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Stability of the Non-Hyperbolic Zero Equilibrium of Two Close-to-Symmetric Systems of Difference Equations with Exponential Terms

Chrysoula Mylona, Nikolaos Psarros, Garyfalos Papaschinopoulos * and Christos Schinas

School of Engineering, Democritus University of Thrace, 67100 Xanthi, Greece; cmylona@ee.duth.gr (C.M.); nikosnpsarros@hotmail.com (N.P.); cschinas@ee.duth.gr (C.S.)

* Correspondence: gpapas@env.duth.gr; Tel.: +30-2541-0-79758

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Abstract: In this paper, we study the stability of the zero equilibria of two close-to-symmetric systems of difference equations with exponential terms in the special case in which one of their eigenvalues is equal to -1 and the other eigenvalue has an absolute value of less than 1. In the present study, we use the approach of center manifold theory.

Keywords: difference equations; stability; center manifold

1. Introduction

There has been interest in difference equations of biological models for a long period of time. Some of the research can be found in [1–8]. The most interesting special cases of such equations are usually those whose characteristic polynomials of their linearizations have characteristic zeros belonging to the unit circle. Some classical results in this direction can be found in [9,10]. The case in which general difference equations of biological models have unity as a characteristic zero has been thoroughly investigated in [6]. Some interesting concrete systems, which naturally extend one of the basic biological models of this type and which appear also in [7], can be found in [11,12]. In [2], the authors obtained results concerning the global behavior of the positive solutions for the difference equation:

$$x_{n+1} = ax_n + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots$$

where a and b are positive constants and the initial values x_{-1} and x_0 are positive numbers, which, as mentioned above, makes this a biological model.

Motivated by some studies of (one-dimensional) difference equations during the second half of 1990s, we started studying some symmetric and related systems of difference equations (see [13–16]). Systems that are obtained from symmetric systems by modifying their parameters are now frequently called close-to-symmetric systems [17–19]. The papers [12,20–28] deal with such systems. It should be noted that many systems of this type, such as those in [12,17,18,25–28], are solvable, which is not so surprising given there are solvable one-dimensional equations of biological models such as that in [7]. In the papers [13–28] are given some study rational systems (see [13–17,25–28]), some nonlinear systems with exponential terms (see [20–24]), and some product-type systems (see [18,19]).

In [22,23], the authors studied analogous results for the following close-to-symmetric systems of difference equations:

$$x_{n+1} = ax_n + by_{n-1}e^{-x_n}, \quad y_{n+1} = cy_n + dx_{n-1}e^{-y_n}$$

and

$$x_{n+1} = ay_n + bx_{n-1}e^{-y_n}, \quad y_{n+1} = cx_n + dy_{n-1}e^{-x_n},$$

respectively, where a, b, c, and d are positive constants and the initial values x_{-1}, x_0, y_{-1}, and y_0 are also positive numbers.

Now in this paper, using the center manifold theorem (see [9,10,29]), we study the stability of the zero equilibrium of the following close-to-symmetric systems of difference equations:

$$x_{n+1} = ax_n + by_n e^{-x_n}, \;\; y_{n+1} = cy_n + dx_n e^{-y_n} \tag{1}$$

and

$$x_{n+1} = ay_n + bx_n e^{-y_n}, \;\; y_{n+1} = cx_n + dy_n e^{-x_n}, \tag{2}$$

where a, b, c, and d are real constants. In [24], the authors studied the instability of Equations (1) and (2). It is known that if the zero equilibrium of Equation (1) (resp. Equation (2)) is hyperbolic, that is, the coefficient matrix of the linearized system of Equation (1) (resp. Equation (2)) has all the eigenvalues inside the unit circle, then it is easy to determine the stability of the zero equilibrium of Equation (1) (resp. Equation (2)) (see Theorem 4.11 of [29]). In the case for which the zero equilibrium of the above systems in non-hyperbolic, the dynamics of the center manifold plays an important role in the determination of the stability of the zero equilibrium of the systems. More precisely, using the center manifold theorem (see [9,10,29]), the dynamics of the systems can be obtained by studying a one-dimensional equation that contains an approximation of the center manifold.

Regarding the asymptotic behavior of the positive solutions of some scalar equations related to the above systems, we note that the most interesting case is when the sum of the coefficients is equal to one (see [6]). For this goal, asymptotic methods and their applications were employed. For some results on the methods and on the existence of specific types of solutions, see, for example, [30–32] and the references therein. Some of the equations related to that in [2] have appeared in mathematical biology (see, e.g., [6–8]). We also note that results concerning symmetric and cyclic systems of difference equations, for study that seemed to have been initiated in [33], are included in the papers [5,13–16,20–24,34–42] and the related references therein. Finally, we note that, because difference equations have several applications in applied sciences, there exists a rich bibliography concerning theory and applications (see [1–42]).

2. Stability of Zero Equilibrium of Equation (1)

In the following, we find conditions for the stability of the zero equilibrium of Equation (1) using center manifold theory.

Proposition 1. *Consider Equation (1), in which a, b, c, and d are real constants and x_0 and y_0 are also real numbers. Suppose that the following relations hold:*

$$(1+a)(1+c) = bd, \;\; -2 < a+c < 0. \tag{3}$$

Then the matrix

$$J = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$$

has one eigenvalue $\lambda_1 = -1$ and the other eigenvalue $\lambda_2 < 1$. Suppose also that one of the following holds: Equation (3) and

$$\frac{-2+\sqrt{4-3c^2}}{3} < a, \;\; -1 < c < 0, \;\; b > \rho_2, \tag{4}$$

where

$$\rho_2 = \frac{(a+1)(a-c)^2 + D}{3a^2+4a+c^2}$$

and

$$D = (a+1)\sqrt{(a-c)^4 - (3a^2+4a+c^2)(3c^2+4c+a^2)}, \tag{5}$$

or Equation (3) and the relations

$$-1 < a,\ c < -\frac{2}{\sqrt{3}},\ m < b < \rho_2, \tag{6}$$

where

$$m = \max\{0, \rho_1\},\ \rho_1 = \frac{(a+1)(a-c)^2 - D}{3a^2 + 4a + c^2},$$

or Equation (3) and the relations

$$\frac{-2+\sqrt{4-3c^2}}{3} < a,\ -\frac{2}{\sqrt{3}} < c < -1,\ m < b < \rho_2. \tag{7}$$

Then the zero equilibrium of Equation (1) is stable.

Proof. The initial system can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \tag{8}$$

where

$$f(x,y) = by(e^{-x} - 1),\ g(x,y) = dx(e^{-y} - 1).$$

Because the characteristic equation of J is $p(\lambda) = \lambda^2 - \lambda(a+c) + ac - bd = 0$ and the conditions for Equation (3) hold, we have that $\lambda_1 = -1$ is an eigenvalue of J. Moreover, we obtain $p(\lambda) = (\lambda+1)(\lambda - a - c - 1)$, and therefore $\lambda_2 = a + c + 1$ is also an eigenvalue of J. Because Equation (3) is satisfied, it is clear that $|\lambda_2| < 1$. We now let

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where T is the matrix that diagonalizes J defined by

$$T = \begin{bmatrix} b & b \\ -1-a & 1+c \end{bmatrix}.$$

Then Equation (8) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1+a+c \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{bmatrix}, \tag{9}$$

where

$$\begin{aligned} \hat{f}(u,v) &= \tfrac{1}{b(a+c+2)}\Big((1+c)b(-(a+1)u + (1+c)v)(e^{-b(u+v)} - 1) - \\ &\quad db^2(u+v)(e^{(a+1)u-(1+c)v} - 1) \Big), \\ \hat{g}(u,v) &= \tfrac{1}{b(2+a+c)}\Big((1+a)b(-(a+1)u + (1+c)v)(e^{-b(u+v)} - 1) + \\ &\quad db^2(u+v)(e^{(a+1)u-(1+c)v} - 1) \Big). \end{aligned} \tag{10}$$

We now let $v = h(u)$ with $h(u) = \psi(u) + O(u^4)$ and $\psi(u) = Au^2 + Bu^3$, where A and B are real numbers, the center manifold. The use of $\psi(u)$ as an approximation of $h(u)$ is justified by

Theorem 7 of [9]. Consequently, according to Theorem 8 of [9] (see also Theorem 5.2 of [29]), and using Equations (9) and (10), the study of the stability of the zero equilibrium of Equation (1) reduces to the study of the stability of the scalar equation

$$u_{n+1} = -u_n + \hat{f}(u_n, \psi(u_n)) = G(u_n). \tag{11}$$

The map G can also be written in the following form:

$$G(u) = \quad -u + (c_1 u + c_2 u^2 + c_3 u^3)(e^{c_4 u + c_5 u^2 + c_6 u^3} - 1) + (c_7 u + c_8 u^2 + c_9 u^3)(e^{c_{10} u + c_{11} u^2 + c_{12} u^3} - 1), \tag{12}$$

where $c_i, i = 1, 2, ..., 12$ are real constants depending on a, c, b, d, and A. In what follows in this section, we use the following constants:

$$\begin{aligned} c_1 &= -\frac{(1+a)(1+c)}{2+a+c}, \; c_2 = A\frac{(1+c)^2}{2+a+c}, \; c_4 = -b, \; c_5 = -bA, \\ c_7 &= -\frac{bd}{2+a+c}, \; c_8 = -A\frac{bd}{2+a+c}, \; c_{10} = a+1, \; c_{11} = -A(c+1). \end{aligned} \tag{13}$$

We need to compute the constant A of the center manifold. From Equation (9), we take

$$h(-u + \hat{f}(u, \psi(u))) = (1+a+c)\psi(u) + \hat{g}(u, \psi(u)),$$

which implies that

$$A(-u + \hat{f}(u, \psi(u)))^2 + B(-u + \hat{f}(u, \psi(u)))^3 = (1+a+c)(Au^2 + Bu^3) + \hat{g}(u, Au^2 + Bu^3).$$

Then we take

$$Au^2 = A(1+a+c)u^2 + \frac{b(a+1)^2 u^2}{2+a+c} + \frac{bd(a+1)u^2}{2+a+c}.$$

Therefore,

$$A = -\frac{b(a+1)(a+d+1)}{(a+c)(2+a+c)}. \tag{14}$$

From Equation (12), we see that $G'(0) = -1$ and

$$G'''(0) = 6c_2c_4 + 6c_1c_5 + 3c_1c_4^2 + 6c_8c_{10} + 6c_7c_{11} + 3c_7c_{10}^2. \tag{15}$$

Then from Equations (13)–(15) and from Equation (3), $d = \frac{(a+1)(c+1)}{b}$, we obtain

$$\begin{aligned} G'''(0) &= \tfrac{3(a+1)(c+1)}{(-a-c)(a+c+2)^2} h(b), \\ h(b) &= (3a^2 + 4a + c^2)b^2 - 2(a+1)(a-c)^2 b + (a+1)^2(3c^2 + 4c + a^2). \end{aligned} \tag{16}$$

We suppose first that Equations (3) and (4) are satisfied, considering that ρ_2 is a root of the quadratic polynomial $h(b)$. We now show that

$$h(b) > 0. \tag{17}$$

We consider

$$\Delta = (a-c)^4 - (3a^2 + 4a + c^2)(3c^2 + 4c + a^2).$$

We can easily prove that

$$\Delta = -2(2+a+c)(a^3 + a^2c + c^3 + 4ac + ac^2). \tag{18}$$

Because $a > \frac{-2+\sqrt{4-3c^2}}{3}$, we obtain $3a^2 + 4a + c^2 > 0$. Moreover, because $c < 0$, we have

$$3a^2c + 4ac + c^3 < 0. \tag{19}$$

Then, if $a < 0$, from the inequality of Equation (19), we obtain

$$a^3 + a^2c + c^3 + 4ac + ac^2 = a(a-c)^2 + 3a^2c + 4ac + c^3 < 0. \tag{20}$$

Equations (3), (18), and (20) imply that $\Delta > 0$. This means that D, as defined in Equation (5), is a real number. Finally, from the third inequality of Equation (4), we have that Equation (17) is true.

Now, if $a > 0$, from Equation (3), we have

$$a^3 + a^2c + c^3 + 4ac + ac^2 = (a^2 + c^2)(a + c) + 4ac < 0, \tag{21}$$

and thus from Equations (3) and (18), we obtain $\Delta > 0$. Thus, D is a real number. Hence from the third inequality of Equation (4), we have that Equation (17) is satisfied. Therefore, from Equations (3), (4), and (16), we obtain $G'''(0) > 0$. This implies that $SG(0) = -G'''(0) - 3/2(G(0)'')^2 < 0$, where $SG(0)$ is the Schwarzian derivative. Hence the zero equilibrium is stable for the scalar Equation (11). Thus, from Theorem 8 of [9] (see also Theorem 5.2 of [29]), the zero equilibrium of the original system (Equation (1)) is stable.

We suppose now that Equations (3) and (6) are satisfied, considering that ρ_1 and ρ_2 are the roots of the quadratic polynomial $h(b)$ in Equation (16). From Equation (6), we obtain $4 - 3c^2 < 0$, and thus $3a^2 + 4a + c^2 > 0$. Therefore, arguing as in the first case, we can prove that D is a real number. Then from Equation (6), we take $h(b) < 0$. Therefore from Equations (3), (6), and (16), we obtain $G'''(0) > 0$. Then $SG(0) < 0$. Hence the zero equilibrium is stable for the scalar Equation (11). Thus, from Theorem 8 of [9] (see also Theorem 5.2 of [29]), the zero equilibrium of the original system (Equation (1)) is stable.

Finally, we suppose that Equations (3) and (7) are satisfied. As we have already seen in the first case from Equation (7), it follows that $3a^2 + 4a + c^2 > 0$. Thus, arguing as in the first case, we can prove that D is a real number. Then from Equation (7), we take $h(b) < 0$. Therefore from Equations (3), (7), and (16), we obtain $G'''(0) > 0$. Then $SG(0) < 0$. Hence the zero equilibrium is stable for the scalar Equation (11). Thus, from Theorem 8 of [9] (see also Theorem 5.2 of [29]), the zero equilibrium of the original system (Equation (1)) is stable. □

3. Stability of Zero Equilibrium of Equation (2)

In the following, we study the stability of the zero equilibrium of Equation (2) using center manifold theory.

Proposition 2. *Consider Equation (2) where a, b, c, and d are arbitrary real constants and the initial values x_0 and y_0 are also real numbers. Suppose that the following relations hold:*

$$(1+b)(1+d) = ac, \quad -2 < b + d < 0. \tag{22}$$

Then the matrix

$$J = \begin{bmatrix} b & a \\ c & d \end{bmatrix}$$

has one eigenvalue $\lambda_1 = -1$ and the other eigenvalue $\lambda_2 < 1$. Suppose also that either Equation (22) and

$$b > 0, \quad -1 < d < 0, \quad \max\left\{\frac{b(d+1)}{d}, -\sqrt{-\frac{b(b+1)(1+d)}{d}}\right\} < a < \min\left\{-\frac{b(b+1)}{d}, \sqrt{-\frac{b(b+1)(1+d)}{d}}\right\} \tag{23}$$

hold, or that Equation (22) and the relations

$$d > 0,\ \ -1 < b < 0,\ \ a > \max\{-\frac{b(b+1)}{d}, \sqrt{-\frac{b(b+1)(1+d)}{d}}\} \tag{24}$$

hold. Then the zero equilibrium of Equation (2) is stable.

Proof. The initial system can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} b & a \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \tag{25}$$

where

$$f(x,y) = bx(e^{-y} - 1),\ \ g(x,y) = dy(e^{-x} - 1).$$

Because the characteristic equation of J is $p(\lambda) = \lambda^2 - \lambda(b+d) + bd - ac = 0$ and the Equation (22) conditions hold, we have that $\lambda_1 = -1$ is an eigenvalue of J. Moreover, we obtain $p(\lambda) = (\lambda+1)(\lambda - b - d - 1)$, and therefore $\lambda_2 = b + d + 1$ is also an eigenvalue of J. Because Equation (22) is satisfied, it is clear that $|\lambda_2| < 1$. We let now

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where T is the matrix that diagonalizes J, defined by

$$T = \begin{bmatrix} a & a \\ -1-b & 1+d \end{bmatrix}.$$

Therefore, Equation (25) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1+b+d \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{bmatrix}, \tag{26}$$

where

$$\begin{aligned} \hat{f}(u,v) &= \tfrac{(1+d)b}{b+d+2}(u+v)(e^{(1+b)u-(1+d)v} - 1) - \tfrac{d}{b+d+2}(-(1+b)u + (1+d)v)(e^{-a(u+v)} - 1), \\ \hat{g}(u,v) &= \tfrac{(1+b)b}{b+d+2}(u+v)(e^{(1+b)u-(1+d)v} - 1) + \tfrac{d}{b+d+2}(-(1+b)u + (1+d)v)(e^{-a(u+v)} - 1). \end{aligned} \tag{27}$$

We now let $v = h(u)$ with $h(u) = \psi(u) + O(u^4)$ and $\psi(u) = Au^2 + Bu^3$, where A and B are real numbers, the center manifold. The use of $\psi(u)$ as an approximation of $h(u)$ is justified by Theorem 7 of [9]. Consequently, according to Theorem 8 of [9] (see also Theorem 5.2 of [29]) and using Equations (26) and (27), the study of the stability of the zero equilibrium of Equation (2) reduces to the study of the stability of the following scalar equation:

$$u_{n+1} = -u_n + \hat{f}(u_n, \psi(u_n)) = G(u_n). \tag{28}$$

The map G can also be written in the following form:

$$G(u) = \quad -u + (c_1u + c_2u^2 + c_3u^3)(e^{c_4u + c_5u^2 + c_6u^3} - 1) + (c_7u + c_8u^2 + c_9u^3)(e^{c_{10}u + c_{11}u^2 + c_{12}u^3} - 1), \tag{29}$$

where $c_i, i = 1, 2, ..., 12$ are real constants that depend on a, b, c, d, and A. In what follows, we use the following constants:

$$
\begin{aligned}
&c_1 = \frac{b(1+d)}{2+b+d}, \ c_2 = A\frac{b(1+d)}{2+b+d}, \ c_4 = 1+b, \ c_5 = -(1+d)A, \\
&c_7 = \frac{d(1+b)}{2+b+d}, \ c_8 = -A\frac{d(1+d)}{2+b+d}, \ c_{10} = -a, \ c_{11} = -aA.
\end{aligned} \tag{30}
$$

We need to compute the constant A. From Equation (26), we take

$$h(-u + \hat{f}(u, \psi(u))) = (1+b+d)\psi(u) + \hat{g}(u, \psi(u)),$$

which implies that

$$A(-u + \hat{f}(u, \psi(u)))^2 + B(-u + \hat{f}(u, \psi(u)))^3 = (1+b+d)(Au^2 + Bu^3) + \hat{g}(u, Au^2 + Bu^3).$$

Then we take

$$Au^2 = A(1+b+d)u^2 + \frac{b(b+1)^2u^2}{2+b+d} + \frac{ad(b+1)u^2}{2+b+d}.$$

Therefore,

$$A = \frac{(b+1)(ad+b+b^2)}{(-b-d)(2+b+d)}. \tag{31}$$

From Equation (29), we see that $G'(0) = -1$, and Equation (15) is satisfied. Then from Equations (15), (30), and (31), we obtain

$$G'''(0) = \frac{6(1+b)(b+b^2+ad)(b-d)(b+bd-ad)}{-(b+d)(b+d+2)^2} + \frac{3(b+1)}{2+b+d}(b(b+1)(d+1)+da^2). \tag{32}$$

Suppose firstly that Equations (22) and (23) hold. We now show that $G'''(0) > 0$. From Equation (23), we have $b - d > 0$. Because $a < -\frac{b}{d}(b+1)$ and $d < 0$, we obtain $ad > -b(b+1)$, and therefore $ad + b(b+1) > 0$. Moreover, because $a > \frac{b}{d}(d+1)$ and $d < 0$, we obtain $b(d+1) > ad$, and therefore $b + bd - ad > 0$. Hence

$$\frac{6(1+b)(b+b^2+ad)(b-d)(b+bd-ad)}{-(b+d)(b+d+2)^2} > 0. \tag{33}$$

In addition, from Equation (23), we have $|a| < \sqrt{-\frac{b}{d}(b+1)(d+1)}$. Hence $a^2d > -b(1+b)(1+d)$, which means that

$$\frac{3(b+1)}{2+b+d}(b(b+1)(d+1)+da^2) > 0. \tag{34}$$

Therefore, from Equations (32), (33), and (34), we have $G'''(0) > 0$.

Finally suppose that Equations (22) and (24) are satisfied. From Equation (24), we have $b - d < 0$. Because $a > \sqrt{-\frac{b}{d}(b+1)(d+1)}$, we have $a^2d + b(1+b)(1+d) > 0$. Moreover, from $a > -\frac{b}{d}(b+1)$, we obtain $ad + b^2 + b > 0$. It is now clear that $b + bd - ad < 0$. Therefore $G'''(0) > 0$.

Hence, we have $SG(0) < 0$, where $SG(0)$ is the Schwarzian derivative. This implies that the zero equilibrium is stable for the scalar Equation (28). Thus, from Theorem 8 of [9] (see also Theorem 5.2 of [29]), the zero equilibrium of the original system (Equation (2)) is also stable. □

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