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An Independent Set of Axioms of MV-Algebras and Solutions of the Set-Theoretical Yang–Baxter Equation

Tahsin Oner [1], Ibrahim Senturk [1,*] and Gulsah Oner [2]

[1] Department of Mathematics, Ege University, İzmir 35100, Turkey; tahsin.oner@ege.edu.tr
[2] Multidisciplinary Vocational School Technical Programme, Dokuz Eylul University, İzmir 35210, Turkey; gulsah.darilmaz@deu.edu.tr
* Correspondence: ibrahim.senturk@ege.edu.tr; Tel.: +90-232-311-2317

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Abstract: The aim of this paper is to give a new equivalent set of axioms for MV-algebras, and to show that the axioms are independent. In addition to this, we handle Yang–Baxter equation problem. In conclusion, we construct a new set-theoretical solution for the Yang–Baxter equation by using MV-algebras.

Keywords: MV-algebras; equivalence of set of axioms; independence; Yang–Baxter equation

1. Introduction

An axiomatic system is composed of a certain undefined or primitive term (or terms) together with a set of statements which are called axioms. Axioms are presupposed to be true. A theorem is any statement that can be deduced from the axioms by using inference rules. So, when an axiomatic system is given, a natural question arises: are its axioms independent? For instance, independence of the Axiom of Choice from Zermelo–Fraenkel set theory's axioms took many decades to be confirmed.

An independent axiomatization of any set of formulas from propositional or predicate logic was difficult to settle down. Many logicians have tried to undertake a solution, but they failed. The first partial success for countable sets was obtained by Tarski [1]. Kreisel [2] worked through research on independent recursive axiomatization.

A set of sentences Γ is *independent* if for all $\varphi \in \Gamma$, φ is not a logical consequence of $\Gamma - \{\varphi\}$, or equivalently, if there is a model of $(\Gamma - \{\varphi\}) \cup \{\neg\varphi\}$, where $\neg$ is the negation operation.

To show that a given finite set of formulas $\{F_1, F_2, ..., F_n\}$ is independent, it is sufficient to find an assignment of truth values that satisfies all the formulas F_j for each $1 \leq i \leq n$, where $j \neq i$, and that does not satisfy F_i. On the contrary, to demonstrate that it is not independent, one shows that one of the formulas F_i is a consequence of the other formulas.

Two sets of formulas are said to be *equivalent* if any formula of one set is a consequence of the other set, and conversely.

Oner and Terziler [3] have exemplified the situation by considering axioms for Boolean algebras; the proof of independence is obtained by using model forming. Additionally, Chajda and Kolařík [4] have proved that the axioms of basic algebras given in Chajda and Emanovský [5] are not independent.

Here, we consider MV-algebras which were originally introduced by Chang [6] and further developed by Chang [7]. A simplified axiomatization of MV-algebras in use today can be found in the monograph Algebraic Foundations of Many-valued Reasoning [8].

Definition 1. *[8] An MV-algebra is an algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ satisfying the following axioms (where A is a nonempty set, $\oplus$ is a binary operation on A, $\neg$ is a unary operation on A, and 0 is a constant element of A):*
$(MV1)$ $x \oplus 0 = x$
$(MV2)$ $\neg\neg x = x$
$(MV3)$ $x \oplus y = y \oplus x$
$(MV4)$ $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
$(MV5)$ $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
$(MV6)$ $x \oplus \neg 0 = \neg 0.$

In [9,10], this axiomatic system for $MV-$algebras was shown not to be independent. Moreover, M. Kolařík [10] proved that if the commutativity axiom is omitted in the above set of axiomatic system, the remaining axioms are independent.

On the other hand, firstly used in theoretical physics [11] and statistical mechanics [12–14], the Yang–Baxter equation has received more and more attention from researchers in a wide range of disciplines in the last years. Intending to apply the Yang–Baxter equation to individual works with different aspects from theoretical to practical in almost all sciences, technology, and industry, researchers have found uncountable and varied applications of this equation in fields such as link invariants, quantum computing, quantum groups, braided categories, knot theory, the analysis of integrable systems, quantum mechanics, etc. ([15,16]). In addition to these, one of the areas which the Yang–Baxter equation was practised is pure mathematics, and there have been many studies in this field, as in other areas. Several researchers have benefited from the axioms of various algebraic structures to solve this equation. Since it is well known that $MV-$algebras which provide an algebraic proof of the completeness theorem of infinite-valued Logics (or Łukasiewicz logics) are important algebraic tools to study in quantum structures, set-theoretical solutions for the Yang–Baxter equation could be examined in $MV-$algebras.

In this paper, we give some theorems in $MV-$algebras and obtain an equivalent set of axioms for $MV-$algebras in Section 2. In Section 3, we prove the independence of this equivalent set of axioms. In the last section, we find some solutions of the set-theoretical Yang–Baxter equation in $MV-$algebras.

2. An Equivalent Set of Axioms for *MV*-Algebras

In this section, we introduce a modified set of axioms for $MV-$algebras. To this end, we add a new axiom and remove axioms $(MV1)$ and $(MV6)$ from the above axiomatization of $MV-$algebras, and we prove that these two sets of axioms are equivalent to each other.

Theorem 1. *Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an $MV-$algebra. Then, there exists a unique element $y = 0 \in A$ such that $y \oplus x = x$ for all $x \in A$.*

Proof. By using $(MV1)$ and $(MV3)$, there exists a $0 \in A$ such that $0 \oplus x = x$ for all $x \in A$. For the uniqueness of this element, we take $x = 0$. Then, by the hypothesis:

$$y \oplus 0 = 0,$$

and since $(MV1)$ implies that $y \oplus 0 = y$, we have

$$y = 0.$$

For $x = 0$, the solution set of y contains only 0. In that case, the intersection sets of y solution values which verify this equation also contains 0. Thus, $y = 0$ is the unique element of A which holds $y \oplus x = x$ for all $x \in A$. □

Theorem 2. *Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an $MV-$algebra. Then*

$$\forall x \in A, x \oplus y = x \Leftrightarrow y = 0.$$

Proof. *Necessity*: It is clear by $(MV3)$ and Theorem 1.
Sufficiency: It follows from $(MV1)$. □

Theorem 3. *$\mathcal{A} = (A, \oplus, \neg, 0)$ is an $MV-$algebra if and only if it satisfies the following axioms:*
$(MV1')$ $x \oplus y = y \Leftrightarrow x = 0$
$(MV2)$ $\neg\neg x = x$
$(MV3)$ $x \oplus y = y \oplus x$
$(MV4)$ $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
$(MV5)$ $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

Proof. Necessity follows from Theorem 1 and [10]. For sufficiency, we have to show that the axioms $(MV1)$ and $(MV6)$ follow from the axioms in Theorem 3. We have $0 \oplus x = x$ by $(MV1')$. For $(MV1)$, it follows from $(MV3)$ that $0 \oplus x = x = x \oplus 0$. Now, we give a deduction of $(MV6)$ as follows:

$$\begin{array}{rcll}
\neg x \oplus 0 & = & \neg x & \text{(by } (MV1)) \\
 & \Rightarrow & \neg(\neg x \oplus 0) = \neg\neg x & \\
 & \Rightarrow & \neg(\neg x \oplus 0) = x & \text{(by } (MV2)) \\
 & \Rightarrow & \neg(\neg x \oplus 0) \oplus 0 = x \oplus 0 = x & \text{(by } (MV1)) \\
 & \Rightarrow & \neg(\neg 0 \oplus x) \oplus x = x & \text{(by } (MV5)) \\
 & \Rightarrow & \neg(\neg 0 \oplus x) = 0 & \text{(by } (MV1')) \\
 & \Rightarrow & \neg\neg(\neg 0 \oplus x) = \neg 0 & \\
 & \Rightarrow & \neg 0 \oplus x = \neg 0 & \text{(by } (MV2)) \\
 & \Rightarrow & x \oplus \neg 0 = \neg 0 & \text{(by } (MV3)).
\end{array}$$

Therefore, the set of axioms $(MV1') - (MV5)$ is an equivalent set of axioms for MV-algebras. □

3. Proof of the Independence of the Equivalent Set of Axioms

Theorem 4. *The axioms $(MV1')$, $(MV2)$, $(MV3)$, $(MV4)$, and $(MV5)$ are independent.*

Proof. To prove this claim, we construct a model for each axiom in which this axiom is false while the others are true. Let $\mathcal{M}$ be our model with the universe $U = \{0, 1, 2\}$. The symbols $\oplus^{\mathcal{M}}$, $\neg^{\mathcal{M}}$, and $0^{\mathcal{M}}$ are interpreted as a binary operation, a unary operation, and a constant, respectively. Let $\mathcal{A} = (\{0, 1, 2\}, \oplus, \neg, 0)$ be an algebra whose operations are defined as in the following tables:

(1) **Independence of $(MV1')$:**
We can define the operations $\oplus^{\mathcal{M}}$ and $\neg^{\mathcal{M}}$ as in the Table 1:

Table 1. Operations tables for the independence of $(MV1')$.

$\oplus$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

x	0	1	2
$\neg x$	0	1	2

The algebra $\mathcal{A}$ is not a model for $(MV1')$ because, if we choose $x = 0$ and $y = 1$, we obtain $x \oplus y = 0 \oplus 1 = 0$, which is a contradiction with $(MV1')$. Clearly, the algebra $\mathcal{A}$ satisfies $(MV2)$ and $(MV3)$. To see that $(MV4)$ and $(MV5)$ are satisfied by the algebra $\mathcal{A}$, it is sufficient to consider the following:

For $(MV4)$:

(i) If $x = 0$, then $0 \oplus (y \oplus z) = 0 = (0 \oplus y) \oplus z$.
(ii) If $x = 1$, then $1 \oplus (y \oplus z) = (1 \oplus y) \oplus z$ is true.
(iii) If $x = 2$, then $2 \oplus (y \oplus z) = (2 \oplus y) \oplus z$ is also true.

As for $(MV5)$:

(i) If $y = 0$, then $\neg(\neg x \oplus 0) \oplus 0 = 0 = \neg(\neg 0 \oplus x) \oplus x$.
(ii) If $y = 1$, then $\neg(\neg x \oplus 1) \oplus 1 = \neg(\neg 1 \oplus x) \oplus x$. It can be checked from the operations in Table 1 that the equation holds for $y = 1$.
(iii) If $y = 2$, then $\neg(\neg x \oplus 2) \oplus 2 = \neg(\neg 2 \oplus x) \oplus x$. It can be also controlled from the operations in Table 1 that the equation holds for $y = 2$.

(2) **Independence of** $(MV2)$**:**
The operations $\oplus^{\mathcal{M}}$ and $\neg^{\mathcal{M}}$ are defined as in the Table 2:

Table 2. Operations Tables for Independence of $(MV2)$.

$\oplus$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

x	0	1	2
$\neg x$	2	2	2

The algebra $\mathcal{A}$ satisfies $(MV1')$, $(MV3)$, $(MV4)$, $(MV5)$ but not $(MV2)$ because $\neg\neg 0 \neq 0$.

(3) **Independence of** $(MV3)$**:**
The operations $\oplus^{\mathcal{M}}$ and $\neg^{\mathcal{M}}$ are defined as in the Table 3:

Table 3. Operations Tables for Independence of $(MV3)$.

$\oplus$	0	1	2
0	0	1	2
1	1	1	1
2	1	1	1

x	0	1	2
$\neg x$	0	1	2

The algebra $\mathcal{A}$ satisfies $(MV1')$, $(MV2)$, $(MV4)$, $(MV5)$, but not $(MV3)$ because $0 \oplus 2 = 2$ but $2 \oplus 0 = 1$.

(4) **Independence of** $(MV4)$**:**
The operations $\oplus^{\mathcal{M}}$ and $\neg^{\mathcal{M}}$ are defined as in the Table 4:

Table 4. Operations tables for the independence of $(MV4)$.

$\oplus$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	1

x	0	1	2
$\neg x$	1	0	2

The algebra $\mathcal{A}$ satisfies $(MV1')$, $(MV2)$, $(MV3)$, $(MV4)$, but not $(MV5)$ because $(1 \oplus 1) \oplus 2 = 1$ but $(1 \oplus 1) \oplus 2 = 2$. So, $(1 \oplus 1) \oplus 2 \neq 1 \oplus (1 \oplus 2)$.

(5) **Independence of** $(MV5)$**:**
The operations $\oplus^{\mathcal{M}}$ and $\neg^{\mathcal{M}}$ are defined as in the Table 5:

Table 5. Operations tables for the independence of $(MV5)$.

$\oplus$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	1

x	0	1	2
$\neg x$	2	1	0

The algebra $\mathcal{A}$ satisfies $(MV1')$, $(MV2)$, $(MV3)$, $(MV5)$, but not $(MV4)$, because $\neg(\neg 0 \oplus 2) \oplus 2 = 1$ but $\neg(\neg 2 \oplus 0) \oplus 0 = 2$. So, $\neg(\neg 0 \oplus 2) \oplus 2 \neq \neg(\neg 2 \oplus 0) \oplus 0 = 2$.

□

4. Solutions of the Yang–Baxter Equation in *MV*-Algebras

Firstly used in theoretical physics by C.N. Yang and in statistical mechanics by R.J. Baxter almost fifty years ago, the Yang–Baxter equation has been studied as the master equation in integrable models in statistical mechanics and quantum field theory. Recent progress in other fields such as C^*- algebras, Hopf algebras, simple Lie algebras, representation theory, conformal field theory, etc. shed light on the significance of the equation, and has drawn attention among many researchers which have used the axioms of these algebraic structures in order to solve this equation.

In this section, we present solutions of the set-theoretical Yang–Baxter equation in $MV-$algebras. Let k be a field and tensor products be defined over the field k. For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$ and by $I : V \to V$ the identity map of the space V; for $R : V \otimes V \to V \otimes V$ a $k-$linear map, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, and $R^{13} = (I \otimes \tau)(R \otimes I)(\tau \otimes I)$.

Definition 2. *[16] A Yang–Baxter operator is $k-$linear map $R : V \otimes V \to V \otimes V$, which is invertible, and it satisfies the braid condition (called the Yang–Baxter equation):*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{1}$$

If R satisfies Equation (1), then both $R \circ \tau$ and $\tau \circ R$ satisfy the quantum Yang–Baxter equation (QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{2}$$

Lemma 1. *[16] Equations (1) and (2) are equivalent to each other.*

To establish a relation between the set-theoretical Yang–Baxter equation and $MV-$algebras, we need the following definition.

Definition 3. *[16] LetX be a set and* $S : X \times X \to X \times X$, $S(u,v) = (u',v')$ *be a map. The map* S *is a solution for the set-theoretical Yang–Baxter equation if it satisfies the following equation:*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \tag{3}$$

which is also equivalent

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}, \tag{4}$$

where

$$\begin{aligned} S^{12} &: X \times X \times X \to X \times X \times X, \;\; S^{12}(u,v,w) = (u',v',w), \\ S^{23} &: X \times X \times X \to X \times X \times X, \;\; S^{23}(u,v,w) = (u,v',w'), \\ S^{13} &: X \times X \times X \to X \times X \times X, \;\; S^{13}(u,v,w) = (u',v,w'). \end{aligned}$$

Now, we give a new method to construct solutions of the set theoretical Yang–Baxter equation by using $MV-$algebras.

Lemma 2. *Let* $(A, \oplus, \neg, 0)$ *be an* $MV-$*algebra. Then,* $S(x,y) = (x \oplus y, 0)$ *is a solution of the set-theoretical Yang–Baxter equation.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(x,y,z) &= (x \oplus y, 0, z), \\ S^{23}(x,y,z) &= (x, y \oplus z, 0). \end{aligned}$$

We verify the equality $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(x,y,z) \in A \times A \times A$. By using $(MV1')$ and $(MV4)$, we get

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(x \oplus y, 0, z) \\ &= S^{12}(x \oplus y, 0 \oplus z, 0) \\ &= S^{12}(x \oplus y, z, 0)) \\ &= ((x \oplus y) \oplus z, 0, 0) \\ &= (x \oplus (y \oplus z), 0, 0) \\ &= (x \oplus (y \oplus z), 0 \oplus 0, 0) \\ &= S^{23}(x \oplus (y \oplus z), 0, 0) \\ &= S^{12}(S^{23}(x, y \oplus z, 0)) \\ &= (S^{23} \circ S^{12} \circ S^{23})(x,y,z). \end{aligned}$$

Hence $S(x,y) = (x \oplus y, 0)$ is a solution of the set-theoretical Yang–Baxter equation in $MV-$algebras. □

Lemma 3. *Let $(A, \oplus, \neg, 0)$ be an $MV-$algebra. Then, $S(x, y) = (\neg y, \neg x)$ is a solution of the set-theoretical Yang–Baxter equation.*

Proof. We define S^{12} and S^{23} as follows:

$$S^{12}(x, y, z) = (\neg y, \neg x, z),$$
$$S^{23}(x, y, z) = (x, \neg z, \neg y).$$

We show that the equality $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ holds for all $(x, y, z) \in A \times A \times A$. By using $(MV2)$, we attain

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\
&= S^{12}(S^{23}(\neg y, \neg x, z) \\
&= S^{12}(\neg y, \neg z, \neg\neg x) \\
&= S^{12}(\neg y, \neg z, x)) \\
&= (\neg\neg z, \neg\neg y, x) \\
&= (z, y, x)
\end{aligned}$$

and

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\
&= S^{23}(S^{12}(x, \neg z, \neg y) \\
&= S^{23}(\neg\neg z, \neg x, \neg y) \\
&= S^{23}(z, \neg x, \neg y)) \\
&= (z, \neg\neg y, \neg\neg x) \\
&= (z, y, x).
\end{aligned}$$

Thus, $S(x, y) = (\neg y, \neg x)$ is a solution of the set-theoretical Yang–Baxter equation in $MV-$algebras. □

Definition 4. *[8] Let A be an $MV-$algebra.The binary relation $\leq$ defined on A as below*

$$x \leq y \quad \textit{if and only if} \quad \neg x \oplus y = 1$$

is a partial order on A. It is called the natural order of A.

Proposition 1. *[8] Let A be an $MV-$algebra with the natural order. If the join and the meet operators are defined as below:*

$$x \vee y = \neg(\neg x \oplus y) \oplus y \ \ \textit{and} \ \ x \wedge y = \neg(\neg x \oplus \neg(\neg x \oplus y)),$$

then the natural order determines a lattice structure on A.

Lemma 4. *[8] Any $MV-$algebra A is a Boolean algebra if and only if the operation $\oplus$ is idempotent on A.*

Lemma 5. *[17] Let $(A, \vee, \wedge, 0, 1, \neg)$ be a Boolean algebra. Then $S(x, y) = (x \vee y, x \wedge y)$ is a solution of the set-theoretical Yang–Baxter equation.*

Proposition 2. *Let $(A, \oplus, \neg, 0)$ be an $MV-$algebra. If the identities $\neg(\neg(x \oplus y) \oplus \neg z) = \neg(\neg x \oplus \neg z) \oplus \neg(\neg y \oplus \neg z)$ or $\neg(\neg x \oplus \neg y) \oplus z = \neg(\neg(x \oplus z) \oplus \neg(y \oplus z))$ are satisfied for all $x, y, z \in A$, then*

$$S(x,y) = (\neg(\neg x \oplus y) \oplus y), \neg(\neg x \oplus \neg(\neg x \oplus y)) \text{ and } S(x,y) = (x \oplus y, \neg(\neg x \oplus \neg y))$$

are solutions for the set-theoretical Yang–Baxter equation in $MV-$algebra A.

Proof. Assume that $\neg(\neg(x \oplus y) \oplus \neg z) = \neg(\neg x \oplus \neg z) \oplus \neg(\neg y \oplus \neg z)$ or $\neg(\neg x \oplus \neg y) \oplus z = \neg(\neg(x \oplus z) \oplus \neg(y \oplus z))$ holds for all $x, y, z \in A$. Now substituting $[x := 1, y := 1]$ or $[x := 0, y := 0]$ in these identities, respectively, we get $z = z \oplus z$ or $\neg z = \neg z \oplus \neg z$ for all $z \in A$. Then, the operation $\oplus$ is idempotent on A and so by Lemma 4, A is a Boolean algebra. From the definitions of meet and join operators, we obtain $S(x,y) = (\neg(\neg x \oplus y) \oplus y, \neg(\neg x \oplus \neg(\neg x \oplus y))) = (x \vee y, x \wedge y)$. Therefore, $S(x,y) = (\neg(\neg x \oplus y) \oplus y, \neg(\neg x \oplus \neg(\neg x \oplus y)))$ is a solution of the Yang–Baxter equation in $MV-$algebra A from Lemma 5. In addition, since the operation is idempotent, we have $(\neg(\neg x \oplus y) \oplus y, \neg(\neg x \oplus \neg(\neg x \oplus y))) = (x \oplus y, \neg(\neg x \oplus \neg y))$. Hence, $S(x,y) = (x \oplus y, \neg(\neg x \oplus \neg y))$ is also a solution of Yang–Baxter equation in $MV-$algebra A. □

Lemma 6. *[8] The below equations hold in every $MV-$algebra A:*

(i) $x \oplus y \oplus \neg(\neg x \oplus \neg y) = x \oplus y$,
(ii) $\neg(\neg x \oplus y) \oplus \neg(\neg(x \oplus \neg y) \oplus \neg y) = x$,
(iii) $\neg(\neg x \oplus \neg y) \oplus \neg(\neg(x \oplus y) \oplus \neg z) = \neg(\neg x \oplus \neg z) \oplus \neg(\neg(x \oplus z) \oplus \neg y)$.

Munduci et al. [8] needed the following proposition for the proof of the above lemma.

Proposition 3. *[8] The following identity*

$$\neg(\neg x \oplus \neg y) \oplus \neg(\neg(x \oplus y) \oplus \neg z) = \neg(\neg(x \oplus y) \oplus \neg(\neg(\neg x \oplus \neg y) \oplus z))$$

holds for each $MV-$algebra.

Theorem 5. *Let $(A, \oplus, \neg, 0)$ be an $MV-$algebra. Then, $S(x,y) = (x \oplus y, \neg(\neg x \oplus \neg y))$ is a solution of the set-theoretical Yang–Baxter equation in $MV-$algebra A.*

Proof. Through the instrumentality of equalities in Lemma 6 and Proposition 3, we obtain the following equalities:

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\
&= S^{12}(S^{23}(x \oplus y, \neg(\neg x \oplus \neg y), z)) \\
&= S^{12}(x \oplus y, \neg(\neg x \oplus \neg y) \oplus z, \neg(\neg\neg(\neg x \oplus \neg y) \oplus \neg z)) \\
&= S^{12}(x \oplus y, \neg(\neg x \oplus \neg y) \oplus z, \neg((\neg x \oplus \neg y) \oplus \neg z)) \\
&= ((x \oplus y) \oplus (\neg(\neg x \oplus \neg y) \oplus z), \neg(x \oplus y) \oplus \neg(\neg(\neg x \oplus \neg y) \oplus z), \\
&\quad \neg((\neg x \oplus \neg y) \oplus \neg z)) \\
&= (((x \oplus y) \oplus \neg(\neg x \oplus \neg y)) \oplus z, \neg(\neg(x \oplus y) \oplus \neg(\neg(\neg x \oplus \neg y) \oplus z)), \\
&\quad \neg((\neg x \oplus \neg y) \oplus \neg z)) \\
&= (x \oplus y \oplus z, \neg(\neg(x \oplus y) \oplus \neg(\neg(\neg x \oplus \neg y) \oplus z)), \neg(\neg x \oplus \neg y \oplus \neg z)),
\end{aligned}$$

and for the other direction of the set-theoretical Yang baxter equation, we attain

$$
\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\
&= S^{23}(S^{12}(x, y \oplus z, \neg(\neg y \oplus \neg z))) \\
&= S^{23}(x \oplus (y \oplus z), \neg(\neg x \oplus \neg(y \oplus z)), \neg(\neg y \oplus \neg z)) \\
&= (x \oplus (y \oplus z), \neg(\neg x \oplus \neg(y \oplus z)) \oplus \neg(\neg y \oplus \neg z), \neg(\neg\neg(\neg x \oplus \neg y \oplus z) \\
&\quad \oplus \neg\neg(\neg y \oplus \neg z))) \\
&= (x \oplus y \oplus z, \neg(\neg x \oplus \neg(y \oplus z)) \oplus \neg(\neg y \oplus \neg z), \neg((\neg x \oplus \neg(y \oplus z)) \\
&\quad \oplus(\neg y \oplus \neg z))) \\
&= (x \oplus y \oplus z, \neg(\neg x \oplus \neg(y \oplus z)) \oplus \neg(\neg y \oplus \neg z), \neg(\neg x \oplus (\neg(y \oplus z) \\
&\quad \oplus(\neg y \oplus \neg z)))) \\
&= (x \oplus y \oplus z, \neg(\neg x \oplus \neg y) \oplus \neg(\neg(x \oplus y) \oplus \neg z), \neg(\neg x \oplus \neg y \oplus \neg z).
\end{aligned}
$$

Therefore, $S(x, y) = (x \oplus y, \neg(\neg x \oplus \neg y))$ is a solution of the set-theoretical Yang–Baxter equation in $MV-$algebra A. $\square$

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