Some Interval Neutrosophic Linguistic Maclaurin Symmetric Mean Operators and Their Application in Multiple Attribute Decision Making

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Abstract: There are many practical decision-making problems in people’s lives, but the information given by decision makers (DMs) is often unclear and how to describe this information is of critical importance. Therefore, we introduce interval neutrosophic linguistic numbers (INLNs) to represent the less clear and uncertain information and give their operational rules and comparison methods. In addition, since the Maclaurin symmetric mean (MSM) operator has the special characteristic of capturing the interrelationships among multi-input arguments, we further propose an MSM operator for INLNs (INLMSM). Furthermore, considering the weights of attributes are the important parameters and they can influence the decision results, we also propose a weighted INLMSM (WINLMSM) operator. Based on the WINLMSM operator, we develop a multiple attribute decision making (MADM) method with INLNs and some examples are used to show the procedure and effectiveness of the proposed method. Compared with the existing methods, the proposed method is more convenient to express the complex and unclear information. At the same time, it is more scientific and flexible in solving the MADM problems by considering the interrelationships among multi-attributes.

Keywords: multiple attribute decision making (MADM); neutrosophic number; Maclaurin symmetric mean; linguistic variables

1. Introduction

The unclear set (FS) theory was put forward by Zadeh [1] in 1965. In this theory, the membership degree (MD) \( T(x) \) is used to describe fuzzy information and it has also been widely used in practice. However, the inadequacies of FS are evident. For example, it is difficult to express the non-membership degree (NMD) \( F(x) \). In order to fix this problem, Intuitionistic FS (IFS) was proposed by Atanassov [2] in 1986. It is made up of two parts: MD and NMD. IFS is an extension and development of Zadeh’ FS and Zadeh’ FS is a special case of IFS [3]. IFS needs to meet two conditions: (1) \( T(x), F(x) \in [0, 1] \); (2) \( 0 \leq T(x) + F(x) \leq 1 \) [2]. Subsequently, the IFS theory was further extended such as Zadeh [4] proposed interval IFS (IIFS). Zwick et al. [5] put forward the triangular IFS while Zeng and Li [6] defined trapezoidal IFS. However, under some circumstances due to the limited cognitive ability of the DMs, they may hesitate in the two choices for accuracy and uncertainty. Since they choose both of them at the same time, this can produce an imprecise or contradictory evaluation result. Therefore, Smarandache [7,8] introduced a concept called neutrosophic set (NS), which included MD, NMD,
and indeterminacy membership degree (IMD) in a non-standard unit interval [9]. Clearly, the NS is the generalization of FS and IFS. Furthermore, Wang [10] proposed the definition of interval NS (INS) which uses the standard interval to express the function of MD, IMD, and NMD. Broumi and Smarandache [11] presented the correlation coefficient of INS.

When dealing with the MADM problems with qualitative information, it is difficult for DMs to describe their own ideas with precise values. Generally, DMs ordinarily uses some linguistic terms (LTs) like “excellent”, “good”, “bad”, “very bad”, or “general” to indicate their evaluations. For example, when we look at a person’s height, we usually describe him as “high” or “very high” by visual inspection, but we will not give the exact value. In order to easily process the qualitative information, Herrera and Herrera-Viedma [12] proposed the LTs to deal with this kind of information instead of numerical computation. However, because LT such as “high” is not with MD, or we can think its MD is 1, which means LTs cannot describe the MD and NMD. Therefore, in order to facilitate DMs to describe the MD and NMD for one LT, Liu and Chen [13] defined the linguistic intuitionistic fuzzy number (LIFN), which combined the advantages of intuitionistic fuzzy numbers (IFNs) and linguistic variables (LVs). Therefore, LIFN can fully express the complex fuzzy information and there is a good prospect in MADM. After that, Ye [14] came up with the single-valued neutrosophic linguistic number (SVNLN). The most striking feature of the SVNLN is that it used LTs to describe the MD, IMD, and NMD. Sometimes, the three degrees are not expressed in a single real number, but is expressed in intervals [15]. And then, Ye [16] defined an interval neutrosophic linguistic set (INLS) and INLNs. INLNs is used to represent three values of MD, IMD, and NMD in the form of intervals. Clearly, INLS is a generalization of FS, IFS, NS, INS, LIFN, and SVNLN. It is general and beneficial for describing practical problems.

The aggregation operators (AOs) are an efficient way to handle MADM problems [17,18]. Many AOs are proposed for achieving some special functions. Yager [19] employed the ordered weighted average (OWA) operator for MADM. Bonferroni [20] proposed the Bonferroni mean (BM) operator, which can capture the correlation between input variables very well. Then BM operators have been extended to process different uncertain information such as IFS [21,22], interval-valued IFS [23], q-Rung Orthopai Fuzzy set [24], and Multi-valued Ns [25]. In addition, Beliakov [26] presented the Heronian mean (HM) operators, which have the same feature as the BM (i.e., they can capture the interrelationship between input parameters). Some HM operators have been proposed [27–30]. Furthermore, Yu [31] gave the comparison of BM with HM. However, since the BM operator and the HM operator can only reflect the relationship between any two parameters, they cannot process the MADM problems, which require the relationship for multiple inputs. In order to solve this shortage, Maclaurin [32] proposed the MSM operator, which has prominent features of capturing the relationship among multiple input parameters. Afterward, Qin and Liu [33] developed some MSM operators for uncertain LVs. Liu and Qin [34] developed some MSM for LIFNs. Liu and Zhang [35] proposed some MSM operators for single valued trapezoidal neutrosophic numbers.

Since the INLNs are superior to other ways of expressing complex uncertain information [16] and the MSM has good flexibility and adaptability, it can capture the relationship among multiple input parameters. However, now the MSM cannot deal with INLNs. Therefore, the objectives of this paper are to extend the MSM and weighted MSM (WMSM) operators to INLNs and to propose the INLMSM operator and the WINLMSM operator, to prove some properties of them and discuss some special cases, to propose a MADM approach with INLNs, and show the advantages of the proposed approach by comparing with other studies.

In Section 2 of this paper, we introduce some basic concepts about NS, INS, INLS, and MSM. In Section 3, we introduce the INLN and its operations including a new scoring function and a comparison method of INLN. In Section 4, we introduce an operator of INLMSM. Additionally, in order to improve flexibility, we propose the INLGMSM operator based on the GMSM operator. Furthermore, we develop the WINLMSM operator and the WINLGMSM operator to compare with operators that lack weight. Afterwards, we use examples to prove our theories. In Section 5, we give...
a MADM method for INLNs. In Section 6, we provide an example to demonstrate the effectiveness of the proposed method. Lastly, we provide the conclusions.

2. Preliminaries

In this section, we will introduce some existing definitions and basic concepts in order to understand this study.

2.1. The NS and INS

**Definition 1** [7–9]. Let \( X \) be a space of points (objects) with a generic element in \( X \) denoted by \( x \). A NS \( A \) in \( X \) is expressed by a MD \( T_\alpha(x) \), an IMD \( I(x) \), and a NMD \( F_\lambda(x) \).

Then a NS \( A \) is denoted below.

\[
A = \{(x, T_\alpha(x), I(x), F_\lambda(x))| x \in X \}
\]  

(1)

\( T_\alpha(x), I(x), \) and \( F_\lambda(x) \) are real standard or non-standard subsets of \([-0, 1]^+\]. That is

\[
T_\alpha: X \to [-0, 1]^+; \ I(x): X \to [-0, 1]^+; \ F_\lambda(x): X \to [-0, 1]^+.
\]

With the condition \(-0 \leq T_\alpha(x) + I(x) + F_\lambda(x) \leq 3^+\).

**Definition 2** [10, 11]. Let \( X \) be a space of points (objects) with a generic element in \( X \) denoted by \( x \). For convenience, the lower and upper ends of \( T, I, F \) are expressed as \( T_\alpha(x), I(x), I^U_\alpha(x), I^L_\alpha(x), F_\lambda(x), F^U_\lambda(x), F^L_\lambda(x) \).

An INS \( A \) in \( X \) is defined below.

\[
A = \left\{(x, \left[T^L_\alpha(x), T^U_\alpha(x)\right], [I^L_\alpha(x), I^U_\alpha(x)], [F^L_\lambda(x), F^U_\lambda(x)])| x \in X \right\}
\]  

(2)

For each point \( x \) in \( X \), we have that \( [T_\alpha(x), T^U_\alpha(x)] \subseteq [0, 1], [I^L_\alpha(x), I^U_\alpha(x)] \subseteq [0, 1], [F^L_\lambda(x), F^U_\lambda(x)] \subseteq [0, 1], \) and \( 0 \leq T^U_\alpha(x) + I^U_\alpha(x) + F^U_\lambda(x) \leq 3 \).

**Definition 3** [10, 11]. An INS \( A \) is contained in the INS \( B \), \( A \subseteq B \), if and only if \( T^L_\alpha(x) \leq T^L_B(x), T^U_\alpha(x) \leq T^U_B(x), I^L_\alpha(x) \geq I^L_B(x), I^U_\alpha(x) \geq I^U_B(x), F^L_\lambda(x) \geq F^L_B(x), \) and \( F^U_\lambda(x) \geq F^U_B(x) \). If \( A \subseteq B \) and \( A \supseteq B \), then \( A = B \).

2.2. LVs

**Definition 4** [36, 37]. Let \( S = \{s_i| i = 0, 1, \ldots, l, l \in N^*\} \) be a LT set (LTS) where \( N^* \) is a set of positive integers and \( s_i \) represents LVs.

Because the LTS is convenient and efficient, it is widely used by DMs in decision making. For instance, when we evaluate the production quality, we can set \( l = 9 \), then \( S \) is given below.

\[
S = \{s_0 = extremely\ bad, s_1 = very\ bad, s_2 = bad, s_3 = slightly\ bad, s_4 = fair, s_5 = slightly\ good, s_6 = good, s_7 = very\ good, s_8 = extremely\ good\}
\]

To relieve the loss of linguistic information in operations, Xu [38, 39] extended LTS \( S \) to continuous LTS \( \widehat{S} = \{s_0| 0 \leq \theta \leq l\} \). About the characteristics of LTS, please refer to References [38–40].

**Definition 5** [13]. Let \( s_\alpha \) and \( s_\beta \) be any two LVs in \( \widehat{S} \). The related operations can be defined below.

\[
s_\alpha \oplus s_\beta = s_{\alpha + \beta - \frac{\alpha + \beta}{2}} \tag{3}
\]

\[
\lambda s_\alpha = s_{\lambda - l(1 - \frac{l}{2})^+}, \lambda > 0 \tag{4}
\]
\[ s_\alpha \otimes s_\beta = s_{\alpha \cdot \beta} \quad (5) \]
\[ (s_\alpha)^\lambda = s_{\alpha^{\lambda}}, \lambda > 0 \quad (6) \]

2.3. MSM Operator

**Definition 6** [15,32]. Let \( x_i (i = 1, 2, \ldots, n) \) be the set of the non-negative real number. An MSM operator of dimension \( n \) is a mapping \( \text{MSM}^{(m)} : (R^+)^n \to R^+ \) and it can be defined below.

\[ \text{MSM}^{(m)}(x_1, \ldots, x_n) = \left( \frac{\sum_{1 \leq i_1 < \cdots < i_m \leq n} \prod_{j=1}^{m} x_{i_j}}{C_m^n} \right)^{\frac{1}{m}} \quad (7) \]

where \( (i_1, i_2, \ldots, i_m) \) traverses all the m-tuple combination of \( (1, 2, \ldots, n) \) and \( C_m^n = \frac{n!}{m!(n-m)!} \) is the binomial coefficient. In addition, \( x_{ij} \) refers to \( i,j \)th element in a particular arrangement.

There are some properties of the \( \text{MSM}^{(m)} \) operator, which are defined below.

1. Idempotency. If \( x_i = x \) for each \( i \), and then \( \text{MSM}^{(m)}(x, x, \ldots, x) = x \);
2. Monotonicity. If \( x_i \leq y_i \) for all \( i \), \( \text{MSM}^{(m)}(x_1, x_2, \ldots, x_n) \leq \text{MSM}^{(m)}(y_1, y_2, \ldots, y_n) \);
3. Boundedness. \( \min \{ x_1, x_2, \ldots, x_n \} \leq \text{MSM}^{(m)} \{ x_1, x_2, \ldots, x_n \} \leq \max \{ x_1, x_2, \ldots, x_n \} \).

Furthermore, the \( \text{MSM}^{(m)} \) operator would degrade some particular forms when \( m \) takes some special values, which are shown as follows.

1. When \( m = 1 \), the \( \text{MSM}^{(1)} \) operator would become the average operator.

\[ \text{MSM}^{(1)}(x_1, x_2, \ldots, x_n) = \left( \frac{\sum_{1 \leq i \leq n} x_i}{C_1^n} \right) = \frac{\sum_{i=1}^{n} x_i}{n} \quad (8) \]

2. When \( m = 2 \), the \( \text{MSM}^{(m)} \) operator would become the following BM operator \( (p = q = 1) \).

\[ \text{MSM}^{(2)}(x_1, \ldots, x_n) = \left( \frac{\sum_{1 \leq i < j \leq n} x_i x_j}{C_2^n} \right)^{\frac{1}{2}} = \left( \frac{2\sum_{1 \leq i < j \leq n} x_i x_j}{n(n-1)} \right)^{\frac{1}{2}} = BM^{1,1}(x_1, \ldots, x_n) \quad (9) \]

3. When \( m = n \), the \( \text{MSM}^{(m)} \) operator would become the geometric mean.

\[ \text{MSM}^{(n)}(x_1, \ldots, x_n) = \left( \prod_{j=1}^{n} x_j \right)^{\frac{1}{n}} \quad (10) \]

**Definition 7** [15]. Let \( x_i (i = 1, 2, \ldots, n) \) be the set of non-negative real numbers and \( p_1, p_2, \ldots, p_m \geq 0 \). A generalized MSM operator of dimension \( n \) is a mapping \( \text{GMSM}^{(m,p_1, p_2, \ldots, p_m)} : (R^+)^n \to R^+ \) and it is defined below.

\[ \text{GMSM}^{(m,p_1, p_2, \ldots, p_m)}(x_1, \ldots, x_n) = \left( \frac{\sum_{1 \leq i_1 < \cdots < i_m \leq n} \prod_{j=1}^{m} x_{i_j}^{p_j}}{C_m^n} \right)^{\frac{1}{\sum_{j=1}^{m} p_j}} \quad (11) \]
where \( (i_1, i_2, \ldots, i_m) \) traverses all the \( m \)-tuple combination of \( \{1, 2, \ldots, n\} \) and \( C_n^m = \frac{n!}{m!(n-m)!} \) is the binomial coefficient.

There are some properties of the \( GMSM^{(m, p_1, p_2, \ldots, p_n)} \) operator below.

1. Idempotency. If \( x_i = x \) for each \( i \), then \( GMSM^{(m, p_1, p_2, \ldots, p_n)}(x, x, \ldots, x) = x \);
2. Monotonicity. If \( x_i \leq y_i \) for all \( i \), \( GMSM^{(m, p_1, p_2, \ldots, p_n)}(x_1, x_2, \ldots, x_n) \leq GMSM^{(m, p_1, p_2, \ldots, p_n)}(y_1, y_2, \ldots, y_n) \);
3. Boundedness. \( \min\{x_1, x_2, \ldots, x_n\} \leq GMSM^{(m, p_1, p_2, \ldots, p_n)}\{x_1, x_2, \ldots, x_n\} \leq \max\{x_1, x_2, \ldots, x_n\} \).

In addition, the \( GMSM^{(m, p_1, p_2, \ldots, p_n)} \) operator would degrade to some particular forms when \( m \) takes some special values, which are shown below.

1. When \( m = 1 \), we have the formula below.
\[
GMSM^{(1, p_1)}(x_1, x_2, \ldots, x_n) = \left( \frac{\sum_{1 \leq i_1 \leq n} x_{i_1}^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}} = \left( \frac{n}{\sum_{j=1}^n x_j^{p_1}} \right)^{\frac{1}{p_1}}
\]
(12)

2. When \( m = 2 \), the \( GMSM^{(m, p_1, p_2, \ldots, p_n)} \) operator would become the following BM operator.
\[
GMSM^{(2, p_1, p_2)}(x_1, \ldots, x_n) = \frac{\sum_{1 \leq i_1 < i_2 \leq n} x_{i_1}^{p_1} x_{i_2}^{p_2}}{C_n^2} \left( \frac{1}{p_1+p_2} \right)^{\frac{1}{p_1+p_2}} = \frac{\sum_{1 \leq i_1 < i_2 \leq n} x_{i_1}^{p_1} x_{i_2}^{p_2}}{n(n-1)} \left( \frac{1}{p_1+p_2} \right)^{\frac{1}{p_1+p_2}}
\]
(13)

3. When \( m = n \), the \( MSM^{(m)} \) operator would become the following formula.
\[
GMSM^{(n, p_1, p_2, \ldots, p_n)}(x_1, \ldots, x_n) = \left( \prod_{j=1}^n x_j^{p_j} \right)^{\frac{1}{1+p_1+p_2+\ldots+p_n}}
\]
(14)

4. When \( p_1 = p_2 = \ldots = p_m = 1 \), the \( GMSM^{(m, p_1, p_2, \ldots, p_n)} \) operator would degenerate to the \( MSM \) operator and the parameter is \( m \) below.
\[
GMSM^{(m, 1, 1, \ldots, 1)}(x_1, \ldots, x_n) = \frac{\sum_{1 \leq i_1 < \ldots < i_m \leq n} \prod_{j=1}^m x_{i_j}^{p_j}}{C_n^m} \left( \frac{1}{p_1+p_2+\ldots+p_n} \right)^{\frac{1}{p_1+p_2+\ldots+p_n}} = MSM^{(m)}(x_1, \ldots, x_n).
\]
(15)

3. INLNs and Operations

**Definition 8** [16,41]. Let \( X \) be a finite universal set. An INLS in \( X \) is defined by the equation below.
\[
A = \left\{ x, \left( x, \langle s_{\theta(x)}, [T_A(x), I_A(x), F_A(x)] \rangle \right) \mid x \in X \right\}
\]
(16)
where \( s_{\theta(x)} \in S \), \( T_A(x) = [T_A^L(x), T_A^U(x)] \subseteq [0, 1] \), \( I_A(x) = [I_A^L(x), I_A^U(x)] \subseteq [0, 1] \), \( F_A(x) = [F_A^L(x), F_A^U(x)] \subseteq [0, 1] \) represent the MD, the IMD, and the NMD of the element \( x \) in \( X \) to the LV \( s_{\theta(x)} \), respectively, with the condition \( 0 \leq T_A^U(x) + I_A^U(x) + F_A^U(x) \leq 3 \) for any \( x \) in \( X \).

Then the seven tuple \( \langle s_{\theta(x)}, ([T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)]) \rangle \) in \( A \) is called an INLN. For convenience, an INLN can be represented as \( a = \langle s_{\theta(a)}, ([T^L(a), T^U(a)], [I^L(a), I^U(a)], [F^L(a), F^U(a)]) \rangle \).

Then we introduced the operational rules of operators of INLNs.
Definition 9 [16,37,42]. Let \( a_1 = \langle s_{\theta(a_1)}, \{T^L(a_1), T^U(a_1)\}, \{I^L(a_1), I^U(a_1)\}, \{F^L(a_1), F^U(a_1)\} \rangle \) and \( a_2 = \langle s_{\theta(a_2)}, \{T^L(a_2), T^U(a_2)\}, \{I^L(a_2), I^U(a_2)\}, \{F^L(a_2), F^U(a_2)\} \rangle \) be two INLNs and \( \lambda \geq 0 \). Then the operation of the INLNs can be expressed by the equation below.

\[
\begin{align*}
\lambda a_1 &= \left\langle s_{\lambda \times \theta(a_1)}, \left[ \left( 1 - (1 - T^L(a_1))^\lambda, 1 - (1 - T^U(a_1))^\lambda \right), \left( (I^L(a_1))^\lambda, (I^U(a_1))^\lambda \right), \left( (F^L(a_1))^\lambda, (F^U(a_1))^\lambda \right) \right] \right\rangle (\lambda > 0) \\
\lambda a_2 &= \left\langle s_{\lambda \times \theta(a_2)}, \left[ \left( (T^L(a_2))^\lambda, (T^U(a_2))^\lambda \right), \left[ 1 - (1 - I^L(a_2))^\lambda, 1 - (1 - I^U(a_2))^\lambda \right], \left[ (F^L(a_2))^\lambda, (F^U(a_2))^\lambda \right] \right] \right\rangle (\lambda > 0) \\
\end{align*}
\]

Example 1. Let \( a_1 = \langle s_3, ([0.1,0.2],[0.2,0.3],[0.4,5.0]) \rangle \) and \( a_2 = \langle s_4, ([0.3,0.5],[0.3,0.4],[0.5,0.6]) \rangle \) be two INLNs and \( S = \{ s_3 = \text{very bad}, s_4 = \text{bad}, s_2 = \text{slightly bad}, s_3 = \text{fair}, s_4 = \text{slightly good}, s_5 = \text{good}, s_6 = \text{very good} \} \), then we have the equations below.

\[
\begin{align*}
a_1 \oplus a_2 &= \langle s_{3+4}, ([0.1+0.3-0.1 \times 0.3,0.2+0.5-0.2 \times 0.5],[0.2 \times 0.3,0.3 \times 0.4],[0.4 \times 0.5,0.5 \times 0.6]) \\
&= \langle s_{7}, ([0.37,0.6],[0.06,0.12],[0.2,0.3]) \rangle \\
\lambda a_1 &= \langle s_{\lambda \times \theta(a_1)}, \left[ \left( 1 - (1 - T^L(a_1))^\lambda, 1 - (1 - T^U(a_1))^\lambda \right), \left( (I^L(a_1))^\lambda, (I^U(a_1))^\lambda \right), \left( (F^L(a_1))^\lambda, (F^U(a_1))^\lambda \right) \right] \rangle (\lambda > 0) \\
\end{align*}
\]

As seen from the above examples, these results are not reasonable because they exceed the range of LTS. In order to overcome these limitations, we will improve these operations by Definition 10.

Definition 10. Let \( a_1 = \langle s_{\theta(a_1)}, \{T^L(a_1), T^U(a_1)\}, \{I^L(a_1), I^U(a_1)\}, \{F^L(a_1), F^U(a_1)\} \rangle \) and \( a_2 = \langle s_{\theta(a_2)}, \{T^L(a_2), T^U(a_2)\}, \{I^L(a_2), I^U(a_2)\}, \{F^L(a_2), F^U(a_2)\} \rangle \) be two INLNs and \( \lambda \geq 0 \). Then the operations of the INLNs can be defined by the equations below.

\[
\begin{align*}
\lambda a_1 &= \left\langle s_{\lambda \times \theta(a_1)}, \left[ \left( 1 - (1 - T^L(a_1))^\lambda, 1 - (1 - T^U(a_1))^\lambda \right), \left( (I^L(a_1))^\lambda, (I^U(a_1))^\lambda \right), \left( (F^L(a_1))^\lambda, (F^U(a_1))^\lambda \right) \right] \right\rangle (\lambda > 0) \\
\lambda a_2 &= \left\langle s_{\lambda \times \theta(a_2)}, \left[ \left( (T^L(a_2))^\lambda, (T^U(a_2))^\lambda \right), \left[ 1 - (1 - I^L(a_2))^\lambda, 1 - (1 - I^U(a_2))^\lambda \right], \left[ (F^L(a_2))^\lambda, (F^U(a_2))^\lambda \right] \right] \right\rangle (\lambda > 0) \\
\end{align*}
\]
Based on the operational rules above, the above example is recalculated as follows.

**Example 2.** Let \( a_1 = \langle s_3, ([0.1, 0.2], [0.2, 0.3], [0.4, 0.5]) \rangle \) and \( a_2 = \langle s_4, ([0.3, 0.5], [0.3, 0.4], [0.5, 0.6]) \rangle \) be two INLNs and \( S = \{ s_0 = \text{very bad}, s_1 = \text{bad}, s_2 = \text{slightly bad}, s_3 = \text{fair}, s_4 = \text{slightly good}, s_5 = \text{good}, s_6 = \text{very good} \} \), then we have the equations below.

\[
\begin{align*}
\text{Let } a_1 \oplus a_2 &= \langle s_{3+4-\frac{3}{4}}, ([0.1 + 0.3 - 0.1 \times 0.3, 0.2 + 0.5 - 0.2 \times 0.5], [0.2 \times 0.3, 0.3 \times 0.4], [0.4 \times 0.5, 0.5 \times 0.6]) \rangle \\
&= \langle s_5, ([0.37, 0.6], [0.06, 0.12], [0.2, 0.3]) \rangle \\
\text{Let } a_1 \otimes a_2 &= \langle s_{3\times4}, ([0.1 \times 0.3, 0.2 \times 0.5], [0.2 + 0.3 - 0.2 \times 0.3, 0.3 + 0.4 - 0.3 \times 0.4], [0.4 + 0.5 - 0.4 \times 0.5, 0.5 + 0.6 - 0.5 \times 0.6]) \rangle \\
&= \langle s_2, ([0.03, 0.1], [0.44, 0.58], [0.7, 0.8]) \rangle 
\end{align*}
\]

From the above example, the results are more reasonable than the previous ones.

In the following definitions, a new scoring function and a comparison method of INLN are described.

**Definition 11.** [37]. Let \( a = \langle s_{\theta(a)}, ([T^L(a), T^U(a)], [I^L(a), I^U(a)], [F^L(a), F^U(a)]) \rangle \) be an INLN. Then the score function of \( a \) can be expressed by the equation below.

\[
S(a) = a \cdot \frac{\theta(a)}{6} \left[ 0.5(T^L(a) + 1 - F^L(a)) + aI^L(a) \right] + (1 - a) \cdot \frac{\theta(a)}{6} \left[ 0.5(T^L(a) + 1 - F^U(a)) + aI^U(a) \right] 
\] (25)

where the values of \( a \in [0, 1] \) reflect the attitudes of the decision makers.

**Definition 12.** [37]. Let \( a \) and \( b \) be two INLNs. Then the INLN comparison method can be expressed by the statements below.

- If \( S(a) > S(b) \), then \( a \succ b \); 
- If \( S(a) = S(b) \), then \( a \sim b \); 
- If \( S(a) < (b) \), then \( a \prec b \).

4. Some Interval Neutrosophic Linguistic MSM Operators

In this section, we will propose INLMSM operators and INLGMMSM operators.

4.1. The INLMSM Operators

**Definition 13.** Let \( a_i = \langle s_{\theta(i)}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)]) \rangle \) \((i = 1, 2, \ldots, n)\) be a set of INLNs. Then the INLMSM operator: \( \Omega^\prime : \Omega \rightarrow \Omega \) is shown below.

\[
\text{INLMSM}^{(m)}(a_1, \ldots, a_n) = \left( \frac{\bigoplus_{1 \leq i_1 < \ldots < i_m \leq n} (\bigotimes_{j=1}^{m} a_{i_j})}{C_n^m} \right) 
\] (29)

\( \Omega \) is a set of INLNs and \( m = 1, 2, \ldots, n \).

According to the operational laws of INLNs in Definition 10, we can get the expression of the INLMSM operator shown below.
Theorem 1. Let \( a_i = (s_{a_i}, (T^L(i_j), T^U(i_j)), (I^L(i_j), I^U(i_j)), (F^L(i_j), F^U(i_j))) \) (\( i = 1, 2, \ldots, n \)) be a set of \( \text{INLN}s \) and \( m = 1, 2, \ldots, n \). Then the value aggregated from Definition 13 is still an \( \text{INLN} \).

\[
\text{INLMSM}^{(m)}(a_1, \ldots, a_n) = \\
\left\langle s \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \left( \left[ \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \left( 1 - \Pi_{m+1}^{\text{up}}(1 - T^L_{i_j(k)}) \right) \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \right]^{\frac{1}{k}} \right) \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \left( 1 - \Pi_{m+1}^{\text{up}}(1 - T^U_{i_j(k)}) \right) \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \right) \right) \right) \right), \quad \text{(30)}
\]

where \( k = 1, 2, \ldots, C_m^n \), \( a_{i_j(k)} \) is the \( ij \)th element of \( k \)th permutation.

**Proof.**

Because

\[
a_{i_j(k)} = \left\langle s_{a_{i_j(k)}}, (T^L(i_j), T^U(i_j)), (I^L(i_j), I^U(i_j)), (F^L(i_j), F^U(i_j)) \right\rangle (j = 1, 2, \ldots, m)
\]

\[
\Rightarrow \prod_{j=1}^{m} a_{i_j(k)} = \left\langle s \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \left( \prod_{j=1}^{m} T^L_{i_j(k)} \prod_{j=1}^{m} T^U_{i_j(k)} \right) \right), \quad \text{(30)}
\]

\[
\Rightarrow \prod_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} \left\langle s \prod_{j=1}^{m} a_{i_j(k)} \right\rangle = \left\langle s \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \left( \prod_{j=1}^{m} T^L_{i_j(k)} \prod_{j=1}^{m} T^U_{i_j(k)} \right) \right) \right), \quad \text{(30)}
\]

\[
\Rightarrow \frac{\prod_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} (m, \otimes, a_{i_j(k)}) \prod_{j=1}^{m} a_{i_j(k)} \right\rangle}{C_m^n} = \left\langle s \prod_{j=1}^{1-\Pi_{m+1}^{\text{up}}(1-\Pi_{m+1}^{\text{up}}(\eta_{j}(i)))) \left( \prod_{j=1}^{m} T^L_{i_j(k)} \prod_{j=1}^{m} T^U_{i_j(k)} \right) \right) \right), \quad \text{(30)}
\]
Therefore, Theorem 1 is kept. □

**Property 1.** Let \( x_i = \langle s_{\alpha_i}([T^I(x_i), T^U(x_i)],[I^L(x_i), I^U(x_i)],[F^L(x_i), F^U(x_i)]) \rangle \) (\( i = 1, 2, \ldots, n \)) and \( y_i = \langle s_{\beta_i}([T^I(y_i), T^U(y_i)],[I^L(y_i), I^U(y_i)],[F^L(y_i), F^U(y_i)]) \rangle \) (\( i = 1, 2, \ldots, n \)) be sets of INLNs. There are four properties of \( \text{INLMSM}^{(m)} \) operator, which is shown below.

1. **Idempotency.** If the INLNs \( x_i = x \) \( \Rightarrow \langle s_{\beta_i}([T^I(x_i), T^U(x_i)],[I^L(x_i), I^U(x_i)],[F^L(x_i), F^U(x_i)] \rangle \rangle \) for each \( i(i = 1, 2, \ldots, n) \) and then \( \text{INLMSM}^{(m)} = x \) \( \Rightarrow \langle s_{\beta_i}((T^I, I^L, F^L, T^U, I^U, F^U)) \rangle \).
2. **Commutativity.** If \( x_i \) is a permutation of \( y_i \) for all \( i \) \( (i = 1, 2, \ldots, n) \) and then \( \text{INLMSM}^{(m)}(x_1, x_2, \ldots, x_n) = \text{INLMSM}^{(m)}(y_1, y_2, \ldots, y_n) \).
3. **Monotonicity.** If \( a_i \leq \beta_i, T^I(x_i) \leq T^I(y_i), T^U(x_i) \leq T^U(y_i), I^L(x_i) \geq I^L(y_i), I^U(x_i) \geq I^U(y_i), F^L(x_i) \geq F^L(y_i) \) and \( F^U(x_i) \geq F^U(y_i) \) for all \( i \) \( (i = 1, 2, \ldots, n) \), then \( x_i \leq y_i \) and \( \text{INLMSM}^{(m)}(x_1, x_2, \ldots, x_n) \leq \text{INLMSM}^{(m)}(y_1, y_2, \ldots, y_n) \).
4. **Boundedness.** \( \min\{x_1, x_2, \ldots, x_n\} \leq \text{INLMSM}^{(m)}(x_1, x_2, \ldots, x_n) \leq \max\{x_1, x_2, \ldots, x_n\} \).

**Proof.**

1. If each \( a_i = x \), then we get the equation below.

\[
\text{INLMSM}^{(m)}(x, x, \ldots, x) = \langle s_{(1-\Pi_{k=1}^n (1-1/\alpha_k))}^{\frac{1}{\alpha}} \left( 1 - \prod_{k=1}^n \left( 1 - \frac{m}{j=1} (1 - F^L_{i_j}(k)) \right) \right)^{\frac{1}{\alpha}} \rangle \]

Therefore, Theorem 1 is kept. □

2. This property is clear and it is now omitted.

3. If \( a_i \leq \beta_i, T^I(x_i) \leq T^I(y_i), T^U(x_i) \leq T^U(y_i), I^L(x_i) \geq I^L(y_i), I^U(x_i) \geq I^U(y_i), F^L(x_i) \geq F^L(y_i) \) and \( F^U(x_i) \geq F^U(y_i) \) for all \( i \), according to Theorem 1. Since

\[
\prod_{j=1}^m a_i \leq \prod_{i=1}^n \left( \frac{\prod_{j=1}^m \beta_i}{\prod_{j=1}^m (1 - F^L_{i_j}(k))} \right)^{\frac{1}{\alpha}} \leq \left( \prod_{k=1}^m \left( 1 - \frac{m}{j=1} (1 - F^L_{i_j}(k)) \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}} \]

\[
\prod_{j=1}^m I^L(x_i) \geq \prod_{i=1}^n I^L(y_i), \prod_{j=1}^m I^U(x_i) \geq \prod_{i=1}^n I^U(y_i), \prod_{j=1}^m F^L(x_i) \geq \prod_{i=1}^n F^L(y_i), \prod_{j=1}^m F^U(x_i) \geq \prod_{i=1}^n F^U(y_i)
\]

then \( l \cdot \left( 1 - \prod_{k=1}^m \left( 1 - \frac{m}{j=1} (1 - F^L_{i_j}(k)) \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}} \leq l \cdot \left( 1 - \prod_{k=1}^m \left( 1 - \frac{m}{j=1} (1 - F^L_{i_j}(k)) \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}} \)

\[
\left( 1 - \prod_{k=1}^m \left( 1 - \frac{m}{j=1} T^L(x_i) \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}} \leq \left( 1 - \prod_{k=1}^m \left( 1 - \frac{m}{j=1} T^L(x_i) \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}} \]
4 According to the idempotency, let \( \min\{x_1, x_2, ..., x_n\} = x_a = \text{INLMSM}^{(m)}(x_a, x_a, ..., x_a) \) and \( \max\{x_1, x_2, ..., x_n\} = x_b = \text{INLMSM}^{(m)}(x_b, x_b, ..., x_b) \). According to the monotonicity, if \( x_a \leq x_i \) and \( x_b \geq x_i \) for all \( i \), then we have \( x_a = \text{INLMSM}^{(m)}(x_a, x_a, ..., x_a) \leq \text{INLMSM}^{(m)}(x_1, x_2, ..., x_n) \) and

\[
\text{INLMSM}^{(m)}(x_1, x_2, ..., x_n) \leq x_b = \text{INLMSM}^{(m)}(x_b, x_b, ..., x_b).
\]

Therefore, we can get the conclusion below.

\[
\text{INLMSM}^{(m)}(x_1, x_2, ..., x_n) \leq \text{INLMSM}^{(m)}(y_1, y_2, ..., y_n)
\]

Furthermore, the \( \text{INLMSM}^{(m)} \) operator would degrade to some particular forms when \( m \) takes some special values.

(1) When \( m = 1 \), we have the formula below.

\[
\text{INLMSM}^{(1)}(x_1, x_2, ..., x_n) = \left( \frac{\pi_{m=1}^n x_i}{c_i} \right) = \left( 1 - \prod_{k=1}^m (1 - T_k^L)^{\frac{1}{2}} \right)^{\frac{1}{2}} \prod_{k=1}^m (1 - T_k^U)^{\frac{1}{2}},
\]

(31)
(2) When $m = 2$, we have the formula below.

$$\text{INLMSM}^{(2)}(x_1, x_2, \ldots, x_n) = \left\langle s_{t_0} \prod_{i=1}^{n} \left( \frac{1 - \Pi_{j=1}^{i-1} \left( 1 - \frac{n_2}{n_1} \left( 1 - \Pi_{k=1}^{i-1} (1 - T_{i,j}(k)) \right)^{n_1} \right)^{n_2} \right) \right\rangle,$$

(32)

(3) When $m = n$, the $\text{INLMSM}^{(m)}$ operator would reduce to the following form.

$$\text{INLMSM}^{(n)}(x_1, \ldots, x_n) = \left\langle s_{\prod_{j=1}^{n} \left( \frac{1 - \Pi_{j=1}^{n} (1 - T_{j})(1 - I_{j})}{n} \right)} \right\rangle,$$

(33)

Definition 14. Let $a_i = \left( s_{\prod_{j=1}^{n} \left( \frac{1 - \Pi_{j=1}^{n} (1 - T_{j})(1 - I_{j})}{n} \right)} \right)$, where $i = 1, 2, \ldots, n$ be a set of INLN. Then the INLGMSM operator: $\Omega^a \rightarrow \Omega$ is shown below.

$$\text{INLGMSM}^{(m,p_1,p_2,\ldots,p_m)}(a_1, \ldots, a_n) = \left( \frac{\oplus_{1 \leq i_1 < \ldots < i_m \leq n} \left( \bigotimes_{j=1}^{m} a^m_{j,i_j} \right)}{C_n^m} \right),$$

(34)

$\Omega$ is a set of INLN and $m = 1, 2, \ldots, n$.

According to the operational laws of INLN in Definition 10, we can get the expression of the INLMSM operator shown below.

Theorem 2. Let $a_i = \left( s_{\prod_{j=1}^{n} \left( \frac{1 - \Pi_{j=1}^{n} (1 - T_{j})(1 - I_{j})}{n} \right)} \right)$, where $i = 1, 2, \ldots, n$ be a set of INLN and $m = 1, 2, \ldots, n$. Then the value aggregated from Definition 14 is still an INLN.
where $k = 1, 2, \ldots, c^n_m$, $a_{i,(k)}$ is the $i$th element of $k$th permutation. Therefore, Theorem 2 is kept. The process of proof is similar to Theorem 1 and is now omitted.

**Property 2.** Let $x_i = \langle s_{a_i}([T^L(x_i), T^U(x_i)], [I^L(x_i), I^U(x_i)], [F^L(x_i), F^U(x_i)]) \rangle$ (i = 1, 2, ..., n) and $y_i = \langle s_{b_i}([T^L(y_i), T^U(y_i)], [I^L(y_i), I^U(y_i)], [F^L(y_i), F^U(y_i)]) \rangle$ (i = 1, 2, ..., n) be two sets of INLNs. There are four properties of $\text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}$ operator shown as follows.

1. **Idempotency.** If the INLNs $x_i$ are a permutation of $y_i$ for all $i$ (i = 1, 2, ..., n), and then $\text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}(x_1, x_2, \ldots, x_n) = \text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}(y_1, y_2, \ldots, y_n)$.

2. **Commutativity.** If $x_i$ is a permutation of $y_i$ for all $i$ (i = 1, 2, ..., n), and then $\text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}(x_1, x_2, \ldots, x_n) = \text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}(y_1, y_2, \ldots, y_n)$.

3. **Monotonicity.** If $x_i \leq y_i$, $T(x_i) \leq T(y_i)$, $T^U(x_i) \leq T^U(y_i)$, $I(x_i) \geq I(y_i)$, $I^U(x_i) \geq I^U(y_i)$, $F^L(x_i) \geq F^L(y_i)$ and $F^U(x_i) \geq F^U(y_i)$ for all $i$ (i = 1, 2, ..., n), then $x_i \leq y_i$ and $\text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}(x_1, x_2, \ldots, x_n) \leq \text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}(y_1, y_2, \ldots, y_n)$.

4. **Boundedness.** $\min \{x_1, x_2, \ldots, x_n\} \leq \text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)} \{x_1, x_2, \ldots, x_n\} \leq \max \{x_1, x_2, \ldots, x_n\}$.

The proofs are similar to Property 1, which are now omitted. Furthermore, the $\text{INLGMSM}^{(m,p_1,p_2,\ldots,p_n)}$ operator would degrade to some particular forms when $m$ takes some special values.

1. **When $m = 1$**, we have the following formula.

\[
\text{INLGMSM}^{(1)}(x_1, x_2, \ldots, x_n) = \left( \frac{a_{i,1} \cdot x_i^P}{x_i^P} \right) \text{ } \vdash \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ }
(3) When \( m = n \), the INLMSM\(^{(m)}\) operator would reduce to the form below.

\[
\text{INLGMSM}^{(a_1, a_2, \ldots, a_n)}(a_1, \ldots, a_n) = \left\langle s_{\Theta^i}, \left( [T^L(a_i), T^{UL}(a_i)], [I^L(a_i), I^{UL}(a_i)], [F^L(a_i), F^{UL}(a_i)] \right) \right\rangle \quad (i = 1, 2, \ldots, n)
\]

4.2. Some Weighted INLMSM Operators

We will introduce two operators, which are the weighted forms of the INLMSM operator and \( \text{INLGMSM} \) operator.

**Definition 15.** Let \( a_i = \langle s_{\Theta^i}, ([T^L(a_i), T^{UL}(a_i)], [I^L(a_i), I^{UL}(a_i)], [F^L(a_i), F^{UL}(a_i)]) \rangle \) \( i = 1, 2, \ldots, n \) be a set of INLNs. Let \( \omega = \langle \omega_1, \omega_2, \ldots, \omega_n \rangle \) \( T \) is the weight vector and satisfies \( \sum_{i=1}^{n} \omega_i = 1 \) with \( \omega_i > 0 \) \( i = 1, 2, \ldots, n \). Each \( \omega_i \) represents the importance of \( a_i \). Then the WINLMSM operator: \( \Omega \rightarrow \Omega \) is defined below.

\[
\text{WINLMSM}^{(n)}(a_1, \ldots, a_n) = \left\langle s_{\Theta}, \left( \prod_{1 \leq i < j \leq n} \left( \sum_{j=1}^{n} \left( a_{ij} \right) \right) \right) \right\rangle,
\]

\( \Omega \) is a set of INLNs and \( m = 1, 2, \ldots, n \).

According to the operational laws of INLNs in Definition 10, we can get the expression of the WINLMSM operator, which is shown below.

**Theorem 3.** Let \( a_i = \langle s_{\Theta^i}, ([T^L(a_i), T^{UL}(a_i)], [I^L(a_i), I^{UL}(a_i)], [F^L(a_i), F^{UL}(a_i)]) \rangle \) \( i = 1, 2, \ldots, n \) be a set of INLNs and \( m = 1, 2, \ldots, n \), then the value aggregated from Definition 15 is still a WINLMSM operator.

\[
\text{WINLMSM}^{(m)}(a_1, \ldots, a_n) = \left\langle s_{\Theta}, \left( \prod_{i=1}^{n} \left( \prod_{j=1}^{n} \left( 1 - \left( 1 - T^{ij}(a_j) \right)^{\omega_{ij}} \right)^{1/\theta} \right) \right) \right\rangle,
\]

\[
\left[ 1 - \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( T^{ij}(a_{ij}) \right)^{\omega_{ij}} \right) \right)^{1/\theta} \right],
\]

\[
\left[ 1 - \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( F^{ij}(a_{ij}) \right)^{\omega_{ij}} \right) \right)^{1/\theta} \right]
\]

where \( k = 1, 2, \ldots, C^n_m \), \( a_{ij(k)} \) is the \( i,j \)th element of \( k \)th permutation. The process of proof is similar to Theorem 1. Now it is omitted.
Property 3. Let \( x_i = \langle s_{a_i}, ([T_i(x_i)], [I_i(x_i)], [F_i(x_i)]) \rangle \) (i = 1, 2, ..., n) and \( y_i = \langle s_{b_i}, ([T_i(y_i)], [I_i(y_i)], [F_i(y_i)]) \rangle \) (i = 1, 2, ..., n) be sets of INLNs. There are some properties of the WINLMSM\(^{(m)}\) operator as shown below.

1. **Reducibility.** When \( \omega = \left( \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right) \), then \( \text{WINLMSM}^{(m)}(a_1, a_2, ..., a_n) = \text{INLMSM}^{(m)}(a_1, a_2, ..., a_n) \).

2. **Monotonicity.** If \( a_i \leq b_i \), \( T_i(x_i) \leq T_i(y_i) \), \( I_i(x_i) \geq I_i(y_i) \), and \( F_i(x_i) \geq F_i(y_i) \) for all \( i \) (i = 1, 2, ..., n), then \( x_i \leq y_i \) and \( \text{WINLMSM}^{(m)}(x_1, x_2, ..., x_n) \leq \text{WINLMSM}^{(m)}(y_1, y_2, ..., y_n) \).

3. **Boundedness.** \( \min\{x_1, x_2, ..., x_n\} \leq \text{WINLMSM}^{(m)}\{x_1, x_2, ..., x_n\} \leq \max\{x_1, x_2, ..., x_n\} \).

**Proof.**

1. If \( \omega = \left( \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right) \), then \( \text{WINLMSM}^{(m)}(a_1, a_2, ..., a_n) = \)

\[
\left\langle s_{1-T^m_I(1-I^m_I)} \cdot \left\langle s_{(1-T^{m_I}(1-(1-T^{m_I})(1-T^{m_I})))} \cdot \left( \left( 1 - T^{m_I}(1-(1-T^{m_I})) \right) \cdot \left( 1 - T^{m_I}(1-(1-T^{m_I})) \right) \right) \right\rangle \right\rangle = \text{INLMSM}^{(m)}(a_1, a_2, ..., a_n).
\]

2. The proofs of Monotonicity and Boundedness are similar to Property 1, which are now omitted.

Furthermore, the \( \text{WINLMSM}^{(m)} \) operator would degrade a particular form when \( m \) takes some special values.

(1) When \( m = 1 \), we have the formula below.

\[
\text{WINLMSM}^{(1)}(a_1, a_2, ..., a_n) = \left\langle s_{1-T^1_I(1-I^1_I)} \cdot \left\langle s_{(1-T^1_I(1-(1-T^1_I)))} \cdot \left( \left( 1 - T^1_I(1-(1-T^1_I)) \right) \cdot \left( 1 - T^1_I(1-(1-T^1_I)) \right) \right) \right\rangle \right\rangle.
\]

(2) When \( m = 2 \), we have the formula below.

\[
\text{WINLMSM}^{(2)}(a_1, a_2, ..., a_n) = \left\langle s_{1-\frac{\sum_{i=1}^{n} (1-T^2_I(1-(1-T^2_I)))}{n}} \cdot \left\langle s_{(1-T^2_I(1-(1-T^2_I)) \cdot \left( \left( 1 - T^2_I(1-(1-T^2_I))) \cdot \left( 1 - T^2_I(1-(1-T^2_I)) \right) \right) \right) \right\rangle \right\rangle.
\]
When \( m = n \), we have the formula below.

\[
\text{WINLMSM}^{(\omega)}(a_1, \ldots, a_n) = \left\{ s_{(1, \ldots, 1)}(1 - \frac{1}{n} s_{(1, \ldots, 1)}{a_i}^{\omega_i}) \right\},
\]

\[
\left( \left[ \prod_{i=1}^{n} \left( 1 - \left( 1 - T_i^j \right)^{a_i} \right) \right]^{\frac{1}{n}}, \left( \prod_{i=1}^{n} \left( 1 - \left( 1 - T_i^j \right)^{a_i} \right) \right) \right]^{\frac{1}{n}},
\]

\[
\left[ 1 - \left( \prod_{i=1}^{n} \left( 1 - \left( T_i^j \right)^{a_i} \right) \right) \right]^{\frac{1}{n}}, \left[ 1 - \left( \prod_{i=1}^{n} \left( 1 - \left( T_i^j \right)^{a_i} \right) \right) \right]^{\frac{1}{n}} \right) \right).
\]

\[
(43)
\]

**Definition 16.** Let \( a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)]) \rangle (i = 1, 2, \ldots, n) \) be a set of INLNs. Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \) \( T \) is the weight vector and it satisfies \( \sum_{i=1}^{n} \omega_i = 1 \) with \( \omega_i > 0 \) \( (i = 1, 2, \ldots, n) \). Each \( \omega_i \) represents the importance of \( a_i \). Then the WINLGM operator: \( \Omega^n \to \Omega \) is defined below.

\[
\text{WINLGM}^{(m, p_1, p_2, \ldots, p_m)}(a_1, \ldots, a_n) = \left( \frac{1}{C_n^m} \prod_{1 \leq i < j \leq m} \left( m \prod_{j=1}^{n} \left( n \omega_{a_i} \cdot a_{i_{(j)}} \right) \right) \right)^{\prod_{1 \leq i < j \leq m} \frac{1}{p_1 + p_2 + \ldots + p_m}} \]

\[
(44)
\]

\( \Omega \) is a set of INLNs and \( m = 1, 2, \ldots, n \).

According to the operational laws of INLNs in Definition 10, we can get the expression of WINLMSM operator shown below.

**Theorem 4.** Let \( a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)]) \rangle (i = 1, 2, \ldots, n) \) be a set of INLNs and \( m = 1, 2, \ldots, n \). Then the value aggregated from Definition 16 is still an WINLMSM.

\[
\text{WINLMSM}^{(m, p_1, p_2, \ldots, p_m)}(a_1, \ldots, a_n) = \left\{ s_{(1, \ldots, 1)}(1 - \frac{1}{n} s_{(1, \ldots, 1)}{a_i}^{\omega_i}) \right\},
\]

\[
\left( \left[ 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - T_i^j \right)^{a_i} \right) \right]^{\frac{1}{n}}, \left( \prod_{i=1}^{n} \left( 1 - \left( 1 - T_i^j \right)^{a_i} \right) \right) \right]^{\frac{1}{n}},
\]

\[
\left[ 1 - \left( \prod_{i=1}^{n} \left( 1 - \left( T_i^j \right)^{a_i} \right) \right) \right]^{\frac{1}{n}}, \left[ 1 - \left( \prod_{i=1}^{n} \left( 1 - \left( T_i^j \right)^{a_i} \right) \right) \right]^{\frac{1}{n}} \right) \right).
\]

\[
(45)
\]

where \( k = 1, 2, \ldots, C_n^m, a_{i_{(k)}} \) is the \( i \)th element of \( k \)th permutation. The process of proof is similar to Theorem 1. It is now omitted.
Property 4. Let $x_i = \langle s_{x_i}, [T^L(x_i), T^U(x_i)], [I^L(x_i), I^U(x_i)], [F^L(x_i), F^U(x_i)] \rangle$ ($i = 1, 2, ..., n$) and 
$y_i = \langle s_{y_i}, [T^L(y_i), T^U(y_i)], [I^L(y_i), I^U(y_i)], [F^L(y_i), F^U(y_i)] \rangle$ ($i = 1, 2, ..., n$) be two sets of INLNs. There are some properties of the \(\text{WINLGSM}^{(m, p_1, p_2, ..., p_m)}\) operator shown below.

1. Reducibility. When \(\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})\). Additionally, \(\text{WINLGSM}^{(m, p_1, p_2, ..., p_m)}(a_1, a_2, ..., a_n) = \text{INLGSM}^{(m, p_1, p_2, ..., p_m)}(a_1, a_2, ..., a_n)\).

2. Monotonicity. If \(a_i \leq \beta_i, T^L(x_i) \leq T^L(y_i), T^U(x_i) \leq T^U(y_i), I^L(x_i) \geq I^L(y_i), I^U(x_i) \geq I^U(y_i)\), then \(x_i \leq y_i\) and \(\text{WINLGSM}^{(m, p_1, p_2, ..., p_m)}(x_1, x_2, ..., x_n) \leq \text{WINLGSM}^{(m, p_1, p_2, ..., p_m)}(y_1, y_2, ..., y_n)\).

3. Boundedness. \(\min \{x_1, x_2, ..., x_n\} \leq \text{WINLGSM}^{(m, p_1, p_2, ..., p_m)}(x_1, x_2, ..., x_n) \leq \max \{x_1, x_2, ..., x_n\}\).

The process of proof is similar to Property 3 and is now omitted.

Furthermore, the \(\text{WINLGSM}^{(m, p_1, p_2, ..., p_m)}\) operator would degrade some particular forms when \(m\) takes some special values.

(1) When \(m = 1\), we have the following formula.

\[
\text{WINLGSM}^{(1, p_1)}(a_1, \ldots, a_n) = \left\{ \begin{array}{l}
\left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - \left( 1 - T^L_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}} \right) \left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - T^U_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}} \\
\left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - I^L_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right) \left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - I^U_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}} \\
\left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - F^L_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right) \left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - F^U_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}}
\end{array} \right\} \right.
\]

(46)

(2) When \(m = 2\), we have the formula below.

\[
\text{WINLGSM}^{(2, p_1, p_2)}(a_1, \ldots, a_n) = \left\{ \begin{array}{l}
\left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - \left( 1 - T^L_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}} \right) \left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - T^U_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}} \\
\left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - I^L_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right) \left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - I^U_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}} \\
\left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - F^L_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right) \left( 1 - \prod_{i=1}^{n} \left( 1 - \left( 1 - F^U_{i(k)} \right)^{n \omega_1} \right)^{p_1} \right)^{\frac{1}{p_1}}
\end{array} \right\} \right.
\]

(47)
The following three attributes can be used to evaluate alternatives by the investment company:

- Symmetry
- 2018
- Step 3

Calculate the score values of the alternatives. We would use Definition 15 and the LTS is $C = \{C_1, C_2, \ldots, C_n\}$, where $\omega_i$ represents the importance of criterion $C_i$. The performance of alternative $d_j$ in criterion $C_i$ is rated by INLNs and the decision matrix is $A = (a_{ij})_{n \times m}$. The objective is to rank the alternatives.

The detailed steps are shown below.

**Step 1** Normalize the decision matrix.

We should normalize the decision-making information in the matrix. The benefit (the bigger the better) and the cost (the smaller the better) are the two possible types. In order to keep the consistency of types, it is necessary to convert the decision matrix $A$ into a standardized matrix $R = (r_{ij})_{m \times n}$.

If $C_i$ is cost type, then $r_{ij} = \langle s_{\theta_{ij}}, ([L^i(r_{ij}), U^i(r_{ij})], [1 - L^i(r_{ij}), 1 - U^i(r_{ij})]), [L^i(r_{ij}), U^i(r_{ij})], [L^i(r_{ij}), U^i(r_{ij})]) \rangle$.

else $r_{ij} = \langle s_{\theta_{ij}}, ([L^i(r_{ij}), U^i(r_{ij})], [1 - L^i(r_{ij}), 1 - U^i(r_{ij})]), [L^i(r_{ij}), U^i(r_{ij})], [L^i(r_{ij}), U^i(r_{ij})]) \rangle$.

**Step 2** Aggregate the criterion values of each alternative. We would use Definition 15 and Definition 16 to aggregate $r_{ij} (j = 1, 2, \ldots, n)$ of the $i$th alternative and get the overall value $z_i$.

**Step 3** Calculate the score values of $z_i (i = 1, 2, \ldots, m)$ according to Definition 11. If two score values are equal, then calculate the accuracy values and certainty values.

**Step 4** According to Step 3 and Definition 12, rank the alternatives.

**6. Illustrative Example**

There are many decision-making problems to be solved in the current society, which requires some decision-making methods.

In this section, we investigate an example (adapted from Ref [43]) about the MADM. In a MADM problem, there are four possible alternatives for an investment company including a car company ($A_1$), a food company ($A_2$), a computer company ($A_3$), and an arms company ($A_4$). The following three attributes can be used to evaluate alternatives by the investment company: the risk ($C_1$), the growth ($C_2$), and the environmental impact ($C_3$) where $C_1$ and $C_2$ are benefit types and $C_3$ is cost type. Then the evaluation values of alternatives are shown in Table 1 where the LTS is $S = \{S_0 = \text{extremely poor (EP)}, S_1 = \text{very poor (VP)}, S_2 = \text{poor (P)}, S_3 = \text{medium (M)}, S_4 = \text{good (G)}, S_5 = \text{very good (VG)}, S_6 = \text{extremely good (EG)}\}$, and the weight vector of criteria
is $\omega = (0.35, 0.25, 0.4)^T$. Now we will use the method proposed in this paper, according to the above
LTs and three criteria. Then we evaluate and sort the four options in Table 1.

### Table 1. Evaluation values of alternatives.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$s_5$, $[0.4,0.5], [0.2,0.3], [0.3,0.4]$</td>
<td>$s_6$, $[0.4,0.6], [0.1,0.2], [0.2,0.4]$</td>
<td>$s_8$, $[0.2,0.3], [0.1,0.2], [0.5,0.6]$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$s_6$, $[0.5,0.7], [0.1,0.2], [0.2,0.3]$</td>
<td>$s_5$, $[0.6,0.7], [0.1,0.2], [0.2,0.3]$</td>
<td>$s_8$, $[0.5,0.7], [0.2,0.2], [0.1,0.2]$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$s_4$, $[0.3,0.5], [0.1,0.2], [0.3,0.4]$</td>
<td>$s_5$, $[0.5,0.6], [0.1,0.3], [0.3,0.4]$</td>
<td>$s_8$, $[0.3,0.4], [0.1,0.2], [0.1,0.2]$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$s_4$, $[0.7,0.8], [0.0,0.1], [0.1,0.2]$</td>
<td>$s_4$, $[0.5,0.7], [0.1,0.2], [0.2,0.3]$</td>
<td>$s_8$, $[0.3,0.4], [0.1,0.2], [0.1,0.2]$</td>
</tr>
</tbody>
</table>

#### 6.1. The Method Based on the WINLMSM Operator

Generally, we can give $m = \frac{1}{2}$, so $m = 1$ and $m = 2$. Then, according to Section 5, we have the
statements below.

1. When $m = 1$, the steps are shown below.

**Step 1** Normalize the decision matrix.

From the example, the risk ($C_1$) and the growth ($C_2$) are benefit types while the environmental
impact ($C_3$) is cost type. We set up the decision matrix as shown below.

$$R = \begin{bmatrix}
(s_5, [0.4,0.5], [0.2,0.3], [0.3,0.4]) & (s_6, [0.4,0.6], [0.1,0.2], [0.2,0.4]) & (s_8, [0.2,0.3], [0.1,0.2], [0.5,0.6]) \\
(s_6, [0.5,0.7], [0.1,0.2], [0.2,0.3]) & (s_5, [0.6,0.7], [0.1,0.2], [0.2,0.3]) & (s_8, [0.5,0.7], [0.2,0.2], [0.1,0.2]) \\
(s_4, [0.3,0.5], [0.1,0.2], [0.3,0.4]) & (s_5, [0.5,0.6], [0.1,0.3], [0.3,0.4]) & (s_8, [0.3,0.4], [0.1,0.2], [0.1,0.2]) \\
(s_4, [0.7,0.8], [0.0,0.1], [0.1,0.2]) & (s_4, [0.5,0.7], [0.1,0.2], [0.2,0.3]) & (s_8, [0.3,0.4], [0.1,0.2], [0.1,0.2])
\end{bmatrix}$$

**Step 2** Aggregate all attribute values of each alternative and get the overall value of each alternative
$a_i$ denoted as $r_i(i = 1, 2, 3, 4)$.

$$r_1 = s_{0.6228}, \quad r_2 = s_{0.8306}, \quad r_3 = s_{0.7462}, \quad r_4 = s_{0.7778}$$

**Step 3** According to Definition 11, we assume $\alpha = 0.7$ and calculate the score values
of $r_i(i = 1, 2, 3, 4)$ below.

$$S(r_1) = s_{0.6228}, \quad S(r_2) = s_{0.8306}, \quad S(r_3) = s_{0.7462}, \quad S(r_4) = s_{0.7778}$$

**Step 4** According to Step 3 and Definition 12, we would get the ranking of the alternatives, which are
$A_2 \succ A_4 \succ A_3 \succ A_1$.

2. When $m = 2$, the steps are shown below.

**Step 1** Normalize the decision matrix.

From the example, the risk ($C_1$) and the growth ($C_2$) are benefit types while the environmental
impact ($C_3$) is cost type. We set up the decision matrix as shown below.

$$R = \begin{bmatrix}
(s_5, [0.4,0.5], [0.2,0.3], [0.3,0.4]) & (s_6, [0.4,0.6], [0.1,0.2], [0.2,0.4]) & (s_8, [0.2,0.3], [0.1,0.2], [0.5,0.6]) \\
(s_6, [0.5,0.7], [0.1,0.2], [0.2,0.3]) & (s_5, [0.6,0.7], [0.1,0.2], [0.2,0.3]) & (s_8, [0.5,0.7], [0.2,0.2], [0.1,0.2]) \\
(s_4, [0.3,0.5], [0.1,0.2], [0.3,0.4]) & (s_5, [0.5,0.6], [0.1,0.3], [0.3,0.4]) & (s_8, [0.3,0.4], [0.1,0.2], [0.1,0.2]) \\
(s_4, [0.7,0.8], [0.0,0.1], [0.1,0.2]) & (s_4, [0.5,0.7], [0.1,0.2], [0.2,0.3]) & (s_8, [0.3,0.4], [0.1,0.2], [0.1,0.2])
\end{bmatrix}$$
Step 2  Aggregate all attribute values of each alternative and get the overall value of each alternative \( a_i \) denoted as \( r_i \) (\( i = 1, 2, 3, 4 \)).

\[
\begin{align*}
  r_1 &= \langle (5.4841, (0.3190, 0.4520), (0.1406, 0.2420), (0.3391, 0.4771)) \rangle,
  r_2 &= \langle (5.3016, (0.5260, 0.6922), (0.1366, 0.2083), (0.1791, 0.2772)) \rangle,
  r_3 &= \langle (4.9567, (0.4224, 0.5587), (0.1077, 0.2741), (0.2494, 0.3754)) \rangle,
  r_4 &= \langle (4.4896, (0.4794, 0.6190), (0.0711, 0.1739), (0.1415, 0.2416)) \rangle
\end{align*}
\]

Step 3  According to Definition 11, we assume \( \alpha = 0.7 \) and calculate the score values of \( r_i \) (\( i = 1, 2, 3, 4 \)). We get the values below.

\[
S(r_1) = s_{0.5695}, S(r_2) = s_{0.7170}, S(r_3) = s_{0.6004}, S(r_4) = s_{0.5765}
\]

Step 4  According to Step 3 and Definition 12, we get the rankings of the alternatives below.

\[
A_2 \succ A_3 \succ A_4 \succ A_1
\]

6.2. The Method Based on the WINLGMSM Operator

When \( m = 1, p = 1 \), the WINLGMSM\(^{(1)} \) operator is the same as the WINLMSM\(^{(1)} \) operator. The steps are omitted here. When \( m = 2 \), the steps are below.

Step 1  Normalize the decision matrix.

From the example, the risk (C\(_1\)) , the growth (C\(_2\)) are benefit types and the environmental impact (C\(_3\)) is cost type, so we set up the matrix as step 1 of Section 6.1.

Step 2  Aggregate all attribute values of each alternative by the WINLMSM\(^{(2)} \) operator and get the overall value of each alternative \( a_i \) denoted as \( r_i \) (\( i = 1, 2, 3, 4 \))

\[
\begin{align*}
  r_1 &= \langle (5.4988, (0.3221, 0.4549), (0.1387, 0.2401), (0.3374, 0.4752)) \rangle,
  r_2 &= \langle (5.3735, (0.5264, 0.6938), (0.1358, 0.2070), (0.1772, 0.2745)) \rangle,
  r_3 &= \langle (5.0083, (0.4296, 0.5610), (0.1069, 0.2702), (0.2449, 0.3721)) \rangle,
  r_4 &= \langle (4.5371, (0.4892, 0.6244), (0.0634, 0.1698), (0.1390, 0.2381)) \rangle
\end{align*}
\]

Step 3  According to Definition 11, we assume \( \alpha = 0.7 \), calculate the score values of \( r_i \) (\( i = 1, 2, 3, 4 \)), and get the values shown below.

\[
S(r_1) = s_{0.5722}, S(r_2) = s_{0.7276}, S(r_3) = s_{0.6087}, S(r_4) = s_{0.5839}
\]

Step 4  According to Step 3 and Definition 12, we get the rankings of the alternatives, which are shown below.

\[
A_2 \succ A_3 \succ A_4 \succ A_1
\]

6.3. Comparative Analysis and Discussion

(1) Based on the results in Sections 6.1 and 6.2, we can show them by using Table 2. From Table 2, we know that there are the same ranking results in two methods when \( m = 1 \) or \( m = 2 \). However, the result when \( m = 1 \) is different from the one when \( m = 2 \). It can be explained that, when \( m = 1 \), the interrelationship between the attributes doesn’t need to be considered when \( m = 2 \). We can consider the interrelationship between two attributes.

(2) Furthermore, we get the comparisons for different values of \( P_1 \) and \( P_2 \) when \( m = 2 \), which are shown in Table 3. From Table 3, we know when \( m = 2 \) and \( P_1 \) and \( P_2 \) are not equal to zero, we can get the same ranking results, i.e., \( A_2 \succ A_4 \succ A_3 \succ A_1 \). However, when \( P_1 = 0 \) or \( P_2 = 0 \), the ranking results are different from the ones when \( P_1 \) and \( P_2 \) are not equal to zero. When \( P_1 = 0 \)
or \( P_2 = 0 \), the interrelationship between the attributes doesn’t need to be considered, so it can get the same ranking results as the ones when \( m = 1 \).

Table 2. Comparison of different operator.

<table>
<thead>
<tr>
<th>Operator</th>
<th>( m )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>( WINLMSM^{(m)} )</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>( A_2 \succ A_3 \succ A_4 \succ A_1 )</td>
</tr>
<tr>
<td>( WINLGM^{(m,p_1,p_2)} )</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( A_2 \succ A_3 \succ A_4 \succ A_1 )</td>
</tr>
</tbody>
</table>

Table 3. Comparisons of different values of \( P_1 \) and \( P_2 \) when \( m = 2 \).

<table>
<thead>
<tr>
<th>Operator</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( S_{(n_j)}(i = 1, 2, 3, 4) )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>( WINLGM^{(m,p_1,p_2)} )</td>
<td>2</td>
<td>1</td>
<td>( S_{(r_1)} = 0.6228 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.8306 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.7462 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.7778 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.6228 )</td>
<td></td>
</tr>
<tr>
<td>2 2</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.8306 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.7462 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.7778 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.5695 )</td>
<td></td>
</tr>
<tr>
<td>2 2</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7170 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
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<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6004 )</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.5765 )</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.5722 )</td>
<td></td>
</tr>
<tr>
<td>2 3</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7276 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6087 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.5839 )</td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.5769 )</td>
<td></td>
</tr>
<tr>
<td>3 1</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7387 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6227 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.6027 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.5745 )</td>
<td></td>
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<tr>
<td>3 2</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7196 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6116 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.6004 )</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.5717 )</td>
<td></td>
</tr>
<tr>
<td>3 3</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7256 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6079 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.5837 )</td>
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<td></td>
<td>( S_{(r_5)} = 0.5733 )</td>
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<tr>
<td>3 2</td>
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<td></td>
<td>( S_{(r_2)} = 0.7256 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6269 )</td>
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</tr>
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<td></td>
<td></td>
<td></td>
<td>( S_{(r_4)} = 0.6280 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_5)} = 0.5806 )</td>
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</tr>
<tr>
<td>3 3</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7226 )</td>
<td>( A_2 \succ A_4 \succ A_3 \succ A_1 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( S_{(r_3)} = 0.6071 )</td>
<td></td>
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<td>( S_{(r_4)} = 0.5997 )</td>
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<td>( S_{(r_5)} = 0.5741 )</td>
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<tr>
<td>3 3</td>
<td></td>
<td></td>
<td>( S_{(r_2)} = 0.7223 )</td>
<td>( A_2 \succ A_3 \succ A_4 \succ A_1 )</td>
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<td>( S_{(r_3)} = 0.6071 )</td>
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<td></td>
<td></td>
<td>( S_{(r_4)} = 0.5909 )</td>
<td></td>
</tr>
</tbody>
</table>
Furthermore, in order to verify the validity of the methods proposed in this paper, we can compare them with methods from Ye [16] and the ranking results are shown in Table 4.

From Table 4, we know that the best choice is $A_2$ for all methods, which is the same as the results produced above. However, the ranking results are different. Compared with the approach proposed by Ye [16], when $m = 1$, our ranking results have the same values as that of Ye [16], but when $m = 2$, our ranking results are different from the Ye method [16]. When $m = 1$, all methods don’t consider the interrelationship. They produce the same results, however, when $m = 2$. Our methods in this paper can take into account the interrelationship while the method by Ye [16] doesn’t consider the interrelationship. Therefore, there are different ranking results. Therefore, our methods are more suitable for the different applications.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Operator</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Methods in this paper</td>
<td>$\text{WINLMSM}^{(m)} m = 1$</td>
<td>$A_2 \succ A_4 \succ A_3 \succ A_1$</td>
</tr>
<tr>
<td></td>
<td>$\text{WINLMSM}^{(m)} m = 2$</td>
<td>$A_2 \succ A_3 \succ A_4 \succ A_1$</td>
</tr>
<tr>
<td></td>
<td>$\text{WINLGMSM}^{(m,p_1,p_2)} m = 1$</td>
<td>$A_2 \succ A_4 \succ A_3 \succ A_1$</td>
</tr>
<tr>
<td></td>
<td>$\text{WINLGMSM}^{(m,p_1,p_2)} m = 2$</td>
<td>$A_2 \succ A_3 \succ A_4 \succ A_1$</td>
</tr>
<tr>
<td>Method in [16]</td>
<td>$\text{INLWAA}$</td>
<td>$A_2 \succ A_4 \succ A_3 \succ A_1$</td>
</tr>
<tr>
<td></td>
<td>$\text{INLWGA}$</td>
<td>$A_2 \succ A_4 \succ A_3 \succ A_1$</td>
</tr>
</tbody>
</table>

From the above comparison results, we can obtain that the methods proposed by this paper are feasible and adaptable for the MADM problems. Additionally, they have better reliability and wider application space than other existing methods.

7. Conclusions

In this study, we propose the concept of INLMSM, which can not only adapt to the cognitive situation of decision maker, but also provide convenience for decision making. We introduce the basic concept of INLMSM and its generalized form, give some operators based on INLMSM, and introduce the theory of weight to investigate WINLMSM and WINLGMSM. Afterwards, we put forward the INLMSM operator, the INLGMSM operator, the WINLMSM operator, and the WINLGMSM operator. In addition, we proved these operators. In addition, we introduce the MADM methods with INLMSM in detail and illustrate their usefulness and effectiveness by showing examples. Finally, we compare other methods to demonstrate our approach. From this paper, we can see that WINLGMSM is more practical and flexible in application and INLMSM can express fuzzy information more conveniently. In further study, we can use the INLMSM operator to solve practical problems and pattern recognition. We should develop other aggregation operators for future research.

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Author Contributions: Yushui Geng proposed extended MSM operators for INLNs; Xingang Wang designed the experiments; Xuemei Li and Kun Yu analyzed the data, and wrote the paper; Peide Liu proposed the ideas of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Compliance with Ethical Standards: (1) Disclosure of potential conflicts of interest. We declare that we have no commercial or associative interests that represent a conflict of interest in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work. (2) Research involving human participants and/or animals. This article does not contain any studies with human participants or animals performed by any of the authors.
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