Electronics 2018, 7, 217; doi:10.3390/electronics7100217
www.mdpi.com/journal/electronics

**Article**

**Weighted Block Sparse Recovery Algorithm for High Resolution DOA Estimation with Unknown Mutual Coupling**

Dandan Meng 1, Xianpeng Wang 1,*, Mengxing Huang 1, Chong Shen 1 and Guoan Bi 2

1 State Key Laboratory of Marine Resource Utilization in South China Sea, College of Information Science and Technology, Hainan University, Haikou 570228, China; mengd0625@163.com (D.M.); huangmx09@163.com (M.H.); chongshen@hainu.edu.cn (C.S.)
2 School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, Singapore; egbi@ntu.edu.sg

*Correspondence: wxpeng1986@126.com; Tel.: +86-155-0185-9293

Received: 30 August 2018; Accepted: 21 September 2018; Published: 25 September 2018

**Abstract:** Based on weighted block sparse recovery, a high resolution direction-of-arrival (DOA) estimation algorithm is proposed for data with unknown mutual coupling. In our proposed method, a new block representation model based on the array covariance vectors is firstly formulated to avoid the influence of unknown mutual coupling by utilizing the inherent structure of the steering vector. Then a weighted $l_1$-norm penalty algorithm is proposed to recover the block sparse matrix, in which the weighted matrix is constructed based on the principle of a novel Capon space spectrum function for increasing the sparsity of solution. Finally, the DOAs can be obtained from the position of the non-zero blocks of the recovered sparse matrix. Due to the use of the whole received data of array and the enhanced sparsity of solution, the proposed method effectively avoids the loss of the array aperture to achieve a better estimation performance in the environment of unknown mutual coupling in terms of both spatial resolution and accuracy. Simulation experiments show the proposed method achieves better performance than other existing algorithms to minimize the effects of unknown mutual coupling.

**Keywords:** DOA estimation; unknown mutual coupling; block sparse recovery; reweighted $l_1$-norm penalty

1. Introduction

The main research objectives of array signal processing include angle estimation, or known as direction-of-arrival (DOA) estimation, and adaptive beamforming, often called spatial filtering or adaptive array processing [1]. In recent decades, there has been great interest in the study of high resolution DOA estimation which has become increasingly important to applications in electronic countermeasures, medical diagnosis, radar, and communication, etc. [2,3]. Meanwhile, inspired by the development of multiple-input multiple-output (MIMO) communication technology, MIMO radar has recently been proposed to obtain more degrees of freedom and high resolution for DOA estimation [4]. It has become a hot spot in the fields of autopilot, target location, and medicine [5–7]. One problem we must consider is that with a fixed array size, as the number of antennas is increased (or the distance between array elements is decreased), this inevitably brings about mutual coupling effect in the received data. In this article, we mainly study DOA estimation with a uniform linear array (ULA) to deal with the effect of unknown mutual coupling.

At present, the existing DOA estimation algorithms for the array are categorized into subspace-based methods and sparse signal recovery (SSR) methods. In the 1980s, the multiple
signals classification (MUSIC) algorithm was proposed to start the use of the subspace technique for DOA estimation [8]. Then, the algorithm of estimating signal parameters via rotational invariant techniques (ESPRIT) algorithm, was reported [9]. The above traditional DOA estimation algorithms are generally known as subspace-based algorithms. Because they are mainly based on eigenvalue decomposition (EVD) or singular value decomposition (SVD) of covariance matrix for DOA estimation, the performance is reduced in the case of low signal to noise ratio (SNR) or a limited number of snapshots [10–13]. In order to deal with the problems associated with traditional DOA estimation methods, the SSR algorithms, such as $l_1$-SVD algorithm [14], sparse Bayesian learning (SBL) algorithm [15,16], $l_1$-sparse representation of array covariance vector (SRACV) algorithm [17], and their derivative algorithms [18] were proposed in the past few years. The research of SSR algorithms has been mainly focused on two core procedures: the sparse representation model and the reconstruction algorithm. Extensive studies have shown that the performance of SSR algorithms is superior compared to the traditional subspace-based DOA estimation algorithms in the environments of low SNR or a small number of snapshots [19–22].

It is known that as the distance between array elements decreases, there may exist mutual coupling in the actual array antennas because space electromagnetic fields interact with each other [23]. In the above-mentioned DOA estimation algorithms, the steering vector depends on the array structure without considering unknown mutual coupling. Hence, the performance of traditional subspace-based algorithms and SSR algorithms will be severely reduced or even invalid under the condition of unknown mutual coupling [24–26]. To solve the DOA estimation problem with unknown mutual coupling, extensive research efforts have been made for performance improvement on the basis of traditional DOA estimation algorithms [27–30]. The banded complex symmetric Toeplitz structure of the mutual coupling matrix (MCM) was used to remove the influence of unknown mutual coupling with the help of auxiliary arrays to enable the ESPRIT algorithm to be directly used for DOA estimation [30]. By exploiting the banded complex symmetric Toeplitz special structure of MCM in [31], the $l_1$-SVD based algorithm was proposed under the condition of unknown mutual coupling. However, both of the above methods in [30,31] have lost the array aperture. In order to make full use of the received data, a new method for avoiding the effect of unknown mutual coupling was reported to be used in the MUSIC algorithm for DOA estimation [32]. Although there is no loss of array aperture, it also has the same limitation as that of subspace-based algorithms. For the SSR method, dealing with the unknown mutual coupling, an effective over-complete dictionary is proposed to avoid the influence of unknown mutual coupling [33]. Since this method takes advantage of the information of the entire received data, the DOA estimation performance is improved. However, this method is not robust since the DOA estimation performance is based on the length of the received data. Meanwhile, the performance of this method is also limited by the use of $l_1$-norm, which is only an approximation of $l_0$-norm.

In this paper, a high resolution DOA estimation method based on a weighted sparse recovery algorithm is proposed to deal with the effect of unknown mutual coupling. Firstly, a new block representation model is constructed by using an over-complete dictionary obtained with the parameterized steering matrix to avoid the influence of unknown mutual coupling. The block sparse matrix can be recovered by proposing the weighted $l_1$-SRACV algorithm, in which the weighted matrix is obtained by the principle of a novel Capon space spectrum function based on the parameterized steering vector for enhancing the sparsity of solution. Finally, the DOAs can be achieved by the position of the non-zero blocks in the recovered block sparse matrix. Computer simulations prove the efficiency and robustness of the proposed method.

Notation: $[\cdot]^T$, $[\cdot]^H$, $[\cdot]^{*}$ and diag{·} denote the transpose, conjugate-transpose, conjugate, and diagonal matrix, respectively. $I_M$ represents an $M \times M$ dimensional unit matrix and $\| \cdot \|_p$ represents the $p$-norm of a matrix or vector. $\text{det}(\cdot)$ is the determinant of a matrix and $\mathbb{E}[\cdot]$ represents the
mathematical expectation of the matrix or vector. $A^{-1}$ denotes the inverse of a matrix and $B^\frac{1}{2}$ means that the $n$th element of a column vector equal to the $l_2$-norm of the $n$th row of matrix $B$.

2. Date Model

Suppose $N$ narrowband far-field source signals $\{s_n(t)\}_{n=1}^N$, impinging on a uniform linear array (ULA) with $M$ array elements, where $t$ is the time index. The spacing between two adjacent array elements is generally assumed to be $d \leq \frac{\lambda}{2}$, where $\lambda$ indicates the signal wavelength. Because the signals come from different and unknown directions, the DOA is denoted as $\theta = [\theta_1, \theta_2, \cdots, \theta_N]$ in the range of $-90^\circ$ to $90^\circ$. If the first array element is regarded as a reference point and there is no unknown mutual coupling, the ideal array output can be expressed as

$$y(t) = A\bar{s}(t) + \bar{n}(t),$$

where $y(t) = [\bar{y}_1(t), \bar{y}_2(t), \cdots, \bar{y}_M(t)]^T$ indicates the received data. $A = [\bar{a}(\theta_1), \bar{a}(\theta_2), \cdots, \bar{a}(\theta_N)]$ denotes array manifold matrix, and its $n$th column vector $\bar{a}(\theta_n) = [1, \varphi(\theta_n), \cdots, \varphi(\theta_n)^{M-1}]^T$ represents the $M \times 1$ steering vector of the $n$th signal with $\varphi(\theta_n) = \exp(-j2\pi fc^{-1}d \sin \theta_n)$, and $c$ is the speed of propagation. In addition, $\bar{s}(t) = [\bar{s}_1(t), \bar{s}_2(t), \cdots, \bar{s}_N(t)]^T$ indicates the vector of a signal with zero mean. The vector $\bar{n}(t) = [\bar{n}_1(t), \bar{n}_2(t), \cdots, \bar{n}_M(t)]^T$ is an independent and narrowband Gaussian white noise.

In practice, when the adjacent array elements are relatively close, the unknown mutual coupling appears due to the interaction of space electromagnetic field. In order to achieve more accurate estimation of DOAs, the effect of unknown mutual coupling has to be properly dealt with. Because the structure of the array steering vector is changed by the unknown mutual coupling, the actual array steering vector takes the following form:

$$a_x(\theta) = DA\bar{a}(\theta),$$

where $D$ is the mutual coupling matrix (MCM) whose element magnitude is related to the distance between array sensors. In this way, the effect of mutual coupling between two closely located array sensors is considered. Based on the above analysis in ULAs, a banded symmetric Toeplitz matrix $[10]$ is adopted to represent the mutual coupling matrix (MCM) in ULA. So $D$ is a banded symmetric Toeplitz matrix, whose specific form is:

$$D = \begin{bmatrix}
1 & d_1 & \cdots & d_{K-1} \\
d_1 & 1 & \cdots & d_{K-1} \\
\vdots & \ddots & \ddots & \cdots \\
d_{K-1} & \cdots & 1 & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
d_{K-1} & \cdots & d_1 & 1 \\
0 & d_{K-1} & \cdots & d_1 \\
d_{K-1} & \cdots & d_1 & 1 \\
\end{bmatrix}_{M \times M},$$

where $d_i (i = 1, 2, \cdots, K-1)$ represents the value of the mutual coupling coefficient between two sensors. When the distance between two array elements is less than or equal to $K$ inter-sensor spacing, the effect of unknown mutual coupling between them cannot be overlooked. Then the array output expression with unknown mutual coupling is converted to:

$$y(t) = DA\bar{s}(t) + \bar{n}(t),$$
which shows that the steering vector has been changed greatly due to the existence of unknown mutual coupling. According to [32], a series of transformations are carried out for the steering vectors with unknown mutual coupling. Then the transformed steering vector can be formulated as the following:

\[ \mathbf{a}_z(\theta) = \mathbf{T}(\theta) \Delta(\theta) \mathbf{v}(\theta), \]  

where,

\[ \mathbf{T}(\theta) = 1 + \sum_{i=1}^{K-1} (d_i \varphi(\theta)^i + d_i \varphi(\theta)^{-i}), \]  

where \( \mathbf{T}(\theta) \) is a constant for each DOA, and the magnitude of its value is related to the mutual coupling. In this paper, we suppose \( \mathbf{T}(\theta) \neq 0, \)

\[ \Delta(\theta) = \begin{bmatrix} 1 & \varphi(\theta) & \cdots & \varphi(\theta)^{K-1} & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \varphi(\theta)^{M-K} & \cdots & \varphi(\theta)^{M-1} \end{bmatrix}_{M \times (2K-1)}, \]  

and

\[ \mathbf{v}(\theta) = [\sigma_1, \cdots, \sigma_{K-1}, 1, \phi_1, \cdots, \phi_{K-1}]^T, \]  

where \( \mathbf{v}(\theta) \) is a \((2K-1) \times 1\) column vector with the Kth element being one, and,

\[ \sigma_z = \frac{\varphi(\theta)^{K-1} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1-i} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1+i}}{\varphi(\theta)^{K-1} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1-i} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1+i}}, \]  

\[ \phi_z = \frac{\varphi(\theta)^{K-1} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1-i} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1+i}}{\varphi(\theta)^{K-1} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1-i} + \sum_{i=1}^{K-1} d_i \varphi(\theta)^{K-1+i}}, \]  

where \( z = 1, 2, \cdots, K - 1. \)

Hence, Equation (4) can be further transformed into:

\[ \mathbf{y}(t) = \mathbf{A}_f \mathbf{\Delta s}(t) + \mathbf{n}(t), \]  

where,

\[ \mathbf{A}_f = [\Delta(\theta_1), \Delta(\theta_2), \cdots, \Delta(\theta_N)], \]  

\[ \mathbf{\Delta} = \begin{bmatrix} \mathbf{T}(\theta_1) \mathbf{v}(\theta_1) & 0 \\ 0 & \mathbf{T}(\theta_2) \mathbf{v}(\theta_2) \\ \vdots & \vdots \\ 0 & 0 \\ \mathbf{T}(\theta_N) \mathbf{v}(\theta_N) \end{bmatrix}. \]  

According to Equation (11), it is known that there are no mutual coupling coefficients in matrix \( \mathbf{A}_f \), which can be regarded as a new array manifold matrix, similar to \( \mathbf{A} \), and where \( n = [1, 2, \cdots, N], \)
and \( Q = 2K - 1 \). Equation (11) only considers the signal reception model at time \( t \). If we consider the total received data of \( T \) snapshots, Equation (11) can be rewritten as:

\[
\mathbf{Y} = \mathbf{A}_f \mathbf{\Delta} \mathbf{S} + \mathbf{N},
\]

where \( \mathbf{Y} = [\mathbf{y}(t_1), \mathbf{y}(t_2), \ldots, \mathbf{y}(t_T)] \) represents \( M \times T \) the received data matrix, \( \mathbf{S} = [\mathbf{s}(t_1), \mathbf{s}(t_2), \ldots, \mathbf{s}(t_T)] \) denotes the signal waveform matrix, and \( \mathbf{N} = [\mathbf{n}(t_1), \mathbf{n}(t_2), \ldots, \mathbf{n}(t_T)] \) represents the noise matrix.

### 3. Sparse Representation Method for DOA Estimation

#### 3.1. l₁-SRACV Algorithm for DOA Estimation with Unknown Mutual Coupling

In this section, the \( l_{1} \)-SRACV algorithm is given for DOA estimation in ULA. Through the study of the new data model in Equation (11), it is found that the block diagonal matrix \( \mathbf{A} \) can be combined with the unknown signal \( \mathbf{s}(t) \) to form a new signal vector \( \mathbf{s}'(t) \). Hence, Equation (11) can be further rewritten as:

\[
\mathbf{y}(t) = \mathbf{A}_f \mathbf{s}'(t) + \mathbf{n}(t),
\]

where \( \mathbf{s}'(t) = \mathbf{A} \mathbf{s}(t) \) is a \( NQ \times 1 \) new signal vector, the \( n \)th block matrix of \( \mathbf{s}'(t) \) is from \( (Qn - Q + 1) \)th to \( (Qn) \)th rows of \( \mathbf{s}(t) \) corresponding to the \( n \)th element of signal vector \( \mathbf{s}(t) \). Therefore, the unknown mutual coupling coefficients are added to the part of unknown signals for avoiding the influence of mutual coupling on DOA estimation.

By using Equation (15), the covariance matrix of array output can be expressed as:

\[
\mathbf{R} = \mathbb{E}[\mathbf{y}(t)\mathbf{y}^H(t)] = \mathbf{A}_f \mathbf{R}_g \mathbf{A}_f^H + \sigma^2 \mathbf{I}_M,
\]

where \( \mathbf{R}_g = \mathbb{E}[\mathbf{s}'(t)\mathbf{s}'^H(t)] \) is the covariance matrix of the new signal \( \mathbf{s}'(t) \). In addition, \( \sigma^2 \) is the power of noise and \( \sigma^2 \mathbf{I}_M \) denotes the noise covariance matrix. At the same time, it can also be shown that \( \mathbf{R}_g \mathbf{A}_f^H \) and \( \mathbf{s}'(t) \) have the same block sparse structure [17]. However, in practice, only limited samples are available. Therefore, the covariance matrix is estimated by using the \( T \) available snapshots, i.e.,

\[
\mathbf{R} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}(t)\mathbf{y}^H(t),
\]

In order to exploit the view of SSR based on Equation (17), an over-complete dictionary is firstly given as \( \mathbf{A}_f(\hat{\theta}_g) = \{\Delta(\hat{\theta}_g)\}_{g=1}^{G} \) (\( G \gg M \)), \( \{\hat{\theta}_g\}_{g=1}^{G} \) represents all possible signal directions and \( G \) is the total number of all possible DOAs. According to the knowledge of linear algebra, each column vector of covariance matrix \( \mathbf{R} \) can be represented linearly by the over-complete dictionary. Then, the \( m \)th column of covariance matrix \( \mathbf{R} \) can be represented as:

\[
r_m = \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}(t)\bar{y}_m^*(t) = \mathbf{A}_f(\hat{\theta}_g) \bar{b}_m + \sigma^2 e_m, \quad m = 1, 2, \cdots, M,
\]

where \( \mathbf{y}_m(t) \) represents the received data of the \( m \)th array sensor at time \( t \). \( \mathbf{A}_f(\hat{\theta}_g) = [\Delta(\hat{\theta}_1), \Delta(\hat{\theta}_2), \cdots, \Delta(\hat{\theta}_G)] \) is a given over-complete dictionary. And \( \{\Delta(\hat{\theta}_g)\}_{g=1}^{G} \) is the \( M \times Q \) block matrix, which has the same structure as \( \Delta(\hat{\theta}) \) in Equation (7). Suppose that the grid points are sufficiently dense; hence, the \( (Qg - Q + 1) \)th to \( (Qg) \)th rows of ideal \( \bar{b}_m \) are non-zero, the position of the non-zero blocks in \( \bar{b}_m \) corresponds to the \( N \) true DOAs. Meanwhile, \( e_m(m = 1, \cdots, M) \) is a \( M \times 1 \) column vector, the \( m \)th element of vector \( e_m \) is one, and the rest are zero. However, Equation (18) only considers one column of the covariance matrix. For a covariance matrix, Equation (18) can be
written as the form of a matrix, and the sparse representation model of the covariance matrix can be obtained as:

$$\mathbf{R} = \mathbf{A}_f(\tilde{\theta})\mathbf{B} + \sigma^2\mathbf{I}_M, \quad \quad (19)$$

where \(\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_M]\). Through the above analysis, it can be obtained that all the ideal \(\{\mathbf{b}_m\}_{m=1}^M\) should have the same block sparse structure. In other words, non-zero block vectors of each ideal \(\{\mathbf{b}_m\}_{m=1}^M\) should appear in the same position of matrix \(\mathbf{B}\). Here, for convenience, a new column vector \(\tilde{\mathbf{b}}^l_g = [b^l_1, b^l_2, \cdots, b^l_M]^T\) is introduced, and \(\{\tilde{\mathbf{b}}^l_g\}_{g=1}^G\) denotes the \(g\)th element of \(\tilde{\mathbf{b}}^l\), which is equal to the \(l_2\)-norm of the \((Q_g - Q + 1)\)th to \((Q_g)\)th rows of matrix \(\mathbf{B}\). Therefore, \(\tilde{\mathbf{b}}^l\) is a sparse vector of \(N\) and satisfies

$$\tilde{b}^l_g \begin{cases} 
\neq 0, & \tilde{\theta}_g \in \theta_1, \theta_2, \cdots, \theta_N \\
= 0, & \tilde{\theta}_g \notin \theta_1, \theta_2, \cdots, \theta_N 
\end{cases}, \quad g = 1, 2, \cdots, G, \quad \quad (20)$$

From the above analysis, it is obviously shown that the sparsity of vector \(\tilde{\mathbf{b}}^l\) can be used to describe the joint block-sparsity of matrix \(\mathbf{B}\). Hence, the DOAs can be achieved by detecting the position of non-zero elements of sparse vector \(\tilde{\mathbf{b}}^l\).

In sparse representation theory, the constraint recovery model that accurately describes the sparsity of signal is the \(l_0\)-norm constraint. It is known that \(l_0\)-norm optimization is a computationally expensive NP-hard problem, and the \(l_1\)-norm penalty optimization is a convex approximation of \(l_0\)-norm optimization. For reducing the computational complexity, we transform the DOA’s estimation problem into a convex optimization. Therefore, the sparse vector \(\tilde{\mathbf{b}}^l\) can be recovered by the following constrained optimization problem:

$$\min \|\tilde{\mathbf{b}}^l\|_1, \quad \text{subject to} \quad \|\mathbf{R} - \mathbf{A}_f(\tilde{\theta})\tilde{\mathbf{b}}^l\|_2^2 \leq \eta, \quad \quad (21)$$

where \(\eta\) represents the regularization parameter, which is used to balance the relationship between data fitting and sparsity of the solution in the \(l_1\)-norm penalty optimization problem. The value of regularization parameter \(\eta\) is related to noise. When the recovered matrix is obtained, the DOAs can be achieved by the position of the non-zero blocks.

3.2. A Reweighted \(l_1\)-SRACV Algorithm for DOA Estimation

Since the \(l_1\)-norm is a convex approximation of \(l_0\)-norm, there exists a difference between the recoveries obtained by these two operations. In the case of the \(l_0\)-norm constraint, the solution for any vector only has a value of zero and one without differentiating the contributions from large and small coefficients to the objective function. To achieve better recovery performance in the \(l_1\)-norm constraint, more constraints should be imposed on the large coefficients to objective function for ensuring the sparse solution of the entire cost function. To avoid this problem, a weighted norm constraint is introduced. By reweighting the \(l_1\)-norm constraint model, both large and small coefficients in the reconstructed signal can be equally constrained as much as possible. That is, the smaller weight penalties are used for large coefficients and the larger weight penalties are used for small coefficients. This adaptive adjustment mechanism enables the reweighted \(l_1\)-norm constraint optimization to better approximate the \(l_0\)-norm penalty minimization. In the following section, the basic principle of Capon algorithm is used to construct a weighted matrix for the reweighted \(l_1\)-norm constraint optimization.

The traditional Capon algorithm is firstly introduced, which is based on the steering vector of the signal and the covariance matrix of the source signal. We first find the minimum value of the following function, and then determine the DOAs with the minimum value for the following function:

$$\hat{f}(\theta) = |\mathbf{\pi}^H(\theta)|\mathbf{R}^{-1}\mathbf{\pi}(\theta), \quad \quad (22)$$
where \( \mathbf{\pi}(\theta) \) denotes the steering vector of source signal and \( \mathbf{R}^{-1} \) represents the inverse of the covariance matrix of the sample. Therefore, \( \mathcal{I}(\theta) \) will tend to be zero when \( \theta \) corresponds to the true DOAs. Following the ideal in [25], it is known that the relationship between the new over-complete dictionary matrix \( \mathbf{A}_J(\tilde{\theta}_g) \) and the covariance matrix can be used to construct the weighted matrix. Then the over-complete dictionary can be expressed as follows, \( \mathbf{A}_J(\tilde{\theta}_g) = [\mathbf{A}_{J1}(\tilde{\theta}_g), \mathbf{A}_{J2}(\tilde{\theta}_g)] \). Suppose that \( \mathbf{A}_{J1}(\tilde{\theta}_g) \) is made up of the \( N \) block matrices corresponding to real DOAs, and \( \mathbf{A}_{J2}(\tilde{\theta}_g) \) is composed with \( G - N \) residual block steering matrices. Now a new function is introduced in the following form:

\[
\hat{\mathbf{w}}_g = \text{det}\{\mathbf{\Delta}^H(\tilde{\theta}_g)\mathbf{R}^{-1}\mathbf{\Delta}(\tilde{\theta}_g)\}, \quad g = 1, 2, \cdots, G,
\]

where \( \hat{\mathbf{w}}_g \) denotes the value of the determinant corresponding to the different possible signals. Then the weights can be formulated as follows:

\[
w_g = \hat{w}_g^2 / \max\{\hat{w}_1^2, \hat{w}_2^2, \cdots, \hat{w}_G^2\},
\]

where \( \hat{w}_g^2 \) denotes the \( l_2 \)-norm of \( \hat{w}_g \). And we define \( \hat{\mathbf{W}} = [w_1, w_2, \cdots, w_G] = [\hat{\mathbf{W}}_1, \hat{\mathbf{W}}_2] \). In Equation (23), the weights \( \hat{w}_g \) are more likely to be zero when \( \{\theta_g\}_{g=1}^{G} \) corresponds to the true DOAs if the number of samples is sufficiently large. \( \hat{\mathbf{W}}_1 \) is assumed to correspond to the true DOAs. Therefore, the weights in \( \hat{\mathbf{W}}_1 \) satisfies \( \hat{\mathbf{W}}_1 \rightarrow 0 \), and the weights in vector \( \hat{\mathbf{W}}_2 \) are smaller than that in vector \( \hat{\mathbf{W}}_1 \). Finally, the weighted matrix is defined as:

\[
\mathbf{W} = \text{diag}(\hat{\mathbf{W}}),
\]

where \( \mathbf{W} \) is a \( G \times G \) diagonal matrix. When the weighted matrix is applied to the above \( l_1 \)-norm minimization problem, the small weights in \( \mathbf{W} \) reserve the large coefficients and the large weights punish the small coefficients who tend to be zero. Such a constraint effectively guarantees the fidelity of signal reconstruction. Through the above-mentioned study, the weighted matrix can enforce the sparsity of the solution. The new reweighted \( l_1 \)-norm minimization problem is as follows:

\[
\min \Vert \mathbf{W}\hat{\mathbf{b}}^l \Vert_1, \quad \text{subject to} \quad \Vert \mathbf{R} - \mathbf{A}_J(\tilde{\theta})\mathbf{B} \Vert_2^2 \leq \eta,
\]

where the regularization parameter \( \eta \) sets the upper limit of noise power that is to be permitted, which will be introduced in detail in Remark 1. Finally, the above optimization problem can be solved by the second order cone (SOC) programming packages such as CVX. After obtaining the sparse vector \( \hat{\mathbf{b}}^l \), DOA estimation of real signals can be obtained by the angle position corresponding to the position of the non-zero values in the sparse vector \( \hat{\mathbf{b}}^l \).

**Remark 1.** In the reweighted \( l_1 \)-norm minimization problem, the regularization parameter \( \eta \) plays an important role in recovering the sparse vector \( \hat{\mathbf{b}}^l \) accurately. The regularization parameter \( \eta \) can also compromise the sparseness of fitting error and resolution. Through the introduction of [14], we find that the selection of parameters is related to the probability distribution of noise covariance. Several studies have shown that the noise approximately follows a \( \chi^2 \) distribution with \( M \times N \) degrees of freedom [17]. The upper limit value of noise power can be computed as the regularization parameter \( \eta \) with a high probability of 99%. The regularization parameter \( \eta \) can be calculated by \( \chi^2_{n}\text{inv}(p, M \times N) \) with the MATLAB. Furthermore, the noise power, that is related to the singular value of the covariance matrix, can be calculated by averaging the square of the \( M - N \) smallest singular values.

**Remark 2.** The main computational complexity of the proposed method is for the construction of the weighted matrix and the solution of the convex optimization problem in Equation (26). The construction of the weighted matrix \( \mathbf{W} \) requires \( O\{M^3 T + GQ^4 M(M - N)(M + Q)\} \) flops, and solving the convex optimization problem
requires $O\{(MQG)^3\}$ flops. Therefore, the proposed method requires more computation than Dai’s method [31] and Wang’s method [33]. Nevertheless, the advantages of the proposed method outweigh the disadvantages. The proposed method can work well in the environment of unknown mutual coupling, and the performance of our proposed method is much better than other methods in terms of accuracy and resolution.

4. Simulation Results

In this section, the DOA estimation performance of the proposed method is demonstrated by our simulation experiments. The methods in [31,33], referred to as Dai’s method and Wang’s method, respectively, and the Cramer-Rao bound (CRB) [17] are chosen to compare with our proposed method. In Dai’s method, a selection matrix is used to remove the influence of an unknown mutual coupling coefficient and in Wang’s method, a novel data model is constructed to avoid the influence of unknown mutual coupling by parameterizing the steering vector. Furthermore, the DOA estimation performance is evaluated by the root mean square error (RMSE) defined as:

$$
\text{RMSE} = \sqrt{\frac{1}{NL} \sum_{l=1}^{N} \sum_{i=1}^{L} (\theta_{l,i} - \theta_{l})},
$$

where $\theta_{l,i}$ denotes the estimated value of $\theta_{l}$ obtained by the $i$th Monte Carlo trial, $L = 200$ is the number of Monte Carlo trials, and $N$ is the total number of source signals.

In the following simulation experiments, we consider a ULA equipped with $M = 8$ sensors that are separated by a half-wavelength. We assume that there are two uncorrelated narrowband far-field signals whose DOAs are $\theta_1 = -6^\circ$ and $\theta_2 = 10^\circ$. The mutual coupling coefficients are set as $d_1 = 0.6864 - j0.1776$, $d_2 = 0.20 - j0.0896$, and $K = 3$. The range from $-90^\circ$ to $90^\circ$ in the spatial domain is discretized with a 0.05$^\circ$ grade resolution for all methods.

Figure 1 represents the spatial spectrum of the proposed method, Wang’s method and Dai’s method for two uncorrelated source signals with $M = 8$, $T = 300$ and SNR = 10 dB. From Figure 1, it can clearly be seen that our proposed method achieves a spectrum that has a lower side-lobe and sharper peaks than those achieved by the other two methods. Therefore, our method has superiority with regard to the spatial resolution. It can be seen clearly from Figure 1 that our proposed method obtains two sharp peaks located very close to the true DOAs; therefore, the proposed method is more accurate than Wang’s method and Dai’s method.

![Figure 1](image)

**Figure 1.** The spatial spectrum of the proposed method, Wang’s method, and Dai’s method.

Figure 2 depicts the RMSE versus SNR for different methods with $M = 8$ and $T = 300$. It can be seen that our proposed method achieves more accurate estimation with a lower RMSE than other two
methods. In addition, the RMSE value achieved by our proposed method is closer to CRB. The main reason is that the weighted matrix is used to ensure that the recovery solution is sparser than other methods. Meanwhile, the proposed method takes full advantage of statistical characteristics. However, Wang’s method has a loss of the array aperture and the performance of Dai’s method is influenced by the data length used for DOA estimation. Therefore, these two methods have a higher RMSE than the proposed method.

![Figure 2. The root mean square error (RMSE) versus signal to noise ratio (SNR) for different methods (M = 8 and T = 300).](image)

Figure 2. The root mean square error (RMSE) versus signal to noise ratio (SNR) for different methods (M = 8 and T = 300).

Figure 3 shows the RMSE versus the number of snapshots achieved by different methods for M = 8 and SNR = 10 dB. It is seen that as the number of snapshots increases, the RMSE values of all methods are reduced and the performance is improved. However, it is evident that the estimation performance of our proposed method is an improvement compared to those achieved by Dai’s method and Wang’s method; its RMSE value is closest to CRB. The performance of the proposed method is much higher than that achieved by the other two methods in the entire range of the number of snapshots. We also observe that the performance of Wang’s method is slightly better than that of Dai’s method.

![Figure 3. The RMSE versus snapshots for different methods (M = 8 and SNR = 10 dB).](image)

Figure 3. The RMSE versus snapshots for different methods (M = 8 and SNR = 10 dB).
Figures 4 and 5 show the DOA estimation performance of the proposed method with different numbers of antennas versus different SNRs or number of snapshots, respectively. From both Figures 4 and 5, we observe that when SNR or the number of snapshots are fixed, the DOA estimation performance of the proposed method is improved through increasing the number of the array elements because a greater diversity gain is achieved as the number of antennas is increased. However, it should be noted that the computational complexity is also increased with the increasing number of antennas for better estimation performance.

![Figure 4. RMSE versus different antennas under different SNRs (T = 300).](image1)

![Figure 5. RMSE versus different antennas under different number of snapshots (SNR = 10 dB).](image2)

Figures 6 and 7 reveal the probability of successful detection versus SNR and the number of snapshots for different methods, respectively. In these two simulation experiments, the successful detection rate is defined as the error between the DOAs estimation and the real DOAs that satisfies $|\hat{\theta}_n - \theta_n| \leq 0.5^\circ$, where $\hat{\theta}_n$ is the estimated value of $\theta_n$. It can be seen from Figures 6 and 7, as the number of snapshots or SNR increases, the probability of detection also increases. When the SNR or the number of snapshots is greater than a certain value, the detection probability of the proposed method reaches 100%. When the SNR is low or the number of snapshots is small, the detection probability of the other two methods is relatively lower. Therefore, the detection probability of the proposed method
is much higher than that achieved by the other two methods in the entire range of SNRs or the number of snapshots.

![Figure 6. The probability of successful detection versus SNR (T = 300 and M = 8).](image)

![Figure 7. The probability of successful detection versus snapshots (SNR = 10 dB and M = 8).](image)

5. Conclusions

In this article, a weighted block sparse recovery algorithm for high resolution DOA estimation under unknown mutual coupling is proposed. In the proposed method, a block sparse representation model based on the array covariance vectors is formulated by parameterizing the steering vector without the influence of mutual coupling, and then the weighted matrix is constructed by the principle of a novel Capon space spectrum estimation algorithm based on the parameterized steering vector. Finally, the DOA is estimated from the position of the non-zero blocks of the recovered block sparse matrix. Due to the fact that our proposed method takes advantage of the whole array aperture and enhances the sparsity of solution by the reweighted $l_1$-norm minimization scheme, the proposed method not only works well but also has a higher spatial resolution. The simulation results have demonstrated that the performance of our proposed method is better than Wang’s method and Dai’s method, both in resolution and accuracy.
Author Contributions: D.M. wrote the manuscript, and X.W. provided the idea of manuscript and performed the simulations; M.H. and C.S. analyzed the data; G.B. given a lot of suggestions for improving the presentation and writing of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (61701144, 61861015), the Program of Hainan Association for Science and Technology Plans to Youth R&D Innovation (QCXM201706), the scientific research projects of University in Hainan Province (Hnky2018ZD-4), the major Science and Technology Project of Hainan Province (ZDK(2016)015), the Scientific Research Setup Fund of Hainan University (KYQD (ZR) 1731), the Natural Science Foundation of Hainan Province (617024) and the collaborative Innovation Fund of Tianjin University & Hainan University (No.HDTDU201810).

Conflicts of Interest: The authors declare no conflict of interest.

References


© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).