Entropy **2013**, *15*, 3746-3761; doi:10.3390/e15093746

OPEN ACCESS

entropy

ISSN 1099-4300

www.mdpi.com/journal/entropy

Article

Combination Synchronization of Three Identical or Different Nonlinear Complex Hyperchaotic Systems

Xiaobing Zhou *, Murong Jiang and Yaqun Huang

School of Information Science and Engineering, Yunnan University, Kunming 650091, China

* Author to whom correspondence should be addressed; E-Mail: zhouxb.cn@gmail.com.

Received: 5 August 2013; in revised form: 3 September 2013 / Accepted: 4 September 2013 / Published: 10 September 2013

Abstract: In this paper, we investigate the combination synchronization of three nonlinear complex hyperchaotic systems: the complex hyperchaotic Lorenz system, the complex hyperchaotic Chen system and the complex hyperchaotic Lü system. Based on the Lyapunov stability theory, corresponding controllers to achieve combination synchronization among three identical or different nonlinear complex hyperchaotic systems are derived, respectively. Numerical simulations are presented to demonstrate the validity and feasibility of the theoretical analysis.

Keywords: combination synchronization; Lyapunov stability theory; complex hyperchaotic Lorenz system; complex hyperchaotic Chen system; complex hyperchaotic Lü system

1. Introduction

Since Fowler *et al.* [1], introduced a complex Lorenz model to generalize the real Lorenz model in 1982, complex chaotic and hyperchaotic systems have attracted increasing attention, due to the fact that systems with complex variables can be used to describe the physics of a detuned laser, rotating fluids, disk dynamos, electronic circuits and particle beam dynamics in high energy accelerators [2]. When applying complex systems in communications, the complex variables will double the number of variables and can increase the content and security of the transmitted information. Many complex chaotic and hyperchaotic systems have been proposed ever since the 1980s. In [3], the authors studied the chaotic unstable limit cycles of complex Van der Pol oscillators. The rich dynamics behaviors of the complex Chen and complex Lü systems were investigated in [4]. By adding state feedback controllers to their complex chaotic systems, complex hyperchaotic Chen, Lorenz and Lü systems were introduced and studied in [5–7], respectively. The authors [8] constructed a complex nonlinear hyperchaotic system by adding a cross-product nonlinear term to the complex Lorenz system. A complex modified hyperchaotic Lü system [9] was proposed by introducing complex variables to its real counterpart.

In 1990 [10], Pecora and Carroll proposed the drive-response concept for constructing the synchronization of coupled chaotic systems. Over the last two decades, synchronization in chaotic systems has been extensively investigated, due to its potential applications in various fields, such as chemical reactions, biological systems and secure communication. Mahmoud *et al.* [11] designed an adaptive control scheme to study the complete synchronization of chaotic complex nonlinear systems with uncertain parameters. The authors achieved phase synchronization and antiphase synchronization of two identical hyperchaotic complex nonlinear systems via an active control technique in [12]. Based on passive theory, the authors studied the projective synchronization of hyperchaotic complex nonlinear systems and its application in secure communications [13]. Liu *et al.* [14] investigated the modified function projective synchronization of general chaotic complex systems described by a unified mathematical expression.

The aforementioned synchronization schemes are based on the usual drive-response synchronization mode, which has one drive system and one response system. Recently, Luo [15] proposed a combination synchronization scheme, which has two drive systems and one response system. This synchronization scheme has advantages over the usual drive-response synchronization, such as being able to provide greater security in secure communication. In secure communication, the transmitted signals can be split into several parts, each part loaded in different drive systems, or can divide time into different intervals, the signals in different intervals being loaded in different drive systems. Thus, the transmitted signals can have stronger anti-attack ability and anti-translated capability than those transmitted by the usual transmission model.

Motivated by the above discussions, this paper aims to study the combination synchronization of three identical or different nonlinear complex hyperchaotic systems. The rest of this paper is organized as follows. Section 2 introduces the scheme of combination synchronization. In Section 3 and Section 4, we investigate combination synchronization among three identical and different complex nonlinear hyperchaotic systems, respectively. Finally, conclusions are given in Section 5.

2. The Scheme of Combination Synchronization

Suppose that there are three nonlinear dynamical systems, two drive systems and one response system. The drive systems are given by:

$$\dot{x} = f(x), \tag{1}$$

and

$$\dot{y} = g(y). \tag{2}$$

The response system is described by:

$$\dot{z} = h(z) + U(x, y, z), \tag{3}$$

where $x = (x_1,\ x_2,\ ...,\ x_n)^T,\ y = (y_1,\ y_2,\ ...,\ y_n)^T, z = (z_1,\ z_2,\ ...,\ z_n)^T$ are the state vectors of systems (1), (2) and (3), respectively, $f(\cdot),\ g(\cdot),\ h(\cdot) : R^n \to R^n$ are three continuous vector functions and $U(\cdot) : R^n \times R^n \times R^n \to R^n$ is a controller vector, which will be designed.

Definition 1 [15]. For drive systems (1) and (2) and response system (3), they are said to be in combination synchronization if there exists three constant matrices, $A,\ B,\ C \in R^n$ and $C \neq 0$, such that:

$$\lim_{t\to+\infty} \parallel Ax + By - Cz \parallel = 0, \tag{4}$$

where $\parallel \cdot \parallel$ represents the matrix norm.

3. Combination Synchronization among Identical Nonlinear Complex Hyperchaotic Systems

In this section, we take the complex hyperchaotic Lorenz system [6] as an example to investigate the combination synchronization among three identical systems.

The first drive system is given by:

$$\begin{cases} \dot{x}_{11} = \alpha(x_{21} - x_{11}) + x_{41}, \\ \dot{x}_{21} = \gamma x_{11} - x_{21} - x_{11}x_{31}, \\ \dot{x}_{31} = \dfrac{1}{2}(\bar{x}_{11}x_{21} + x_{11}\bar{x}_{21}) - \beta x_{31} + x_{41}, \\ \dot{x}_{41} = \dfrac{1}{2}(\bar{x}_{11}x_{21} + x_{11}\bar{x}_{21}) - \sigma x_{41}, \end{cases} \tag{5}$$

and the second drive system is described as follows:

$$\begin{cases} \dot{x}_{12} = \alpha(x_{22} - x_{12}) + x_{42}, \\ \dot{x}_{22} = \gamma x_{12} - x_{22} - x_{12}x_{32}, \\ \dot{x}_{32} = \dfrac{1}{2}(\bar{x}_{12}x_{22} + x_{12}\bar{x}_{22}) - \beta x_{32} + x_{42}, \\ \dot{x}_{42} = \dfrac{1}{2}(\bar{x}_{12}x_{22} + x_{12}\bar{x}_{22}) - \sigma x_{42}. \end{cases} \tag{6}$$

The response system takes the following form:

$$\begin{cases} \dot{x}_{13} = \alpha(x_{23} - x_{13}) + x_{43} + U_1 + iU_2, \\ \dot{x}_{23} = \gamma x_{13} - x_{23} - x_{13}x_{33} + U_3 + iU_4, \\ \dot{x}_{33} = \dfrac{1}{2}(\bar{x}_{13}x_{23} + x_{13}\bar{x}_{2}) - \beta x_{33} + x_{43} + U_5, \\ \dot{x}_{43} = \dfrac{1}{2}(\bar{x}_{13}x_{23} + x_{13}\bar{x}_{23}) - \sigma x_{43} + U_6, \end{cases} \tag{7}$$

where α, β, γ and σ are positive parameters, $x_{11} = u_1 + iu_2,\ x_{21} = u_3 + iu_4,\ x_{12} = v_1 + iv_2,\ x_{22} = v_3 + iv_4,\ x_{13} = w_1 + iw_2,\ x_{23} = w_3 + iw_4$ are complex variables and $i = \sqrt{-1}$; $u_i,\ v_i,\ w_i\ (i = 1, 2, 3, 4)$, $x_{31} = u_5,\ x_{41} = u_6,\ x_{32} = v_5,\ x_{42} = v_6,\ x_{33} = w_5,\ x_{43} = w_6$ are real variables. The overbar represents a complex conjugate function. $U_1,\ U_2,\ U_3,\ U_4,\ U_5$ and U_6 are real controllers to be determined.

For the convenience of our discussions, we assume $A = diag(l_1, l_2, l_3, l_4),\ B = diag(m_1, m_2, m_3, m_4),\ C = diag(k_1, k_2, k_3, k_4)$ in our synchronization scheme.

We define error states between systems (5), (6) and (7) as:

$$\begin{cases} e_1 + ie_2 = k_1x_{13} - l_1x_{11} - m_1x_{12}, \\ e_3 + ie_4 = k_2x_{23} - l_2x_{21} - m_2x_{22}, \\ \qquad e_5 = k_3x_{33} - l_3x_{31} - m_3x_{32}, \\ \qquad e_6 = k_4x_{43} - l_4x_{41} - m_4x_{42}, \end{cases} \tag{8}$$

such that:

$$\begin{cases} \lim\limits_{t\to\infty} \parallel k_1x_{13} - l_1x_{11} - m_1x_{12} \parallel = 0, \\ \lim\limits_{t\to\infty} \parallel k_2x_{23} - l_2x_{21} - m_2x_{22} \parallel = 0, \\ \lim\limits_{t\to\infty} \parallel k_3x_{33} - l_3x_{31} - m_3x_{32} \parallel = 0, \\ \lim\limits_{t\to\infty} \parallel k_4x_{43} - l_4x_{41} - m_4x_{42} \parallel = 0. \end{cases} \tag{9}$$

Thus, we have the following error dynamical system:

$$\begin{cases} \dot{e}_1 + i\dot{e}_2 = k_1\dot{x}_{13} - l_1\dot{x}_{11} - m_1\dot{x}_{12}, \\ \dot{e}_3 + i\dot{e}_4 = k_2\dot{x}_{23} - l_2\dot{x}_{21} - m_2\dot{x}_{22}, \\ \qquad \dot{e}_5 = k_3\dot{x}_{33} - l_3\dot{x}_{31} - m_3\dot{x}_{32}, \\ \qquad \dot{e}_6 = k_4\dot{x}_{43} - l_4\dot{x}_{41} - m_4\dot{x}_{42}. \end{cases} \tag{10}$$

Substituting Equations (5)–(7) in Equation (10) and separating the real and imaginary parts yields:

$$\begin{cases} \dot{e}_1 = k_1[\alpha(w_3 - w_1) + w_6] - l_1[\alpha(u_3 - u_1) + u_6] - m_1[\alpha(v_3 - v_1) + v_6] + k_1U_1, \\ \dot{e}_2 = k_1\alpha(w_4 - w_2) - l_1\alpha(u_4 - u_2) - m_1\alpha(v_4 - v_2) + k_1U_2, \\ \dot{e}_3 = k_2(\gamma w_1 - w_3 - w_1w_5) - l_2(\gamma u_1 - u_3 - u_1u_5) - m_2(\gamma v_1 - v_3 - v_1v_5) + k_2U_3, \\ \dot{e}_4 = k_2(\gamma w_2 - w_4 - w_2w_5) - l_2(\gamma u_2 - u_4 - u_2u_5) - m_2(\gamma v_2 - v_4 - v_2v_5) + k_2U_4, \\ \dot{e}_5 = k_3(w_1w_3 + w_2w_4 - \beta w_5 + w_6) - l_3(u_1u_3 + u_2u_4 - \beta u_5 + u_6) \\ \quad - m_3(v_1v_3 + v_2v_4 - \beta v_5 + v_6) + k_3U_5, \\ \dot{e}_6 = k_4(w_1w_3 + w_2w_4 - \sigma w_6) - l_4(u_1u_3 + u_2u_4 - \sigma u_6) - m_4(v_1v_3 + v_2v_4 - \sigma v_6) + k_4U_6. \end{cases} \tag{11}$$

Then, we obtain the following results.

Theorem 1. If the controllers are chosen as follows:

$$\begin{cases} U_1 = -\dfrac{1}{k_1}\{(k_1w_1 - l_1u_1 - m_1v_1) + [k_1\alpha(w_3 - w_1) + k_1w_6 - l_1\alpha(u_3 - u_1) - l_1u_6 \\ \qquad - m_1\alpha(v_3 - v_1) - m_1v_6] - \alpha(k_1w_2 - l_1u_2 - m_1v_2)\}, \\ U_2 = -\dfrac{1}{k_1}\{(k_1w_2 - l_1u_2 - m_1v_2) + [k_1\alpha(w_4 - w_2) - l_1\alpha(u_4 - u_2) - m_1\alpha(v_4 - v_2)] \\ \qquad + \alpha(k_1w_1 - l_1u_1 - m_1v_1) - \gamma(k_2w_3 - l_2u_3 - m_2v_3)\}, \\ U_3 = -\dfrac{1}{k_2}\{(k_2w_3 - l_2u_3 - m_2v_3) + [k_2(\gamma w_1 - w_3 - w_1w_5) - l_2(\gamma u_1 - u_3 - u_1u_5) \\ \qquad - m_2(\gamma v_1 - v_3 - v_1v_5)] + \gamma(k_1w_2 - l_1u_2 - m_1v_2) - \beta(k_2w_4 - l_2u_4 - m_2v_4)\}, \\ U_4 = -\dfrac{1}{k_2}\{(k_2w_4 - l_2u_4 - m_2v_4) + [k_2(\gamma w_2 - w_4 - w_2w_5) - l_2(\gamma u_2 - u_4 - u_2u_5) \\ \qquad - m_2(\gamma v_2 - v_4 - v_2v_5)] + \beta(k_2w_3 - l_2u_3 - m_2v_3) - \sigma(k_3w_5 - l_3u_5 - m_3v_5)\}, \\ U_5 = -\dfrac{1}{k_3}\{(k_3w_5 - l_3u_5 - m_3v_5) + [k_3(w_1w_3 + w_2w_4 - \beta w_5 + w_6) - l_3(u_1u_3 + u_2u_4 - \beta u_5 + u_6) \\ \qquad - m_3(v_1v_3 + v_2v_4 - \beta v_5 + v_6)] + \sigma(k_2w_4 - l_2u_4 - m_2v_4) - \alpha(k_4w_6 - l_4u_6 - m_4v_6)\}, \\ U_6 = -\dfrac{1}{k_4}\{(k_4w_6 - l_4u_6 - m_4v_6) + [k_4(w_1w_3 + w_2w_4 - \sigma w_6) - l_4(u_1u_3 + u_2u_4 - \sigma u_6) \\ \qquad - m_4(v_1v_3 + v_2v_4 - \sigma v_6)] + \alpha(k_3w_5 - l_3u_5 - m_3v_5)\}, \end{cases} \tag{12}$$

then driven systems (5) and (6) will achieve combination synchronization with response system (7).

Proof. Construct the following Lyapunov function:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2). \tag{13}$$

Taking the time derivative of V along the trajectory of error dynamical system (11) yields:

$$\begin{aligned} \dot{V} =& e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + e_6\dot{e}_6 \\ =& e_1\{k_1[\alpha(w_3 - w_1) + w_6] - l_1[\alpha(u_3 - u_1) + u_6] - m_1[\alpha(v_3 - v_1) + v_6] + k_1U_1\} \\ &+ e_2[k_1\alpha(w_4 - w_2) - l_1\alpha(u_4 - u_2) - m_1\alpha(v_4 - v_2) + k_1U_2] \\ &+ e_3[k_2(\gamma w_1 - w_3 - w_1w_5) - l_2(\gamma u_1 - u_3 - u_1u_5) - m_2(\gamma v_1 - v_3 - v_1v_5) + k_2U_3] \\ &+ e_4[k_2(\gamma w_2 - w_4 - w_2w_5) - l_2(\gamma u_2 - u_4 - u_2u_5) - m_2(\gamma v_2 - v_4 - v_2v_5) + k_2U_4] \\ &+ e_5[k_3(w_1w_3 + w_2w_4 - \beta w_5 + w_6) - l_3(u_1u_3 + u_2u_4 - \beta u_5 + u_6) - m_3(v_1v_3 + v_2v_4 - \beta v_5 + v_6) + k_3U_5] \\ &+ e_6[k_4(w_1w_3 + w_2w_4 - \sigma w_6) - l_4(u_1u_3 + u_2u_4 - \sigma u_6) - m_4(v_1v_3 + v_2v_4 - \sigma v_6) + k_4U_6]. \end{aligned} \tag{14}$$

Substituting Equation (12) into Equation (14), then:

$$\begin{aligned}
\dot{V} =& e_1\{k_1[\alpha(w_3 - w_1) + w_6] - l_1[\alpha(u_3 - u_1) + u_6] - m_1[\alpha(v_3 - v_1) + v_6] - [k_1w_1 - l_1u_1 - m_1v_1 \\
&+ k_1\alpha(w_3 - w_1) + k_1w_6 - l_1\alpha(u_3 - u_1) - l_1u_6 - m_1\alpha(v_3 - v_1) - m_1v_6 - \alpha(k_1w_2 - l_1u_2 - m_1v_2)]\} \\
&+ e_2\{k_1\alpha(w_4 - w_2) - l_1\alpha(u_4 - u_2) - m_1\alpha(v_4 - v_2) - [k_1w_2 - l_1u_2 - m_1v_2 \\
&+ k_1\alpha(w_4 - w_2) - l_1\alpha(u_4 - u_2) - m_1\alpha(v_4 - v_2) + \alpha(k_1w_1 - l_1u_1 - m_1v_1) - \gamma(k_2w_3 - l_2u_3 - m_2v_3)]\} \\
&+ e_3\{k_2(\gamma w_1 - w_3 - w_1w_5) - l_2(\gamma u_1 - u_3 - u_1u_5) - m_2(\gamma v_1 - v_3 - v_1v_5) - [k_2w_3 - l_2u_3 - m_2v_3 \\
&+ k_2(\gamma w_1 - w_3 - w_1w_5) - l_2(\gamma u_1 - u_3 - u_1u_5) - m_2(\gamma v_1 - v_3 - v_1v_5) \\
&+ \gamma(k_1w_2 - l_1u_2 - m_1v_2) - \beta(k_2w_4 - l_2u_4 - m_2v_4)]\} \\
&+ e_4\{k_2(\gamma w_2 - w_4 - w_2w_5) - l_2(\gamma u_2 - u_4 - u_2u_5) - m_2(\gamma v_2 - v_4 - v_2v_5) - [k_2w_4 - l_2u_4 - m_2v_4 \\
&+ k_2(\gamma w_2 - w_4 - w_2w_5) - l_2(\gamma u_2 - u_4 - u_2u_5) - m_2(\gamma v_2 - v_4 - v_2v_5) \\
&+ \beta(k_2w_3 - l_2u_3 - m_2v_3) - \sigma(k_3w_5 - l_3u_5 - m_3v_5)]\} + e_5\{k_3(w_1w_3 + w_2w_4 - \beta w_5 + w_6) \\
&- l_3(u_1u_3 + u_2u_4 - \beta u_5 + u_6) - m_3(v_1v_3 + v_2v_4 - \beta v_5 + v_6) - [k_3w_5 - l_3u_5 - m_3v_5 \\
&+ k_3(w_1w_3 + w_2w_4 - \beta w_5 + w_6) - l_3(u_1u_3 + u_2u_4 - \beta u_5 + u_6) - m_3(v_1v_3 + v_2v_4 - \beta v_5 + v_6) \\
&+ \sigma(k_2w_4 - l_2u_4 - m_2v_4) - \alpha(k_4w_6 - l_4u_6 - m_4v_6)]\} + e_6\{k_4(w_1w_3 + w_2w_4 - \sigma w_6) \\
&- l_4(u_1u_3 + u_2u_4 - \sigma u_6) - m_4(v_1v_3 + v_2v_4 - \sigma v_6) - [k_4w_6 - l_4u_6 - m_4v_6 + k_4(w_1w_3 + w_2w_4 - \sigma w_6) \\
&- l_4(u_1u_3 + u_2u_4 - \sigma u_6) - m_4(v_1v_3 + v_2v_4 - \sigma v_6) + \alpha(k_3w_5 - l_3u_5 - m_3v_5)]\} \\
=& e_1(-e_1 + \alpha e_2) + e_2(-e_2 - \alpha e_1 + \gamma e_3) + e_3(-e_3 - \gamma e_2 + \beta e_4) + e_4(-e_4 - \beta e_3 + \sigma e_5) \\
&+ e_5(-e_5 - \sigma e_4 + \alpha e_6) + e_6(-e_6 - \alpha e_5) \\
=& - e_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 - e_6^2.
\end{aligned} \tag{15}$$

Since $\dot{V} \leq 0$ as $t \to \infty$, according to the Lyapunov stability theory, we know $e_i \to 0 (i = 1, 2, 3, 4, 5, 6)$, ***i.e.***, $\lim_{t\to\infty} \parallel e \parallel = 0$. Therefore, drive systems (5) and (6) will achieve combination synchronization with the response system (7).

This completes the proof.

The following corollaries can be easily obtained from Theorem 1.

Corollary 1. (i) Suppose that $l_1 = l_2 = l_3 = l_4 = 0$ and $k_1 = k_2 = k_3 = k_4 = 1$, and if the controllers are chosen as follows:

$$\begin{cases}
U_1 = - \{(w_1 - m_1v_1) + [\alpha(w_3 - w_1) + w_6 - m_1\alpha(v_3 - v_1) - m_1v_6] - \alpha(w_2 - m_1v_2)\}, \\
U_2 = - \{(w_2 - m_1v_2) + [\alpha(w_4 - w_2) - m_1\alpha(v_4 - v_2)] + \alpha(w_1 - m_1v_1) - \gamma(w_3 - m_2v_3)\}\}, \\
U_3 = - \{(w_3 - m_2v_3) + [(\gamma w_1 - w_3 - w_1w_5) - m_2(\gamma v_1 - v_3 - v_1v_5)] + \gamma(w_2 - m_1v_2) - \beta(w_4 - m_2v_4)\}, \\
U_4 = - \{(w_4 - m_2v_4) + [(\gamma w_2 - w_4 - w_2w_5) - m_2(\gamma v_2 - v_4 - v_2v_5)] + \beta(w_3 - m_2v_3) - \sigma(w_5 - m_3v_5)\}, \\
U_5 = - \{(w_5 - m_3v_5) + [(w_1w_3 + w_2w_4 - \beta w_5 + w_6) - m_3(v_1v_3 + v_2v_4 - \beta v_5 + v_6)] \\
\qquad + \sigma(w_4 - m_2v_4) - \alpha(w_6 - m_4v_6)\}, \\
U_6 = - \{(w_6 - m_4v_6) + [(w_1w_3 + w_2w_4 - \sigma w_6) - m_4(v_1v_3 + v_2v_4 - \sigma v_6)] + \alpha(w_5 - m_3v_5)\},
\end{cases} \tag{16}$$

then drive system (6) will achieve projective synchronization with response system (7).

(ii) Suppose that $m_1 = m_2 = m_3 = m_4 = 0$ and $k_1 = k_2 = k_3 = k_4 = 1$, and if the controllers are chosen as follows:

$$\begin{cases} U_1 = -\{(w_1 - l_1u_1) + [\alpha(w_3 - w_1) + w_6 - l_1\alpha(u_3 - u_1) - l_1u_6] - \alpha(w_2 - l_1u_2)\}, \\ U_2 = -\{(w_2 - l_1u_2) + [\alpha(w_4 - w_2) - l_1\alpha(u_4 - u_2)] + \alpha(w_1 - l_1u_1) - \gamma(w_3 - l_2u_3)\}, \\ U_3 = -\{(w_3 - l_2u_3) + [(\gamma w_1 - w_3 - w_1w_5) - l_2(\gamma u_1 - u_3 - u_1u_5)] + \gamma(w_2 - l_1u_2) - \beta(w_4 - l_2u_4)\}, \\ U_4 = -\{(w_4 - l_2u_4) + [(\gamma w_2 - w_4 - w_2w_5) - l_2(\gamma u_2 - u_4 - u_2u_5)] + \beta(w_3 - l_2u_3) - \sigma(w_5 - l_3u_5)\}, \\ U_5 = -\{(w_5 - l_3u_5) + [(w_1w_3 + w_2w_4 - \beta w_5 + w_6) - l_3(u_1u_3 + u_2u_4 - \beta u_5 + u_6)] \\ \qquad + \sigma(w_4 - l_2u_4) - \alpha(w_6 - l_4u_6)\}, \\ U_6 = -\{(w_6 - l_4u_6) + [(w_1w_3 + w_2w_4 - \sigma w_6) - l_4(u_1u_3 + u_2u_4 - \sigma u_6)] + \alpha(w_5 - l_3u_5)\}, \end{cases} \tag{17}$$

then drive system (5) will achieve projective synchronization with response system (7).

Corollary 2. Suppose that $l_1 = l_2 = l_3 = l_4 = 0, m_1 = m_2 = m_3 = m_4 = 0$ and $k_1 = k_2 = k_3 = k_4 = 1$, and if the controllers are chosen as follows:

$$\begin{cases} U_1 = -[w_1 + \alpha(w_3 - w_1) + w_6 - \alpha w_2], \\ U_2 = -[w_2 + \alpha(w_4 - w_2) + \alpha w_1 - \gamma w_3], \\ U_3 = -(w_3 + \gamma w_1 - w_3 - w_1w_5 + \gamma w_2 - \beta w_4), \\ U_4 = -(w_4 + \gamma w_2 - w_4 - w_2w_5 + \beta w_3 - \sigma w_5), \\ U_5 = -(w_5 + w_1w_3 + w_2w_4 - \beta w_5 + w_6 + \sigma w_4 - \alpha w_6), \\ U_6 = -(w_6 + w_1w_3 + w_2w_4 - \sigma w_6 + \alpha w_5), \end{cases} \tag{18}$$

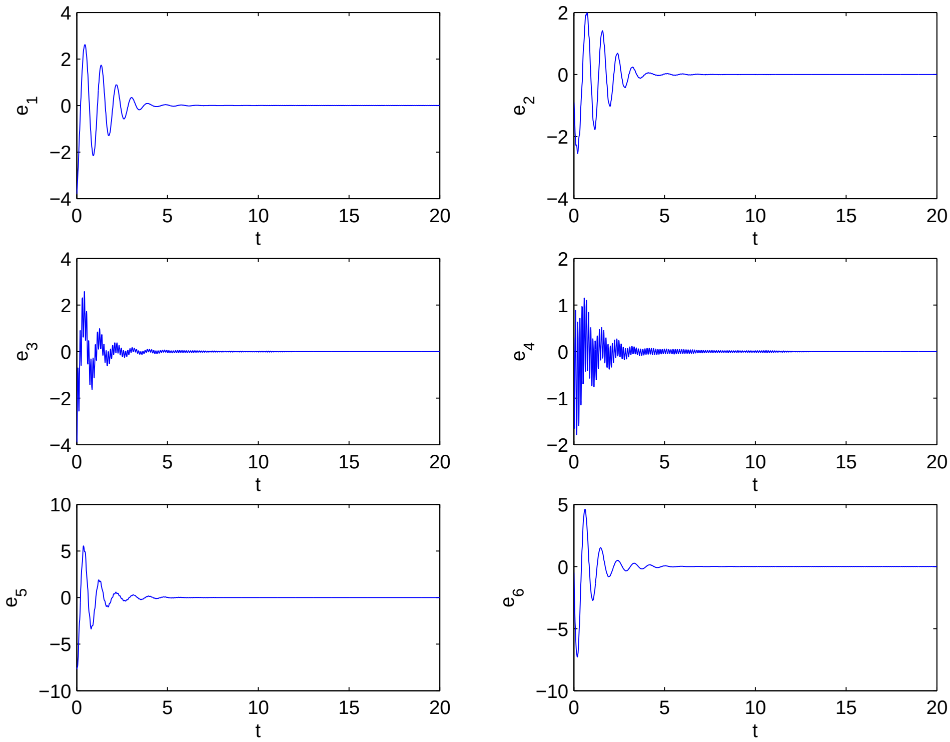
then system (7) is stabilized to the equilibrium, $O(0, 0, 0, 0, 0, 0)$.

Remark 1: The proofs of Corollary 1 and Corollary 2 are similar to those of theorem 1, so we omitted them.

In the following, numerical experiments are given to demonstrate our results. The fourth-order Runge-Kutta method is used with a time step size of 0.001. The system parameters are given as $\alpha = 8,\ \beta = 5,\ \gamma = 50$ and $\sigma = 15$, so that the complex Lorenz system exhibits hyperchaotic behavior. We assume $k_1 = k_2 = k_3 = k_4 = 1$, $l_1 = l_2 = l_3 = l_4 = 1$ and $m_1 = m_2 = m_3 = m_4 = 1$, and the initial states for drive systems (5) and (6) and response system (7) are arbitrarily given by $(x_{11}(0), x_{21}(0), x_{31}(0), x_{41}(0)) = (4 - 0.3i, 2.2 - 0.8i, 4.9, 1.1), (x_{12}(0), x_{22}(0), x_{32}(0), x_{42}(0)) = (4.4 - 0.6i, 3.3 - 1.4i, 5.3, 1.4)$ and $(x_{13}(0), x_{23}(0), x_{33}(0), x_{43}(0)) = (4.6 - 1.8i, 1.6 - 1.9i, 2.5, 2)$, *i.e.*, $(u_1(0), u_2(0), u_3(0), u_4(0), u_5(0), u_6(0)) = (4, -0.3, 2.2, -0.8, 4.9, 1.1)$, $(v_1(0), v_2(0), v_3(0), v_4(0), v_5(0), v_6(0)) = (4.4, -0.6, 3.3, -1.4, 5.3, 1.4)$ and $(w_1(0), w_2(0), w_3(0), w_4(0), w_5(0), w_6(0)) = (4.6, -1.8, 1.6, -1.9, 2.5, 2)$, respectively. The corresponding numerical results are shown in Figures 1 and 2. Figure 1 displays the time response of the combination synchronization errors, $e_1, e_2, e_3, e_4, e_5 and e_6$. The errors converge to zero, which implies that systems (5), (6) and (7) have achieved combination synchronization. Figures 2 depicts the time responses of the states, $u_1 + v_1$ *versus* w_1, $u_2 + v_2$ *versus* w_2, $u_3 + v_3$ *versus* w_3, $u_4 + v_4$ *versus* w_4, $u_5 + v_5$ *versus* w_5 and $u_6 + v_6$ *versus* w_6, respectively. Next, suppose that $k_1 = k_2 = k_3 = k_4 = 1$, $l_1 = l_2 = l_3 = l_4 = 0$ and $m_1 = m_2 = m_3 = m_4 = 0$. The time evolution of the states,

$w_1, w_2, w_3, w_4, w_5, w_6$, of system (7) with controller (18) are displayed in Figure 3, which illustrates that system (7) is stabilized to the equilibrium, $O(0, 0, 0, 0, 0, 0)$.

Figure 1. Combination synchronization errors, e_1, e_2, e_3, e_4, e_5 and e_6, between drive systems (5) and (6) and response system (7).



4. Combination Synchronization among Different Nonlinear Complex Hyperchaotic Systems

In this section, we investigate the combination synchronization among three different nonlinear complex hyperchaotic systems. The hyperchaotic complex Lorenz system [6] and the hyperchaotic complex Chen system [5], respectively, describe the drive systems:

$$\begin{cases} \dot{x}_1 = \alpha_1(x_2 - x_1) + x_4, \\ \dot{x}_2 = \gamma_1 x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = \dfrac{1}{2}(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \beta_1 x_3 + x_4, \\ \dot{x}_4 = \dfrac{1}{2}(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \sigma_1 x_4, \end{cases} \tag{19}$$

and

$$\begin{cases} \dot{y}_1 = \alpha_2(y_2 - y_1), \\ \dot{y}_2 = (\gamma_2 - \alpha_2) y_1 - y_1 y_3 + \gamma_2 y_2 + y_4, \\ \dot{y}_3 = \dfrac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - \beta_2 y_3 + y_4, \\ \dot{y}_4 = \dfrac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - d_2 y_4, \end{cases} \tag{20}$$

Figure 2. Time responses for states $u_i + v_i$ *versus* w_i, $i = 1, 2, ..., 6$, respectively.

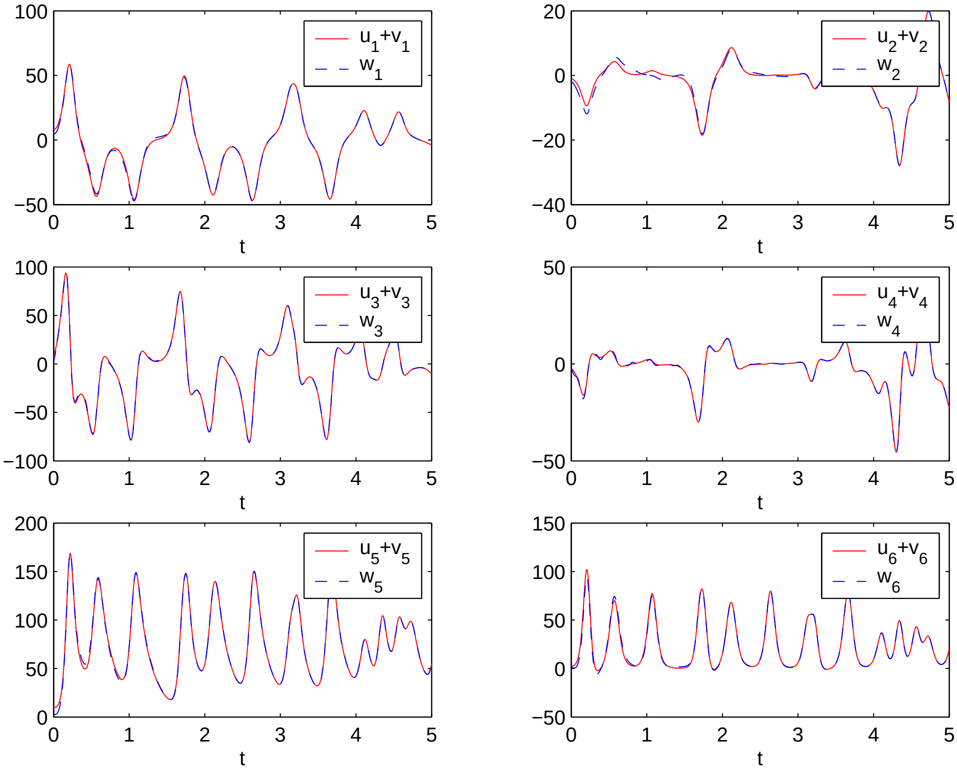
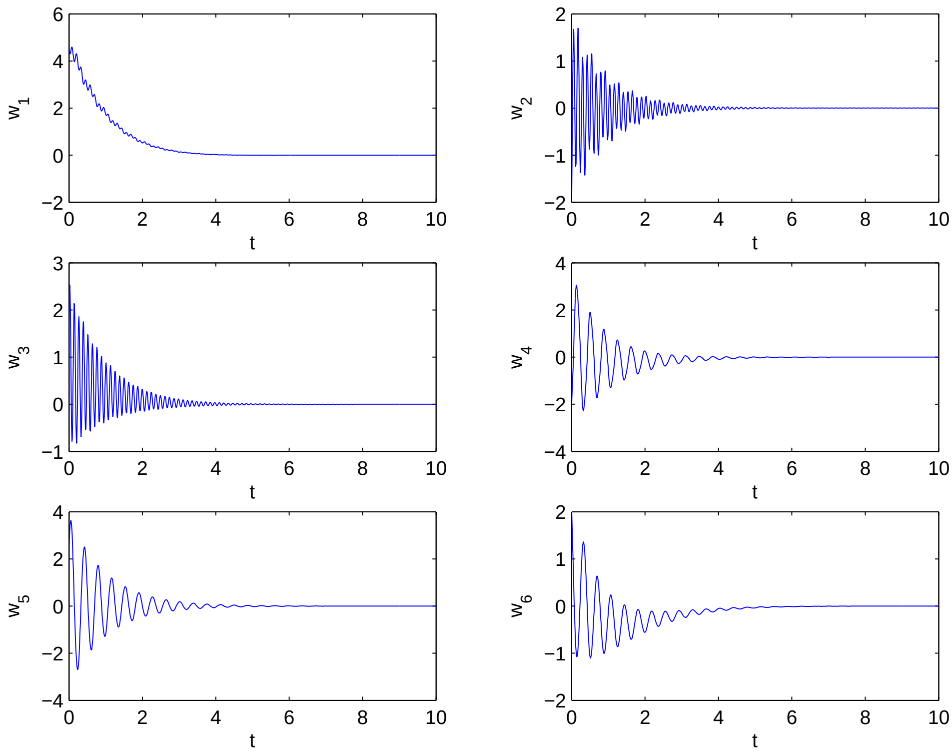


Figure 3. Time evolution of the states for system (7).



and the hyperchaotic complex Lü system system [7] is the response system given by:

$$\begin{cases} \dot{z}_1 = \rho_3(z_2 - z_1) + z_4 + U_1 + iU_2, \\ \dot{z}_2 = \nu_3 z_2 - z_1 z_3 + z_4 + U_3 + iU_4, \\ \dot{z}_3 = \frac{1}{2}(\bar{z}_1 z_2 + z_1 \bar{z}_2) - \mu_3 z_3 + U_5, \\ \dot{z}_4 = \frac{1}{2}(\bar{z}_1 z_2 + z_1 \bar{z}_2) - \sigma_3 z_4 + U_6, \end{cases} \tag{21}$$

where α_1, β_1, γ_1, σ_1, α_2, β_2, γ_2, d_2, ρ_3, ν_3, μ_3 and σ_3 are positive parameters, $x_1 = u_1 + iu_2,\ x_2 = u_3 + iu_4, y_1 = v_1 + iv_2,\ y_2 = v_3 + iv_4,\ z_1 = w_1 + iw_2,\ z_2 = w_3 + iw_4$ are complex variables and $i = \sqrt{-1}$; $u_i,\ v_i,\ w_i(i = 1, 2, 3, 4)$, $x_3 = u_5,\ x_4 = u_6,\ y_3 = v_5,\ y_4 = v_6,\ z_3 = w_5$ and $z_4 = w_6$ are real variables. The overbar represents the complex conjugate function. $U_1,\ U_2,\ U_3,\ U_4,\ U_5$ and U_6 are real control functions to be determined.

For the convenience of the following discussions, we assume $A = diag(l_1, l_2, l_3, l_4)$, $B = diag(m_1, m_2, m_3, m_4)$ and $C = diag(k_1, k_2, k_3, k_4)$ in our synchronization scheme.

We define error states between drive systems (19)–(20) and response system (21) as:

$$\begin{cases} e_1 + ie_2 = k_1 z_1 - l_1 x_1 - m_1 y_1, \\ e_3 + ie_4 = k_2 z_2 - l_2 x_2 - m_2 y_2, \\ e_4 = k_3 z_3 - l_3 x_3 - m_3 y_3, \\ e_5 = k_4 z_4 - l_4 x_4 - m_4 y_4, \end{cases} \tag{22}$$

such that:

$$\begin{cases} \lim_{t\to\infty} \parallel k_1 z_1 - l_1 x_1 - m_1 y_1 \parallel = 0, \\ \lim_{t\to\infty} \parallel k_2 z_2 - l_2 x_2 - m_2 y_2 \parallel = 0, \\ \lim_{t\to\infty} \parallel k_3 z_3 - l_3 x_3 - m_3 y_3 \parallel = 0, \\ \lim_{t\to\infty} \parallel k_4 z_4 - l_4 x_4 - m_4 y_4 \parallel = 0. \end{cases} \tag{23}$$

Separating the real and imagery parts of Equation (22) gets the following:

$$\begin{cases} e_1 = (k_1 w_1 - l_1 u_1 - m_1 v_1), \\ e_2 = (k_1 w_2 - l_1 u_2 - m_1 v_2), \\ e_3 = (k_2 w_3 - l_2 u_3 - m_2 v_3), \\ e_4 = (k_2 w_4 - l_2 u_4 - m_2 v_4), \\ e_5 = (k_3 w_5 - l_3 u_5 - m_3 v_5), \\ e_6 = (k_4 w_6 - l_4 u_6 - m_4 v_6). \end{cases} \tag{24}$$

Thus, we have the following error dynamical system:

$$\begin{cases} \dot{e}_1 = k_1[\rho_3(w_3 - w_1) + w_6] - l_1[\alpha_1(u_3 - u_1) + u_6] - m_1\alpha_2(v_3 - v_1) + k_1U_1, \\ \dot{e}_2 = k_1\rho_3(w_4 - w_2) - l_1\alpha_1(u_4 - u_2) - m_1\alpha_2(v_4 - v_2) + k_1U_2, \\ \dot{e}_3 = k_2(-w_1w_5 + \nu_3w_3 + w_6) - l_2(\gamma_1u_1 - u_3 - u_1u_5) - m_2[(\gamma_2 - \alpha_2)v_1 - v_1v_5 + \gamma_2v_3 + v_6] + k_2U_3, \\ \dot{e}_4 = k_2(-w_2w_5 + \nu_3w_4) - l_2(\gamma_1u_2 - u_4 - u_2u_5) - m_2[(\gamma_2 - \alpha_2)v_2 - v_2v_5 + \gamma_2v_4] + k_2U_4, \\ \dot{e}_5 = k_3(w_1w_3 + w_2w_4 - \mu_3w_5) - l_3(u_1u_3 + u_2u_4 - \beta_1u_5 + u_6) - m_3(v_1v_3 + v_2v_4 - \beta_2v_5 + v_6) + k_3U_5, \\ \dot{e}_6 = k_4(w_1w_3 + w_2w_4 - \sigma_3w_6) - l_4(u_1u_3 + u_2u_4 - \sigma_1u_6) - m_4(v_1v_3 + v_2v_4 - d_2v_6) + k_4U_6. \end{cases} \tag{25}$$

Similar to Section 3, we have the following results.

Theorem 2. If the controllers are chosen as:

$$\begin{cases} U_1 = -\dfrac{1}{k_1}\{(k_1w_1 - l_1u_1 - m_1v_1) + [k_1\rho_3(w_3 - w_1) + k_1w_6 - l_1\alpha_1(u_3 - u_1) - l_1u_6 - m_1\alpha_2(v_3 - v_1)] \\ \qquad - \alpha_1(k_1w_2 - l_1u_2 - m_1v_2)\}, \\ U_2 = -\dfrac{1}{k_1}\{(k_1w_2 - l_1u_2 - m_1v_2) + [k_1\rho_3(w_4 - w_2) - l_1\alpha_1(u_4 - u_2) - m_1\alpha_2(v_4 - v_2)] \\ \qquad + \alpha_1(k_1w_1 - l_1u_1 - m_1v_1) - \alpha_2(k_2w_3 - l_2u_3 - m_2v_3)\}, \\ U_3 = -\dfrac{1}{k_2}\{(k_2w_3 - l_2u_3 - m_2v_3) + [k_2(-w_1w_5 + \nu_3w_3 + w_6) - l_2(\gamma_1u_1 - u_3 - u_1u_5) - m_2(\gamma_2 - \alpha_2)v_1 \\ \qquad + m_2v_1v_5 - m_2\gamma_2v_3 - m_2v_6] + \alpha_2(k_1w_2 - l_1u_2 - m_1v_2) - \beta_1(k_2w_4 - l_2u_4 - m_2v_4)\}, \\ U_4 = -\dfrac{1}{k_2}\{(k_2w_4 - l_2u_4 - m_2v_4) + [k_2(-w_2w_5 + \nu_3w_4) - l_2(\gamma_1u_2 - u_4 - u_2u_5) - m_2(\gamma_2 - \alpha_2)v_2 \\ \qquad + m_2v_2v_5 - m_2\gamma_2v_4] + \beta_1(k_2w_3 - l_2u_3 - m_2v_3) - \beta_2(k_3w_5 - l_3u_5 - m_3v_5)\}, \\ U_5 = -\dfrac{1}{k_3}\{(k_3w_5 - l_3u_5 - m_3v_5) + [k_3(w_1w_3 + w_2w_4 - \mu_3w_5) - l_3(u_1u_3 + u_2u_4 - \beta_1u_5 + u_6) \\ \qquad - m_3(v_1v_3 + v_2v_4 - \beta_2v_5 + v_6)] + \beta_2(k_2w_4 - l_2u_4 - m_2v_4) - \gamma_1(k_4w_6 - l_4u_6 - m_4v_6)\}, \\ U_6 = -\dfrac{1}{k_4}\{(k_4w_6 - l_4u_6 - m_4v_6) + [k_4(w_1w_3 + w_2w_4 - \sigma_3w_6) - l_4(u_1u_3 + u_2u_4 - \sigma_1u_6) \\ \qquad - m_4(v_1v_3 + v_2v_4 - d_2v_6)] + \gamma_1(k_3w_5 - l_3u_5 - m_3v_5)\}, \end{cases} \tag{26}$$

then drive systems (19) and (20) will achieve combination synchronization with response system (21).

Corollary 3. (i) Suppose that $l_1 = l_2 = l_3 = l_4 = 0$ and $k_1 = k_2 = k_3 = k_4 = 1$, and if the controllers are chosen as follows:

$$\begin{cases} U_1 = -\{(w_1 - m_1v_1) + [\rho_3(w_3 - w_1) + w_6 - m_1\alpha_2(v_3 - v_1)] - \alpha_1(w_2 - m_1v_2)\}, \\ U_2 = -\{(w_2 - m_1v_2) + [\rho_3(w_4 - w_2) - m_1\alpha_2(v_4 - v_2)] + \alpha_1(w_1 - m_1v_1) - \alpha_2(w_3 - m_2v_3)\}, \\ U_3 = -\{(w_3 - m_2v_3) + [(-w_1w_5 + \nu_3w_3 + w_6) - m_2(\gamma_2 - \alpha_2)v_1 + m_2v_1v_5 - m_2\gamma_2v_3 - m_2v_6] \\ \qquad + \alpha_2(w_2 - m_1v_2) - \beta_1(w_4 - m_2v_4)\}, \\ U_4 = -\{(w_4 - m_2v_4) + [(-w_2w_5 + \nu_3w_4) - m_2(\gamma_2 - \alpha_2)v_2 + m_2v_2v_5 - m_2\gamma_2v_4] \\ \qquad + \beta_1(w_3 - m_2v_3) - \beta_2(w_5 - m_3v_5)\}, \\ U_5 = -\{(w_5 - m_3v_5) + [(w_1w_3 + w_2w_4 - \mu_3w_5) - m_3(v_1v_3 + v_2v_4 - \beta_2v_5 + v_6)] \\ \qquad + \beta_2(w_4 - m_2v_4) - \gamma_1(w_6 - m_4v_6)\}, \\ U_6 = -\{(w_6 - m_4v_6) + [(w_1w_3 + w_2w_4 - \sigma_3w_6) - m_4(v_1v_3 + v_2v_4 - d_2v_6)] + \gamma_1(w_5 - m_3v_5)\}, \end{cases} \tag{27}$$

then drive system (20) will achieve projective synchronization with response system (21).

(ii) Suppose that $m_1 = m_2 = m_3 = m_4 = 0$ and $k_1 = k_2 = k_3 = k_4 = 1$, and if the controllers are chosen as follows:

$$\begin{cases} U_1 = -\{(w_1 - l_1u_1) + [\rho_3(w_3 - w_1) + w_6 - l_1\alpha_1(u_3 - u_1) - l_1u_6] - \alpha_1(w_2 - l_1u_2)\}, \\ U_2 = -\{(w_2 - l_1u_2) + [\rho_3(w_4 - w_2) - l_1\alpha_1(u_4 - u_2)] + \alpha_1(w_1 - l_1u_1) - \alpha_2(w_3 - l_2u_3)\}, \\ U_3 = -\{(w_3 - l_2u_3) + [(-w_1w_5 + \nu_3w_3 + w_6) - l_2(\gamma_1u_1 - u_3 - u_1u_5)] + \alpha_2(w_2 - l_1u_2) \\ \qquad - \beta_1(w_4 - l_2u_4)\}, \\ U_4 = -\{(w_4 - l_2u_4) + [(-w_2w_5 + \nu_3w_4) - l_2(\gamma_1u_2 - u_4 - u_2u_5)] + \beta_1(w_3 - l_2u_3) - \beta_2(w_5 - l_3u_5)\}, \\ U_5 = -\{(w_5 - l_3u_5) + [(w_1w_3 + w_2w_4 - \mu_3w_5) - l_3(u_1u_3 + u_2u_4 - \beta_1u_5 + u_6)] + \beta_2(w_4 - l_2u_4) \\ \qquad - \gamma_1(w_6 - l_4u_6)\}, \\ U_6 = -\{(w_6 - l_4u_6) + [(w_1w_3 + w_2w_4 - \sigma_3w_6) - l_4(u_1u_3 + u_2u_4 - \sigma_1u_6)] + \gamma_1(w_5 - l_3u_5)\}, \end{cases} \tag{28}$$

then drive system (19) will achieve projective synchronization with response system (21).

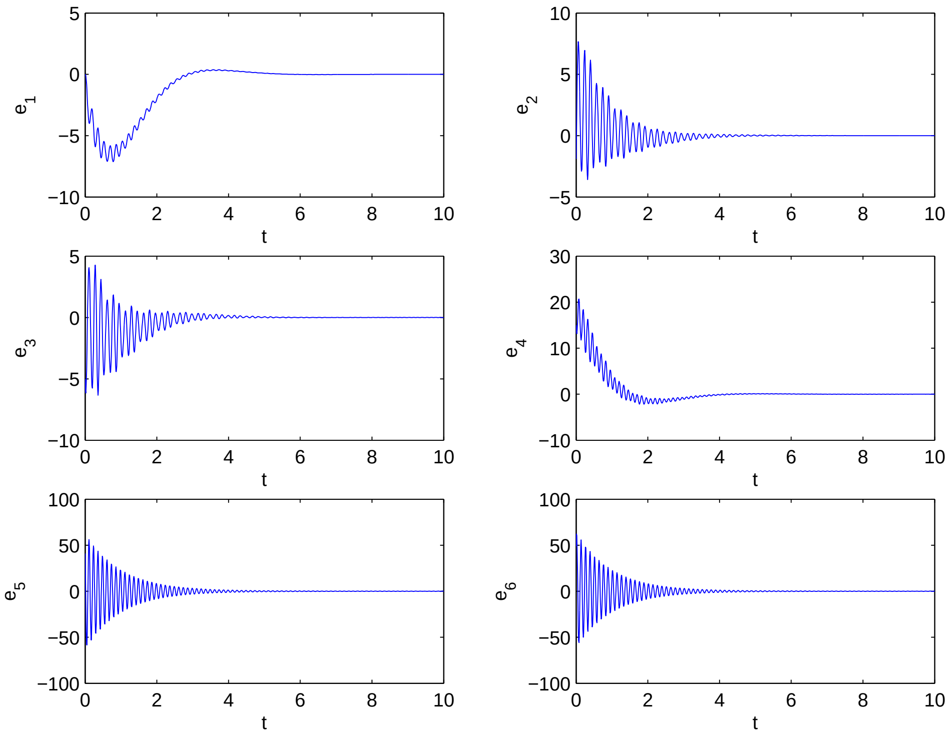
Corollary 4. Suppose that $l_1 = l_2 = l_3 = l_4 = 0$, $m_1 = m_2 = m_3 = m_4 = 0$ and $k_1 = k_2 = k_3 = k_4 = 1$, and if the controllers are chosen as follows:

$$\begin{cases} U_1 = -\{w_1 + [\rho_3(w_3 - w_1) + w_6] - \alpha_1w_2\}, \\ U_2 = -[w_2 + \rho_3(w_4 - w_2) + \alpha_1w_1 - \alpha_2w_3], \\ U_3 = -[w_3 + (-w_1w_5 + \nu_3w_3 + w_6) + \alpha_2w_2 - \beta_1w_4], \\ U_4 = -[w_4 + (-w_2w_5 + \nu_3w_4) + \beta_1w_3 - \beta_2w_5], \\ U_5 = -[w_5 + (w_1w_3 + w_2w_4 - \mu_3w_5) + \beta_2w_4 - \gamma_1w_6], \\ U_6 = -[w_6 + (w_1w_3 + w_2w_4 - \sigma_3w_6) + \gamma_1w_5], \end{cases} \tag{29}$$

then system (21) is stabilized to the equilibrium, $O(0, 0, 0, 0, 0, 0)$.

In what follows, numerical experiments are given to demonstrate our results. The fourth-order Runge-Kutta method is used with a time step size of 0.001. The system parameters are given as

Figure 4. Combination synchronization errors, e_1, e_2, e_3, e_4, e_5 and e_6, between drive systems (19), (20) and response system (21).



$\alpha_1 = 8, \beta_1 = 5, \gamma_1 = 50$, $\sigma_1 = 15$, $\alpha_2 = 36, \beta_2 = 4, \gamma_2 = 25$, $d_2 = 5$, $\rho_3 = 42, \nu_3 = 25, \mu_3 = 6$ and $\sigma_3 = 5$, so that the three complex nonlinear hyperchaotic systems exhibit hyperchaotic behaviors, respectively.

First, we assume $k_1 = k_2 = k_3 = k_4 = 1$, $l_1 = l_2 = l_3 = l_4 = 1$ and $m_1 = m_2 = m_3 = m_4 = 1$, and the initial states for the drive systems and response systems are arbitrarily given by $(x_1(0), x_2(0), x_3(0), x_4(0)) = (2.0 - i, 5.8 - 2i - 12, -16), (y_1(0), y_2(0), y_3(0), y_4(0)) = (1.7 + 2.3i, 0.1 - 14i, -16, -18)$ and $(z_1(0), z_2(0), z_3(0), z_4(0)) = (3.6 - 0.6i, 0.9 - i, 13, 15)$, ***i.e.***, $(u_1(0), u_2(0), u_3(0), u_4(0), u_5(0), u_6(0)) = (2.0, -1, 5.8, -2, -12, -16)$, $(v_1(0), v_2(0), v_3(0), v_4(0), v_5(0), v_6(0)) = (1.7, 2.3, 0.1, -14, -16, -18)$ and $(w_1(0), w_2(0), w_3(0), w_4(0), w_5(0), w_6(0)) = (3.6, -0.6, 0.9, -1, 13, 15)$, respectively. The corresponding numerical results are shown in Figures 4 and 5. Figure 4 displays the time response of the combination synchronization errors, $e_1, e_2, e_3, e_4, e_5 and e_6$. The errors converge to zero, which implies that systems (19), (20) and (21) have achieved combination synchronization. Figure 5 depicts the time responses of the states, $u_1 + v_1$ *versus* w_1, $u_2 + v_2$ *versus* w_2,$u_3 + v_3$ *versus* w_3,$u_4 + v_4$ *versus* w_4,$u_5 + v_5$ *versus* w_5 and $u_6 + v_6$ *versus* w_6, respectively. When $k_1 = k_2 = k_3 = k_4 = 1$, $l_1 = l_2 = l_3 = l_4 = 0$ and $m_1 = m_2 = m_3 = m_4 = 0$, the time evolution of the states, $w_1, w_2, w_3, w_4, w_5, w_6$, of system (21) with controller (29) are displayed in Figure 6, which means that system (21) is stabilized to the equilibrium, $O(0, 0, 0, 0, 0, 0)$.

Figure 5. Time responses for states $u_i + v_i$ *versus* w_i, $i = 1, 2, ..., 6$, respectively.

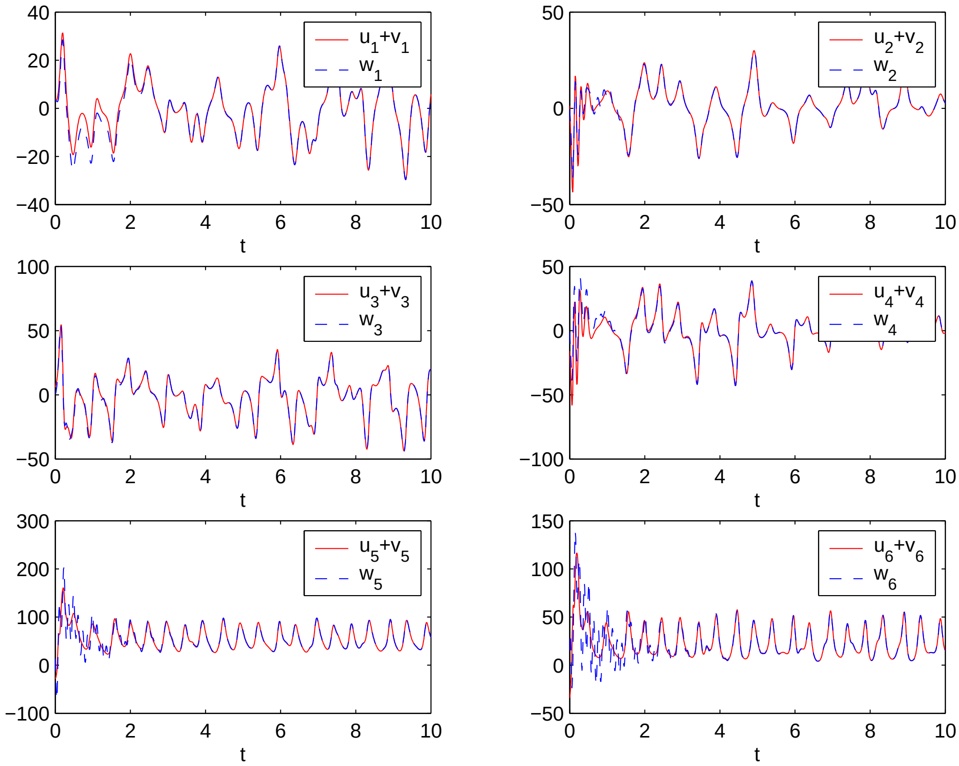
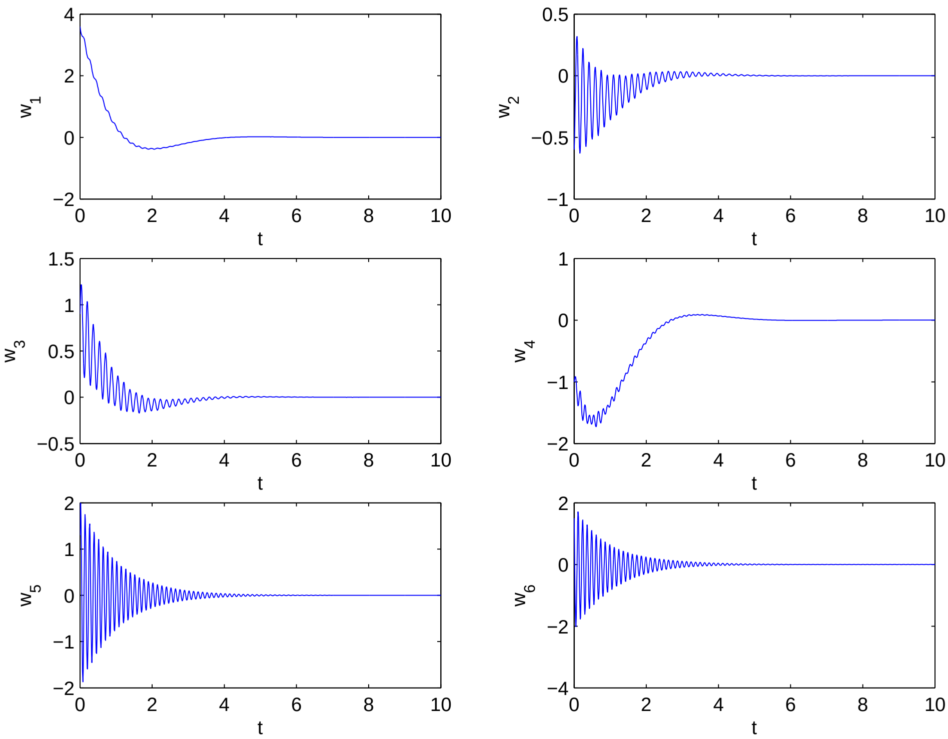


Figure 6. Time evolution of the states for system (21).



5. Conclusions

This paper investigates the combination synchronization of three nonlinear complex hyperchaotic systems: the complex hyperchaotic Lorenz system, the complex hyperchaotic Chen system and the complex hyperchaotic Lü system. Based on the Lyapunov stability theory, corresponding controllers to achieve combination synchronization among three identical or different nonlinear complex hyperchaotic systems are derived, respectively. Numerical simulations are conducted to illustrate the validity and feasibility of the theoretical analysis. When applying the complex systems in communications, the complex variables will double the number of variables and can increase the content and security of the transmitted information. Furthermore, combination synchronization between two drive systems and one response system has obvious advantages over synchronization between one drive system and one response system. Thus combination synchronization of complex nonlinear systems can find better applications in security communication. However, in practical chaotic synchronization, mismatched parameters exist, and the external disturbances are always unavoidable [17]. In our future work, we will investigate robust combination synchronization in the existence of mismatched parameters and external disturbances.

Acknowledgments

The authors sincerely thank the referees for their helpful comments. This work was supported by the the Natural Science Foundation of Yunnan Province under grant No. 2009CD019 and the Natural Science Foundation of China under grants No. 61065008 and No. 11161055.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Fowler, A.C.; Gibbon, J.D.; McGuinness, M.J. The complex Lorenz equations. *Phys. D* **1982**, *4*, 139–163.
2. Mahmoud, E.E. Dynamics and synchronization of new hyperchaotic complex Lorenz system. *Math. Comput. Model.* **2012**, *55*, 1951–1962.
3. Mahmoud, G.M.; Farghaly, A.A.M. Chaos control of chaotic limit cycles of real and complex van der Pol oscillators. *Chaos Solitons Fractals* **2004**, *21*, 915–924.
4. Mahmoud, G.M.; Bountis, T.; Mahmoud, E.E. Active control and global synchronization of the complex Chen and Lü systems. *Int. J. Bifur. Chaos* **2007**, *17*, 4295–4308.
5. Mahmoud, G.M.; Mahmoud, E.E.; Ahmed, M.E.A. hyperchaotic complex Chen system and its dynamics. *Int. J. Appl. Math. Stat.* **2007**, *12*, 90–100.
6. Mahmoud, G.M.; Ahmed, M.E.; Mahmoud, E.E. Analysis of hyperchaotic complex Lorenz systems. *Int. J. Mod. Phys. C* **2008**, *19*, 1477–1494.
7. Mahmoud, G.M.; Mahmoud, E.E.; Ahmed, M.E. On the hyperchaotic complex Lü system. *Nonlinear Dyn.* **2009**, *58*, 725–738.

8. Mahmoud, G.M.; Al-Kashif, M.A.; Farghaly, A.A. Chaotic and hyperchaotic attractors of a complex nonlinear system. *J. Phys. A Math. Theor.* **2008**, *41*, e055104.
9. Mahmoud, G.M.; Ahmed, M.E.; Sabor, N. On autonomous and nonautonomous modified hyperchaotic complex Lü systems. *Int. J. Bifur. Chaos* **2011**, *21*, 1913–1926.
10. Pecora, L.M.; Carroll, T.L. Synchronization in chaotic systems. *Phys. Rev. Lett.* **1990**, *64*, 821–824.
11. Mahmoud, G.M.; Mahmoud, E.E. Complete synchronization of chaotic complex nonlinear systems with uncertain parameters. *Nonlinear Dyn.* **2010**, *62*, 875–882.
12. Mahmoud, G.M.; Mahmoud, E.E. Phase and antiphase synchronization of two identical hyperchaotic complex nonlinear systems. *Nonlinear Dyn.* **2010**, *61*, 141–152.
13. Mahmoud, G.M.; Mahmoud, E.E.; Arafa, A.A. On projective synchronization of hyperchaotic complex nonlinear systems based on passive theory for secure communications. *Phys. Scr.* **2013**, *87*, e055002.
14. Liu, P.; Liu, S.; Li, X. Adaptive modified function projective synchronization of general uncertain chaotic complex systems. *Phys. Scr.* **2012**, *85*, e035005.
15. Luo, R.Z.; Wang, Y.L.; Deng, S.C. Combination synchronization of three classic chaotic systems using active backstepping design. *Chaos* **2011**, *21*, e043114.
16. Luo, R.Z.; Wang, Y.L. Finite-time stochastic combination synchronization of three different chaotic systems and its application in secure communication. *Chaos* **2012**, *22*, e023109.
17. Ahn, C.K.; Jung, S.T.; Kang, S.K.; Joo, S.C. Adaptive $H\infty$ synchronization for uncertain chaotic systems with external disturbance. *Commun. Nonlinear Sci. Numer. Simulat.* **2010**, *15*, 2168–2177.