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Article

Metrization Theorem for Uniform Loops with the Invertibility Property

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Abstract: In this paper, we have proved a metrization theorem that gives the sufficient conditions for a uniform IP-loop X to be metrizable by a left-invariant metric. It is shown that by consideration of topological IP-loop dual to X we obtain an analogical theorem for the case of the right-invariant metric.

Keywords: quasigroup; topological IP-loop; left-invariant uniformity; invariant metrics

1. Introduction

The theory of topological groups [1–6] is a currently well-established field. In contrast, the theory of topological quasigroups and loops [7–12] is not as well developed. This is because many of the constructions and conditions one takes for granted in the former theory do not extend easily to the latter. In our recently published papers [13,14], we dealt with the study of uniform loops with the invertibility property (uniform IP-loops, for short). In [13] we proved that in every locally compact topological IP-loop with topology induced by a left-invariant uniformity there exists at least one regular left Haar measure and in [14] we proved that this measure is essentially unique.

In this contribution, we extend our study concerning uniform IP-loops. Our aim is to provide a metrization theorem for uniform IP-loops. A metrization theorem for topological groups is proved in [2] (see also [1,3]). In this theorem, the sufficient conditions for a topological group X to be metrizable by a left-invariant metric are given. Our purpose in this paper is to prove an analogy of this metrizable theorem for the case that X does not have a group structure, X is only a quasigroup. A topological quasigroup is not metrizable by a left-invariant metric in general what we illustrate by presented example. IP-loops are a special case of quasigroups. We give here the conditions ensuring that an IP-loop with a left-invariant uniform topology is metrizable by a left-invariant metric. By consideration of the topological IP-loop $\overset{\frown}{X}$ dual to X we obtain that an IP-loop with a right-invariant uniform topology is metrizable by a right-invariant metric.

2. Basic Definitions and Facts

First, we recall the definitions of basic notions and some known facts which will be used in the following.

A quasigroup is a groupoid $(X, \cdot)$ in which for every two elements $a, b \in X$ every of the equations $ax = b$ and $ya = b$ has a unique solution in X. Instead of $a \cdot b$ we write ab. If a quasigroup X contains an identity element, then X is called a loop. Evidently, every associative loop is a group. A quasigroup $(X, \cdot)$ is called an IP-quasigroup (or a quasigroup with the invertibility property), if there exist mappings $f_P : X \to X$ and $f_L : X \to X$ such that, for any $x, y \in X$, it holds $(xy)f_P(y) = x$ and

$f_L(x)(xy) = y$. An IP-quasigroup with an identity element is called an IP-loop. In any IP-loop every element x in X has an inverse $x^{-1} \in X$ and it holds $(xy)^{-1} = y^{-1}x^{-1}$ for every x, $y \in X$. It is easy to see that IP-loop is a groupoid $(X, \cdot)$ with an identity element and with the following property: for each $x \in X$ there exists an element $x^{-1} \in X$ such that $(yx)x^{-1} = y$ and $x^{-1}(xy) = y$ for every $y \in X$.

The theory of quasigroups and loops was established by Bruck [7,8], see also [9–11]. Moufang loops [15,16] are a very important case of IP-loops. Another interesting example of an IP-loop is the octonion (Cayley) algebra $(O, +, \cdot)$ [17]. Every octonion x can be written in the form:

$$x = x_0e_0+x_1e_1+x_2e_2+x_3e_3+x_4e_4+x_5e_5+x_6e_6+x_7e_7, \tag{1}$$

where $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ are the unit octonions, e_0 is the scalar element (the real number 1) and $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are real coefficients. By linearity and distributivity, multiplication of octonions is completely determined once given a multiplication table for the unit octonions (see, e.g., [17]). The conjugate of an octonion:

$$x = x_0+x_1e_1+x_2e_2+x_3e_3+x_4e_4+x_5e_5+x_6e_6+x_7e_7 \tag{2}$$

is given by:

$$x^* = x_0-x_1e_1-x_2e_2-x_3e_3-x_4e_4-x_5e_5-x_6e_6-x_7e_7. \tag{3}$$

Conjugation is an involution of O and satisfies $(xy)^* = y^*x^*$. This octonionic multiplication is neither commutative ($e_ie_j = -e_je_i$ if i, j are distinct and non-zero) nor associative. On other hand, the nonzero elements of O form an IP-loop. The norm of the octonion x is defined as:

$$\|x\| = \sqrt{xx^*} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}, \tag{4}$$

where x^* is the conjugate of x. So this norm agrees with standard Euclidean norm on $\Re^8$. The existence of a norm on O implies the existence of inverses for all nonzero elements of O. The inverse of x, $x \neq 0$, is given by the equality $x^{-1} = \frac{x^*}{\|x\|^2}$. It satisfies $xx^{-1} = x^{-1}x = 1$. The couple $(O^n, \cdot)$, where the operation $\cdot$ is defined by the equality:

$$(o_1^1, o_2^1, \ldots, o_n^1)\cdot(o_1^2, o_2^2, \ldots, o_n^2) = (o_1^1o_1^2, o_2^1o_2^2, \ldots, o_n^1o_n^2), \tag{5}$$

is also an IP-loop.

A metric d on a groupoid $(X, \cdot)$ is said to be left-invariant, if $d(ax, ay) = d(x, y)$ for every $a, x, y \in X$. A right-invariant metric is defined analogously and a metric is said to be bi-invariant if it is both left and right invariant. The term "invariant" hence means that the distance is unchanged when you translate by a fixed element a. If X is abelian, then both left and right invariance implies bi-invariance and we simply say that d is invariant.

Example 1. *Let O_1 be a set of all octonions with a unit norm. Let d be the metric induced by the norm $\|\ \|$ on O_1 . Let x, y, a be octonions with a unit norm:*

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7,$$

$$y = y_0 + y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5 + y_6e_6 + y_7e_7,$$

$$a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7.$$

Since:

$$d(ax, ay) = \sqrt{\sum_{i=0}^{7} a_i^2 \sum_{k=0}^{7} (x_k - y_k)^2)} = \sqrt{\left(\sum_{i=0}^{7} a_i^2\right) \cdot \sum_{k=0}^{7} (x_k - y_k)^2}$$

$$= \sqrt{\sum_{i=0}^{7} a_i^2} \cdot \sqrt{\sum_{k=0}^{7} (x_k - y_k)^2} = \sqrt{\sum_{k=0}^{7} (x_k - y_k)^2} = d(x, y),$$

the metric d is left-invariant. Analogously we obtain that d is right-invariant.

Example 2. *Let $x, y, a \in O_1^n$, where O_1 is a set of octonions with a unit norm, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $a = (a_1, \ldots, a_n)$. Put:*

$$\overline{d}(x, y) = \sqrt{\sum_{i=1}^{n} d(x_i, y_i)} = \sqrt{\sum_{i=1}^{n} \sum_{k=0}^{7} (x_{i_k} - y_{i_k})^2},$$

where d is the metric from Example 1. It is easy to verify that $\overline{d}$ is also a metric. Since d is left-invariant, we obtain:

$$\overline{d}(ax, ay) = \sqrt{\sum_{i=1}^{n} d(a_i x_i, a_i y_i)} = \sqrt{\sum_{i=1}^{n} d(x_i, y_i)} = \overline{d}(x, y).$$

This means that the metric $\overline{d}$ is left-invariant. Analogously we obtain that the metric $\overline{d}$ is right-invariant.

A topological IP-loop is an IP-loop $(X, \cdot)$ with Hausdorff topology such that the transformation $X \times X \to X$ defined by $(x, y) \to x^{-1}y$ is continuous. A topological IP-loop is said to be connected, totally disconnected, compact, locally compact, etc., if the corresponding property holds for its underlying topological space. Topological IP-loops are studied, e.g., in [18–21].

In the following section, we will deal with topological IP-loops with a left-invariant uniformity. Therefore, we recall the definition of a uniform topology and remind the facts which will be further used.

A uniformity of a set X is non-empty system W of subsets of the Cartesian product $X \times X$ such that the following conditions are satisfied:

(1) Every element of W contains the diagonal $\Delta = \{(x, x); x \in X\}$.
(2) If $U \in W$, then $U^{-1} = \{(y, x); (x, y) \in U\} \in W$.
(3) If $U \in W$, then there exists $V \in W$ such that $V \circ V \subset U$, where $V \circ V = \{(x, y)$; there exists $z \in X$ such that (x, z) and (z, y) are in $V\}$.
(4) If $U, V \in W$, then $U \cap V \in W$.
(5) If $U \in W$ and $U \subset V \subset X \times X$, then $V \in W$.

The above described couple (X, W) is called a uniform space and its elements entourages. Every uniform space X becomes a topological space by defining a subset U of X to be open if and only if for every $x \in U$ there exists an entourage V such that $V[x] = \{y \in X; (x, y) \in V\}$ is a subset of U. A basis of a uniformity W is any system B of entourages of W such that every entourage of W contains a set belonging to B. Every uniform space has a basis of entourages consisting of symmetric entourages. It is known (see [22,23]) that any system B of subsets of the Cartesian product $X \times X$ is a basis of some uniformity of X if and only if the following conditions are satisfied:

(6) Every element of B contains the diagonal $\Delta = \{(x, x); x \in X\}$.
(7) If $U \in \mathsf{B}$, then there exists $V \in \mathsf{B}$ such that $V \subset U^{-1}$.
(8) If $U \in \mathsf{B}$, then there exists $V \in \mathsf{B}$ such that $V \circ V \subset U$.
(9) If $U, V \in \mathsf{B}$, then there exists $M \in \mathsf{B}$ such that $M \subset U \cap V$.

We will use throughout this paper the following notations. If E and F are any two subsets of a groupoid $(X, \cdot)$, then EF is the set of all elements of the form xy, where $x \in E$ and $y \in F$. If $x \in X$, it is customary to write xE and Ex in place of $\{x\}E$ and $E\{x\}$, respectively. The set xE (or Ex) is called a left translation (or right translation) of E by the element x.

A uniformity of a groupoid $(X, \cdot)$ is called left-invariant, if it has a left-invariant basis B, i.e., a basis B such that, for every $a \in X$, it holds $(a, a)\,\mathsf{B} = \mathsf{B}$, where $(a, a)\,(x, y) = (ax, ay)$. A right-invariant uniformity is defined analogously. An invariant uniformity is a left-invariant uniformity which also satisfies a right-invariant condition.

A uniform topology is a generalization of a metric topology because if (X, d) is a metric space, then the system $\mathsf{B} = \{U_\varepsilon;\, \varepsilon > 0\}$, where $U_\varepsilon = \{(x, y); d(x, y) < \varepsilon\}$, is a basis of some uniformity of X. If a metric d of a quasigroup $(X, \cdot)$ is left-invariant, then a uniformity induced by the metric d is left-invariant, too. Indeed, for all $a \in X$ and any $U_\varepsilon \in \mathsf{B}$, we have:

$$U_\varepsilon = \{(x,y);\, d(x,y) < \varepsilon\} = \{(at,av); d(at,av) < \varepsilon\} = \{(at,av); d(t,v) < \varepsilon\} = (a,a)\,\{(t,v); d(t,v) < \varepsilon\} = (a,a)U_\varepsilon.$$

Analogously, if a metric d of a quasigroup $(X, \cdot)$ is right-invariant, then a uniformity induced by the metric d is also right-invariant.

3. Results

The following example shows that a topological quasigroup is not metrizable by invariant metric, in general.

Example 3. *Let $\Re$ be the set of all real numbers and $\circ$ be a binary operation on $\Re$ defined, for every $a, b \in \Re$, by the prescription $a \circ b = \frac{a+b}{2}$. Evidently, the couple $(\Re, \circ)$ is a topological quasigroup with standard topology. We shall show that this topological quasigroup is not metrizable topological space by an invariant metric. Suppose that there is a left-invariant metric d in $(\Re, \circ)$ and the metric topology induced by d coincides with standard topology. So the metric d and standard metric on $\Re$ are equivalent. We will derive a contradiction. Let $x, y \in \Re$, $x \neq y$. Since d is left-invariant, for every element a in $\Re$, we have:*

$$d(x,y) = d(a \circ x, a \circ y) = d\Big(\frac{a}{2} + \frac{x}{2}, \frac{a}{2} + \frac{y}{2}\Big)$$
$$= d\Big(a \circ \Big(\frac{a}{2} + \frac{x}{2}\Big), a \circ \Big(\frac{a}{2} + \frac{y}{2}\Big)\Big) = d\Big(\frac{a}{2} + \frac{a}{2^2} + \frac{x}{2^2}, \frac{a}{2} + \frac{a}{2^2} + \frac{y}{2^2}\Big)$$
$$= d\Big(\frac{a}{2} + \frac{a}{2^2} + \frac{a}{2^3} + \frac{x}{2^3}, \frac{a}{2} + \frac{a}{2^2} + \frac{a}{2^3} + \frac{y}{2^3}\Big) = \dots$$
$$= d\Big(\frac{a}{2} + \frac{a}{2^2} + \frac{a}{2^3} + \dots + \frac{a}{2^n} + \frac{x}{2^n}, \frac{a}{2} + \frac{a}{2^2} + \frac{a}{2^3} + \dots + \frac{a}{2^n} + \frac{y}{2^n}\Big)$$

for any $n \in N$. Put $x_n = \frac{a}{2} + \frac{a}{2^2} + \frac{a}{2^3} + \dots + \frac{a}{2^n} + \frac{x}{2^n}$, $y_n = \frac{a}{2} + \frac{a}{2^2} + \frac{a}{2^3} + \dots + \frac{a}{2^n} + \frac{y}{2^n}$. The sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ converge in the standard metric and then also in the metric d to element a. We derive a contradiction:

$$d(x,y) = \lim_{n\to\infty} d(x_n, y_n) = d(a,a) = 0.$$

This means that the topological quasigroup $(\Re, \circ)$ with standard topology is not metrizable topological space by left-invariant metric, and also by right-invariant metric because the operation $\circ$ is commutative.

Proposition 1. *Let $(X, \cdot)$ be an IP-loop and $\{U_n\}_{n=1}^{\infty}$ be a sequence of subsets of $X \times X$ such that:*

(1) $U_0 = X \times X$;
(2) $\Delta \subset U_n$, for $n = 0, 1, 2, \dots$;
(3) $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$, for $n = 0, 1, 2, \dots$;
(4) $(a, a)\, U_n = U_n$, for $n = 0, 1, 2, \dots$, and every $a \in X$.

Then there exists a non-negative real-valued function $d : X \times X \to \Re$ *with the following properties:*
(5) $d(x, z) \le d(x, y) + d(y, z)$, *for every elements* $x, y, z \in X$;
(6) $d(x, y) = d(ax, ay)$, *for every elements* $a, x, y \in X$;
(7) $U_{n+1} \subset U_n \subset \{(x, y); d(x, y) < 2^{-n}\} \subset U_{n-1}$, *for* $n = 1, 2, \ldots$.
If, moreover, every set U_n *is symmetric, then in* $(X, \cdot)$ *there exists a pseudometric d with the above properties.*

Proof. Let us define a non-negative real-valued function $f : X \times X \to \Re$ as follows:

$$f(x,y) = \begin{cases} 2^{-n}, & \text{if } (x,y) \in U_{n-1} - U_n; \\ 0, & \text{if } (x,y) \in U_n \quad \text{for every } n. \end{cases}$$

Then the searched function d has the following form:

$$d(x,y) = \inf\left\{ \sum_{i=0}^{n} f(x_i,\, x_{i+1});\; x_0 = x,\; x_{n+1} = y \right\},$$

where the infimum is taken over all elements $x_1, x_2, \ldots, x_n \in X$.

We shall prove that the function d has the required properties. For every $x, y, z \in X$, we have:

$$d(x,z) = \inf\left\{ \sum_{i=0}^{n} f(x_i, x_{i+1});\; x_1, x_2, \ldots, x_n \in X,\; x_0 = x,\; x_{n+1} = z \right\}$$

$$\le \inf\{ f(x, x_1) + \ldots + f(x_k, y) + f(y, y_1) + \ldots + f(y_m, z);\; x_1, \ldots, x_k, y_1, \ldots, y_m \in X \}$$

$$= \inf\left\{ \sum_{i=0}^{k} f(x_i, x_{i+1});\; x_1, \ldots, x_k \in X, x_0 = x, x_{k+1} = y \right\} + \inf\left\{ \sum_{j=0}^{m} f(y_j, y_{j+1}); y_1, \ldots, y_m \in X, y_0 = y, y_{m+1} = z \right\}$$

$$= d(x,y) + d(y,z).$$

This means that the first property holds.

Now, we shall prove the second property. Since $(X, \cdot)$ is an IP-loop, the condition $(a,a)U_n = U_n$ is equivalent to the condition $(a^{-1}, a^{-1})U_n = U_n$, for $n = 0, 1, 2, \ldots$, and every $a \in X$. Therefore $(x,y) \in U_{n-1} - U_n$ if and only if $(x,y) \in (a^{-1}, a^{-1})U_{n-1} - (a^{-1}, a^{-1})U_n$. Thus, $(x,y) \in U_{n-1} - U_n$ if and only if $(ax, ay) \in U_{n-1} - U_n$.

Since:

$$f(ax,ay) = \begin{cases} 2^{-n}, & \text{if } (ax,ay) \in U_{n-1} - U_n; \\ 0, & \text{if } (ax,ay) \in U_n \text{ for every } n, \end{cases}$$

we get $f(ax, ay) = f(x, y)$, for every $a, x, y \in X$.

Let ε be any positive real number. Then there are elements $x_1^\varepsilon, x_2^\varepsilon, \ldots, x_{n_\varepsilon}^\varepsilon \in X$ such that:

$$d(x,y) + \varepsilon > f(x, x_1^\varepsilon) + f(x_1^\varepsilon, x_2^\varepsilon) + \ldots + f(x_{n_\varepsilon}^\varepsilon, y)$$

$$= f(ax, ax_1^\varepsilon) + f(ax_1^\varepsilon, ax_2^\varepsilon) + \ldots + f(ax_{n_\varepsilon}^\varepsilon, ay) \ge d(ax, ay),$$

so $d(x, y) \ge d(ax, ay)$.

Analogously, there are elements $z_1^\varepsilon, z_2^\varepsilon, \ldots, z_{m_\varepsilon}^\varepsilon \in X$ such that:

$$d(ax,ay) + \varepsilon > f(ax, z_1^\varepsilon) + f(z_1^\varepsilon, z_2^\varepsilon) + \ldots + f(z_{m_\varepsilon}^\varepsilon, ay).$$

The couple $(X, \cdot)$ is an IP-loop, and therefore there exist the elements $r_1^\varepsilon, r_2^\varepsilon, \ldots, r_{m_\varepsilon}^\varepsilon \in X$ such that $z_i^\varepsilon = ar_i^\varepsilon$, for $i = 1, 2, \ldots, m_\varepsilon$. Hence:

$$d(ax, ay) + \varepsilon > f(ax, ar_1^\varepsilon) + f(ar_1^\varepsilon, ar_2^\varepsilon) + \ldots + f(ar_{m_\varepsilon}^\varepsilon, ay)$$

$$= f(x, r_1^\varepsilon) + f(r_1^\varepsilon, r_2^\varepsilon) + \ldots + f(r_{m_\varepsilon}^\varepsilon, y) \geq d(x, y),$$

so $d(ax, ay) \geq d(x, y)$. Therefore, we can conclude that $d(ax, ay) = d(x, y)$, for every $a, x, y \in X$.

It remains to prove the third property. The inclusions $U_{n+1} \subset U_n$, $n = 0, 1, 2, \ldots$, are the consequence of the previous two conditions. If $(x, y) \in U_n$, then $(x, y) \in U_k$, $k = 0, 1, \ldots, n$, and $d(x, y) \leq f(x, y) \leq 2^{-n-1} < 2^{-n}$. This means that $U_n \subset \{(x, y); d(x, y) < 2^{-n}\}$.

Let us prove the inclusion $\{(x, y); d(x, y) < 2^{-n}\} \subset U_{n-1}$, for $n = 1, 2, \ldots$. If $d(x, y) < 2^{-n}$, then there are the elements $x_0, x_1, \ldots, x_k, x_{k+1}$, where $x_0 = x$ and $x_{k+1} = y$, such that:

$$f(x_0, x_1) + f(x_1, x_2) + \ldots + f(x_k, x_{k+1}) < 2^{-n}.$$

By induction on k we prove that $(x_0, x_{k+1}) \in U_{n-1}$.

For $k = 0$ we have $f(x_0, x_1) = 2^{-m} < 2^{-n}$ and therefore $(x_0, x_1) \in U_{m-1} \subset U_{n-1}$. If $k = 1$, we have $f(x_0, x_1) + f(x_1, x_2) < 2^{-n}$, what implies the inequalities $f(x_0, x_1) \leq 2^{-(n+1)}$, $f(x_1, x_2) \leq 2^{-(n+1)}$. Hence we get $(x_0, x_1) \in U_n$, $(x_1, x_2) \in U_n$. Since $(x_2, x_2) \in U_n$, we have $(x_0, x_2) \in U_n \circ U_n \circ U_n \subset U_{n-1}$.

Assume that it holds for some $k \geq 1$, i.e., $f(z_0, z_1) + f(z_1, z_2) + \ldots + f(z_k, z_{k+1}) < 2^{-n}$ $\Rightarrow \exists\, i \in \{1, \ldots, k\}$ such that $(z_0, z_i) \in U_n$, and $(z_i, z_{k+1}) \in U_n$. Now, if $f(x_0, x_1) + f(x_1, x_2) + \ldots + f(x_{k+1}, x_{k+2}) < 2^{-n}$, then $f(x_0, x_1) = 2^{-n}$ and $f(x_1, x_2) + \ldots + f(x_{k+1}, x_{k+2}) < 2^{-n}$. There exists $i \in \{2, \ldots, k+1\}$ such that $(x_1, x_i) \in U_n$, and $(x_i, x_{k+2}) \in U_n$. So we can conclude that $(x_0, x_{k+2}) \in U_n \circ U_n \circ U_n \subset U_{n-1}$.

Since, by means of the second condition, we have $(x, x) \in U_n$, for $n = 0, 1, 2, \ldots$, it holds $f(x, x) = 0$, for every $x \in X$, and so $d(x, x) = 0$, for every $x \in X$. If $\{U_n\}_{n=0}^{\infty}$ is a sequence of symmetric subsets of $X \times X$, then $f(x, y) = f(y, x)$, for every $x, y \in X$, and therefore $d(x, y) = d(y, x)$, for every $x, y \in X$. The constructed function d is a pseudometric. The proof is completed. □

Theorem 1. *Let $(X, \cdot)$ be an IP-loop. A uniform space (X, W) is pseudometrizable by a left-invariant pseudometric if and only if the uniformity W of a set X has a left-invariant countable base. A uniform space (X, W) is metrizable by a left-invariant metric if and only if the corresponding topological space is Hausdorff and the uniformity W of X has a left-invariant countable base.*

Proof. If a uniformity W of a set X has a left-invariant countable base, then we can construct a system of symmetric subsets of a set $X \times X$ satisfying the condition of the preceding proposition. Therefore, the uniform space (X, W) is pseudometrizable by a left-invariant pseudometric. The second part of assertion of the theorem is obvious. □

If $(X, \cdot)$ is any topological IP-loop, one can consider the topological IP-loop $(\hat{X}, \circ)$ dual to X. The topological IP-loop $\hat{X}$ has, by definition, the same elements and same topology as X,the product $\circ$ in $\hat{X}$ is defined , for every $x, y \in \hat{X}$, by $x \circ y = y \cdot x$. Since, for every $x, y \in \hat{X}$, it holds:

$$x^{-1} \circ (x \circ y) = x^{-1} \circ (y \cdot x) = (y \cdot x) \cdot x^{-1} = y$$

and:

$$(x \circ y) \circ y^{-1} = (y \cdot x) \circ y^{-1} = y^{-1} \cdot (y \cdot x) = x,$$

we see that $(\hat{X}, \circ)$ is in fact an IP-loop.

Let $(X, \cdot)$ be a right-invariant uniform IP-loop. Consider the topological IP-loop $(\hat{X}, \circ)$ dual to X. If $\mathbf{B}$ is a right-invariant base of the uniformity W of $(X, \cdot)$, then $\mathbf{B}$ is a left-invariant base of a uniformity

W of the groupoid $(\hat{X}, \circ)$. Indeed, for every $x, y \in U$, where U is any entourage in **B**, and for every $a \in X$, we have:

$$(a,a) \circ (x,y) = (a \circ x,\, a \circ y\,) = (x \cdot a, y \cdot a) = (x,y) \cdot (a,a) = (x,y).$$

Thus the topological IP-loop $(\hat{X}, \circ)$ has a topology induced by a left-invariant uniformity. If d is a left-invariant metric in $(\hat{X}, \circ)$, then, for every $a, x, y \in X$, it holds $d(xa, ya) = d(a \circ x, a \circ y) = d(x, y)$, and hence the metric d is right-invariant in $(X, \cdot)$.

From the preceding considerations we obtain the following assertion.

Theorem 2. *Let $(X, \cdot)$ be an IP-loop. A uniform space (X, W) is metrizable by a right-invariant metric if and only if the corresponding topological space is Hausdorff and the uniformity W of X has a right-invariant countable base.*

4. Discussion

In the paper we provide a metrization theorem for uniform IP-loops. This theorem is a generalization of metrization theorem for a topological group X for the case that X does not have a group structure, X is only a quasigroup.

The proof of metrization theorem for topological groups [2] is based on the existence a decrease sequence $\{U_i; i = 0, 1, 2, \ldots\}$ of open neighborhoods of the identity e of a topological group G with the properties $U_i^{-1} = U_i$ and $U_{i+1}^2 \subset U_i$, for $i = 0, 1, 2, \ldots$. This sequence permits us to define open neighborhoods $U(r)$ for all dyadic rational numbers r with the property $U(\frac{m}{2^n}) \cdot U\left(\frac{1}{2^n}\right) \subset U\left(\frac{m+1}{2^n}\right)$ for all nonnegative integers m, n. If we put $f(x) = \inf\{r;\, r$ is a nonnegative dyadic rational and $x \in U(r)\}$, $x \in G$, then the function $N : G \to \Re_0^+$, $N(x) = \sup\{\,|f(yx) - f(y)\,|;\, y \in G\}$ is a continuous pseudo-norm on G. These results are achieved, inter alia, by using the associativity of the group operation. An operation of an IP-loop may not be associative. Thus the construction of metrizability of topological groups is not possible to extend on IP-loops fully.

It is necessary to observe that many results of the paper work for any uniform loop. This should provide a possibility of metrizability of any uniform loops.

5. Conclusions

In the paper we have proved a non-associative version of the classical Birkhoff-Kakutani theorem stating that a topological group is metrizable if and only if it has a countable base of neighborhoods at the identity. We have provided the sufficient conditions ensuring that an IP-loop X with a left-invariant uniform topology is metrizable by a left-invariant metric. Finally, it has been shown that by consideration of topological IP-loop dual to X we obtain an analogical theorem for the case of the right-invariant metric.

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