameters.

Test Index	P_c	P_m	Pop_size	Expected Value	Variance	Optimal Objective	Relative Error
1	0.4	0.3	30	78.52	35.23	78.52	2.13%
2	0.4	0.4	50	78.12	38.64	78.12	2.63%
3	0.5	0.6	40	78.54	34.68	78.54	2.11%
4	0.5	0.5	50	78.59	37.14	78.59	2.04%
5	0.6	0.7	40	80.23	39.83	80.23	0.00%
6	0.6	0.5	50	78.00	38.60	78.00	2.78%
7	0.7	0.6	40	78.08	36.59	78.08	2.68%
8	0.6	0.7	50	77.42	37.27	77.42	3.50%
9	0.7	0.7	60	78.28	35.91	78.28	2.43%
10	0.8	0.7	50	78.05	36.30	78.05	2.72%

(2) Sensitivity w.r.t parameter λ

In the proposed model, we use the m_λ measure to characterize the feature of the involved decision makers. Then, we are particularly interested in investigating the sensitivity analysis of the near-optimal solution with respect to parameter λ. In detail, we discretize this parameter into the following numbers, i.e., $\lambda = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$, and the critical parameters in the genetic algorithm are taken as follows: $P_c = 0.6$, $P_m = 0.7$ and $Pop\_size = 40$. We list the computational results in Table 4. In particular, to further show a straightforward overview, we also give Figure 2 to show the variation of the objective function with respect to different parameters λ. Typically, in the experimental results, the returned optimal objective values take almost an increasing tendency when we enhance the parameter λ except for $\lambda = 0.7$ at which an opposite tendency occurs in comparison to the adjacent values. As expected, when we take different parameters λ, the optimal solutions can be different. For instance, when $\lambda = 0.4$; the optimal solution turns out to be $x_2 = 0.29$, $x_3 = 0.39$, $x_6 = 0.13$, $x_8 = 0.19$; if $\lambda = 0.9$, the optimal solution is changed to $x_2 = 0.36$, $x_3 = 0.30$, $x_9 = 0.23$, $x_{16} = 0.11$, which is typically different from the solution of $\lambda = 0.4$.

Table 4. The computational results with different λ.

λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Opti.Obj	76.00	76.65	77.30	77.95	78.60	79.25	79.90	77.97	80.23	81.85	82.50

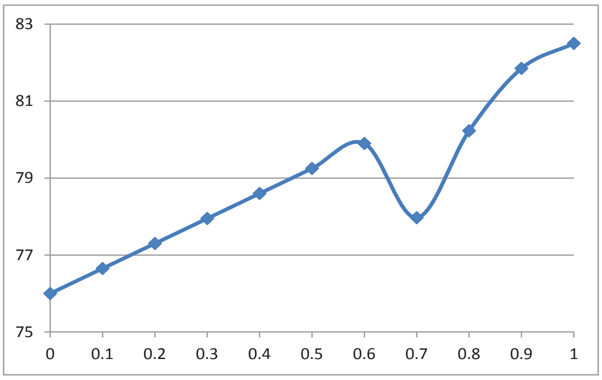

Figure 2. Variation of the optimal objective with parameter λ.

(3) Sensitivity w.r.t parameter $\bar{V}$

In our proposed variance-constrained Model (9), the parameter $\bar{V}$ is regarded as the upper bound of variance to denote the risk-averse degree in the decision-making process. Practically, since the variance can be used to denote the risk of an investment strategy, a small parameter corresponds to a risk-averse decision. In order to show the influence of this parameter on the optimal solution, we here especially implement a set of experiments to test this performance. Specifically, we take a total of ten cases for parameter $\bar{V}$, i.e., $\bar{V} = 40, 41, 42, 43, 44, 45, 46, 47, 48, 49$. In addition, we set $\lambda = 0.8$ in the m_λ measure, and $P_c = 0.6$, $P_m = 0.7$, $Pop\_size = 40$ in the genetic algorithm. The computational results are listed in Table 5. Clearly, different parameters $\bar{V}$ might produce different optimal solutions. For instance, if we take $\bar{V} = 43$, the optimal solution turns out to be $x_2 = 0.10$, $x_6 = 0.42$, $x_8 = 0.48$; on the other hand, when we set $\bar{V} = 48$, the outputted optimal solution is $x_3 = 0.11$, $x_6 = 0.37$, $x_8 = 0.52$.

Table 5. The computational results with different $\bar{V}$.

$\bar{V}$	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**
Opti.Obj	80.23	77.98	76.58	77.88	78.66	77.22	77.03	77.05	78.34	78.13

6. Conclusions

This paper proposed a new model for the portfolio selection problem in the fuzzy environment. To balance the optimism and pessimism in the decision-making process, we defined a new concept, called the m_λ measure, to scale the chance of a fuzzy event. Then, the expected value and variance of the fuzzy variable are defined based on the m_λ measure. Some properties of the expected value and variance were also investigated, e.g., linearity. Based on these definitions, we developed three risk-guaranteed models for the portfolio selection problem. Since the proposed models are nonlinear, we in particular designed a genetic algorithm to search for near-optimal solutions. A set of numerical experiments was also implemented to show the performance of the proposed variance-constrained model and algorithm.

In this paper, we propose some basic reliable models for the portfolio selection problem in fuzzy environment. Here, we need to mention that the proposed model framework can be also suitable for a variety of practical problems so as to optimize the decision risks. In the further study, we will investigate their real-world applications of the proposed models by real cases.

Author Contributions: Yuan Feng conceived of and designed the experiments. Li Wang analyzed the data and wrote the paper. Xinhong Liu performed the experiments.

Conflicts of Interest: The authors declare no conflict of interest.

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