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Article

Modified Gravity Models Admitting Second Order Equations of Motion

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Abstract: The aim of this paper is to find higher order geometrical corrections to the Einstein–Hilbert action that can lead only to second order equations of motion. The metric formalism is used, and static spherically-symmetric and Friedmann–Lemaître space-times are considered, in four dimensions. The Fulling, King, Wybourne and Cummings (FKWC) basis is introduced in order to consider all of the possible invariant scalars, and both polynomial and non-polynomial gravities are investigated.

Keywords: modified gravities; non-polynomial gravities; higher order corrections; regular cosmological solutions; FLRW space-times, static spherically-symmetric space-times

1. Introduction

Most of the equations of motion describing physical effects are second order, that is we need to specify either an initial and a final position in space-time to describe the dynamics between them or we need an initial position and velocity to describe how the system will evolve. Concerning general relativity (GR), for which the gravitational field $g_{\mu\nu}(x)$ is encoded into the geometry of space-time:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \,, \tag{1}$$

the Einstein field equations, describing the dynamics of the geometry, are also second order ones.

However, it is well known that two of the simplest solutions of GR, the Schwarzschild metric and the Friedmann–Lemaître one, suffer from the existence of singularities. When one is dealing with ordinary matter, this is a general fact. Furthermore, this theory alone is not able to describe dark energy, even though the inclusion of a suitable cosmological constant is sufficient. However, then, other problems arise, like the cosmological constant one [1] and the coincidence problem. Therefore, one can think about modifying the Einstein equations, in the hopes to describe dark energy and to cure singularities. In order to do so, one can add higher order invariant scalars, like $R_{\mu\nu}R^{\mu\nu}$, to the Einstein–Hilbert action to have high energy corrections that could describe what really happens around the singularities [2,3]. With regard to the dark energy issue, see [4–9].

Within a higher order modified gravity model, the equations of motion will no longer be second order ones: there will be more than two initial conditions to specify in order to find the dynamics; so to keep the physical sense of what is an equation of motion, it is necessary to introduce new fields to which these additional initial conditions would apply, such that at the end, the theory would involve at least two dynamical fields, with second order equations of motion for all of them. By doing so, we face an important problem, that is the presence of Ostrogradsky instabilities (see, for example, [4]): the new field defined in this way can carry negative kinetic energy, such that the Hamiltonian of the theory is not bounded from below and can reach arbitrarily negative energies, which would make this theory impossible to quantize in a satisfying way [4]. Additionally, there are no general rules to avoid this problem, although a well-known class of modified gravity, equivalent to GR plus a scalar field, the $f\left(R\right)$ one, might not suffer from this problem [10].

Moreover, with a new field involved in the dynamics of gravity, this last one would not be a fully geometrical theory anymore, which is yet one of the most important implications of general relativity. Nevertheless, it is possible to find second order equations of motion from the addition into the Einstein–Hilbert action of higher order scalars [11]. In this way, the Ostrogradsky instability may be avoided, and there are no additional fields involved in the dynamics, so these corrections can be said to be "geometrical" ones. These kinds of modifications are the Lovelock scalars, but it turns out that in four dimensions, the only higher order scalar made of contractions of curvature tensors (only) that leads to second order equations [11] is the so-called Gauss–Bonnet invariant:

$$\mathcal{E}_4 = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma}{}^{\delta}R^{\alpha\beta\gamma}{}_{\delta}, \tag{2}$$

which is however a total derivative in four dimensions [12], and then, it does not contribute to the equation of motion:

$$\sqrt{-g}\mathcal{E}_4 = \partial_\alpha\Bigg(- \sqrt{-g}\,\epsilon^{\alpha\beta\gamma\delta}\;\epsilon_{\rho\sigma}{}^{\mu\nu}\Gamma_{\mu\beta}{}^{\rho}\left(\frac{1}{2}R_{\delta\gamma\nu}{}^{\sigma} - \frac{1}{3}\Gamma_{\lambda\gamma}{}^{\sigma}\Gamma_{\nu\delta}{}^{\lambda}\right)\Bigg). \tag{3}$$

This result is background independent, which means that if we want to find a second order correction for all possible metrics in four dimensions, then this unique term does not contribute to the dynamics. That is why, in order to find significant corrections to general relativity that could cure some of its problems, we will search for additional terms that will give second order equations for only some specific metrics: the most studied ones, which suffer from singularities, the Friedmann–Lemaître–Robertson–Walker (FLRW) space-time describing the large-scale dynamics of the universe and the static spherically-symmetric space-time describing neutral non-rotating stars and black holes.

We note however that our way to find second order corrections is not at all the only possible one. There are other formulations of GR than the metric one, where the equations of motion are found by varying the action with respect to the metric field only. In the spirit of gauge theories, one can also vary the action with respect to the connections and independently with respect to the metric. Then, it is possible to find second order corrections with no *a priori* background structures [13].

In some sense, our approach is similar to Horndeski's theory, which is the most general one leading to second order equations of motion for gravity described by a metric $g_{\mu\nu}$ coupled with a scalar field ϕ and its first two derivatives [14]. This theory involves non-linear higher order derivatives of the scalar field, like $\left(\Box\phi\right)^2$, and yet leads to second order. Moreover, if all of the matter fields are minimally coupled with the same metric $\widetilde{g}_{\mu\nu}\left(g_{\mu\nu},\phi\right)$, one can expect the equivalence principle to hold [15], which is also a fundamental feature of GR that one wants to keep.

Briefly, the outline of the paper is as follows. First, we consider all of the independent scalar invariants built from the metric field and its derivatives, for example of the form $\left(\Box R\right)^2$, and see if some linear combinations of them or, in the spirit of [16] and [17], if some roots of these combinations could lead to second order differential equations for FLRW space-time and the static spherically-symmetric one. The basis of the independent scalars that are needed has been presented in [18], but for specific backgrounds, we will show that this basis may be reduced. Furthermore, we will start to exhibit, order by order for FLRW, the existence of polynomial and non-polynomial gravity models that give second order equations and polynomial corrections to the Friedmann equation. Finally, we will investigate the static spherically-symmetric space-times.

2. Order Six FKWC Basis

The basis of all independent invariant geometrical scalars involving $2n$ derivatives of the metric are separated into different classes, depending on how many covariant derivatives act on curvature tensors. For order six ($n = 3$), the first class, which does not involve explicitly covariant derivatives, from $\mathcal{L}_1$ to $\mathcal{L}_8$, is denoted by $\mathcal{R}^0_{6,3}$: these scalars are built with six derivatives of the metric and by the contraction of three curvature tensors. The two other classes, $\mathcal{R}^0_{\{2,0\}}$ and $\mathcal{R}^0_{\{1,1\}}$, contain scalars that involve, respectively, a curvature tensor contracted with two covariant derivatives acting on another curvature tensor (from $\mathscr{L}_1$ to $\mathscr{L}_4$) and two covariant derivatives, each acting on one curvature tensor (from $\mathscr{L}_5$ to $\mathscr{L}_8$):

$$\begin{cases} \mathcal{L}_1 = R^{\mu\nu\alpha\beta}R_{\alpha\beta\sigma\rho}R^{\sigma\rho}{}_{\mu\nu}\ , & \mathcal{L}_2 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\sigma}{}_{\nu\rho}R^{\beta\rho}{}_{\mu\sigma} \\ \mathcal{L}_3 = R^{\mu\nu\alpha\beta}R_{\alpha\beta\nu\sigma}R^{\sigma}{}_{\mu}\ , & \mathcal{L}_4 = RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = RT \\ \mathcal{L}_5 = R^{\mu\nu\alpha\beta}R_{\mu\alpha}R_{\nu\beta}\ , & \mathcal{L}_6 = R^{\mu\nu}R_{\nu\alpha}R^{\alpha}{}_{\mu} \\ \mathcal{L}_7 = RR^{\mu\nu}R_{\mu\nu} = RS\ , & \mathcal{L}_8 = R^3 \end{cases}$$

$$\begin{cases} \mathscr{L}_1 = R\Box R & , \quad \mathscr{L}_2 = R_{\mu\nu}\Box R^{\mu\nu} \\ \mathscr{L}_3 = R^{\mu\nu\alpha\beta}\nabla_\nu\nabla_\beta R_{\mu\alpha} & , \quad \mathscr{L}_4 = R^{\mu\nu}\nabla_\mu\nabla_\nu R \\ \mathscr{L}_5 = \nabla_\sigma R_{\mu\nu}\nabla^\sigma R^{\mu\nu} & , \quad \mathscr{L}_6 = \nabla_\sigma R_{\mu\nu}\nabla^\nu R^{\mu\sigma} \\ \mathscr{L}_7 = \nabla_\sigma R_{\mu\nu\alpha\beta}\nabla^\sigma R^{\mu\nu\alpha\beta} & , \quad \mathscr{L}_8 = \nabla_\sigma R\nabla^\sigma R \end{cases}$$

Recall that our aim is to see that if we consider all of these scalars, there are quite natural modified gravity Lagrangian densities that we can expect to lead to second order equations of motion and that actually do. There are linear combinations of all of the scalars of the basis, but also, for example, the square-root of the $\mathcal{R}^0_{\{1,1\}}$-class or cubic-roots of the $\mathcal{R}^0_{6,3}$ one, even if we are not going to study this last one because we search here for high energy geometrical correction to the Einstein–Hilbert action. We write down this fact as:

$$\mathscr{L} = \sum \Big(R^3 + R\nabla\nabla R + \nabla R\nabla R\Big) + \sqrt{\sum\left(\nabla R\nabla R\right)} + \sqrt[3]{\sum\left(R^3\right)}.$$

Because inside the same class, the scalars have approximatively the same terms in their expansions, we can indeed expect to cancel higher order derivatives for some specific combinations of them and then to have second order equations of motion.

Now, let us write down some definitions that allow us to find relations between these scalars, coming from the fact that we are going to restrict our study to specific backgrounds, the FLRW metric and the static spherically-symmetric one, both in four dimensions.

For both of them, there are relations between the scalars coming from the Lovelock theorem (that are not taken into account in [18]) and, also, for FLRW, relations coming from the fact that this is a conformally-invariant flat metric. All of these relations are written in the Appendix A.1, and we need to express them to define the Weyl tensor, as:

$$W_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{1}{2}(R_{\mu\alpha}g_{\nu\beta} - R_{\mu\beta}g_{\nu\alpha} + R_{\nu\beta}g_{\mu\alpha} - R_{\nu\alpha}g_{\mu\beta}) + \frac{1}{6}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})R. \quad (4)$$

Additionally, the following rank two tensor is null in four dimensions because of the Lovelock theorem:

$$L_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\mathcal{E}_4 + 2Q_{\mu\nu} - 4P_{\mu\nu} + 4R^{\alpha}_{\ \nu\mu}{}^{\gamma}R_{\alpha\gamma} + 2RR_{\mu\nu} = 0, \quad (5)$$

where $Q_{\mu\nu} = R_{\mu\eta\alpha}{}^{\beta}R_{\nu}{}^{\eta\alpha}{}_{\beta}$ and $P_{\mu\nu} = R_{\nu\gamma}R^{\gamma}_{\ \mu}$. Indeed, if we vary the Lagrangian associated with the Gauss–Bonnet invariant with respect to the metric field, we find:

$$\delta\big(\sqrt{-g}\mathcal{E}_4\big) = \sqrt{-g}\ \delta g^{\mu\nu}L_{\mu\nu}. \quad (6)$$

However, as we saw in Equation 3, this Lagrangian can be written as a total derivative in four dimensions, which means that its contribution to the equations of motion is identically zero.

3. Friedmann–Lemaître Space-Time

3.1. Order 6

We start with the Friedmann–Lemaître cosmological metric describing the dynamics of the universe at a very large scale, in the simplest manner:

$$ds^2 = -dt^2 + a(t)^2\big(dr^2 + r^2 d\Omega^2\big), \quad (7)$$

where $a(t)$ is the scale factor and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric of the two-sphere. This metric is conformally invariant, and from the relations written in the Appendix A.1, we can choose the following reduced basis of all independent order six scalar invariants: $(\mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_7, \mathscr{L}_1, \mathscr{L}_3, \mathscr{L}_5, \mathscr{L}_8)$. We note that there is one scalar less than for a general conformally-invariant space-time coming from the particular metric of FLRW for which there is the additional relation:

$$\mathcal{L}_1 = \frac{1}{3}\big(- \mathcal{L}_7 + 2\mathcal{L}_4\big). \tag{8}$$

3.1.1. Linear Combination: H^6 Correction

With this metric, we can right the most general order six linear combination of all of the independent scalars:

$$\begin{aligned} J =& \sum \Big(v_i \mathcal{L}_i + x_i \mathscr{L}_i \Big) \\ =& \frac{3}{a(t)^6} \Bigg(\sigma_1(v_i,x_i)\, \dot{a}(t)^6 + \sigma_2(v_i,x_i)\, a(t)\, \dot{a}(t)^4\, \ddot{a}(t) + \sigma_3(v_i,x_i)\, a(t)^2\, \dot{a}(t)^2\, \ddot{a}(t)^2 \\ & + \sigma_4(v_i,x_i)\, a(t)^3\, \ddot{a}(t)^3 + \sigma_5(v_i,x_i)\, a(t)^2\, \dot{a}(t)^3\, a^{(3)}(t) + \sigma_6(v_i,x_i)\, a(t)^3\, \dot{a}(t)\, \ddot{a}(t)\, a^{(3)}(t) \\ & + \sigma_7(v_i,x_i)\, a(t)^4\, a^{(3)}(t)^2 + \sigma_8(v_i,x_i)\, a(t)^3\, \dot{a}(t)^2\, a^{(4)}(t) + \sigma_9(v_i,x_i)\, a(t)^4\, \ddot{a}(t)\, a^{(4)}(t) \Bigg), \end{aligned}$$

where the expressions of the σ_j in terms of (v_i, x_i) are presented in the Appendix A.2. Setting all of them to zero allows us to check that our list of scalars is a basis. We can then impose $v_1 = v_2 = v_3 = v_5 = v_8 = x_2 = x_4 = x_6 = x_7 = 0$ to take into account the algebraic relations that we have found.

Moreover, in this section, we are interested in linear combinations of order six scalars that lead to second order equations of motion. Therefore, we can also consider equivalence relations (up to boundary terms) between the scalars, and there are three of them that remain after considering the previous algebraic relations:

$$\int d^4x\sqrt{-g}\mathscr{L}_3 = -\frac{1}{12}\int d^4x\sqrt{-g}\ \mathscr{L}_8 \qquad ; \qquad \int d^4x\sqrt{-g}\mathscr{L}_1 = -\int d^4x\sqrt{-g}\ \mathscr{L}_8$$

and

$$\int d^4x\sqrt{-g}\mathscr{L}_5 = \frac{1}{6}\int d^4x\sqrt{-g}\ \big(2\mathscr{L}_8 + 3\mathcal{L}_4 - 12\mathcal{L}_6 + \mathcal{L}_7\big).$$

We check that there is no other one by deriving the equations of motion for the scale factor considering the Lagrangian $L = a(t)^3\, J$, which we substitute into the generalized Euler–Lagrange equation for third order Lagrangians:

$$-\frac{d^3}{dt^3}\Big(\frac{\partial L}{\partial a^{(3)}}\Big) + \frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial \ddot{a}}\Big) - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{a}}\Big) + \frac{\partial L}{\partial a} = 0. \tag{9}$$

Finally, we only need to consider the combination:

$$J = v_4\mathcal{L}_4 + v_6\mathcal{L}_6 + v_7\mathcal{L}_7 + x_8\mathscr{L}_8,$$

and after deriving the equation of motion and imposing a simultaneous cancellation of the higher order terms, we find that there is only one linear combination leading to second order equations:

$$J_1 = -7\mathcal{L}_4 + 2(6\mathcal{L}_6 + \mathcal{L}_7) = 72H(t)^4\left(3\dot{H}(t) + 2H(t)^2\right), \tag{10}$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. We note that this linear combination only involves contractions of curvature tensors, which are the kinds of corrections expected to follow from quantum field theory. Therefore, considering the following action that could represent a natural high energy geometrical correction to the Einstein–Hilbert action for the FLRW space-time,

$$S_1 = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}\left[R + \nu\Big(-7\,RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + 12\,R^{\mu\nu}R_{\nu\alpha}R^{\alpha}_{\ \mu} + 2\,RR^{\mu\nu}R_{\mu\nu}\Big)\right] + \mathscr{L}_m\right), \tag{11}$$

where $\mathscr{L}_m$ is the Lagrangian density for matter, one finds the acceleration equation:

$$3H(t)^2 + 2\dot{H}(t) - 36\,\nu\,H(t)^4\big(H(t)^2 + 2\dot{H}(t)\big) = -8\pi p, \tag{12}$$

where p is the cosmic pressure. Then, the equation of the conservation of energy, with ρ the cosmic energy density,

$$\frac{d\rho}{dt} + 3H(t)\big(\rho + p\big) = 0, \tag{13}$$

gives the following modified Friedmann equation:

$$3H(t)^2 - 36\,\nu\,H(t)^6 = 8\pi\rho. \tag{14}$$

One can solve it and choose only the solutions that reduce to the standard equation when ρ is small. However, the only solution we are going to see in this work for FLRW space-time is coming from a non-polynomial correction involving order eight scalars; but, the fact that S_1 is unique and second order could be a sufficient reason to study its cosmological solutions.

3.1.2. Non-Polynomial Gravity: H^3 Correction

Now, we want to consider all of the order six linear combinations that are perfect squares, in order to consider non-polynomial corrections that are the square-roots of these squares. Rewrite the most general linear combination in terms of the Hubble parameter $H(t)$:

$$\begin{aligned} J =& \sum_{i=4,6,7} v_i\mathcal{L}_i + \sum_{j=1,3,5,8} x_j\mathscr{L}_j \\ =& 3\Bigg(\widetilde{\sigma}_1(v_i,x_i)\;H(t)^6 + \widetilde{\sigma}_2(v_i,x_i)\;H(t)^4\dot{H}(t) + \widetilde{\sigma}_3(v_i,x_i)\;H(t)^2\dot{H}(t)^2 \\ &+ \widetilde{\sigma}_4(v_i,x_i)\;\dot{H}(t)^3 + \widetilde{\sigma}_5(v_i,x_i)\;H(t)^3\ddot{H}(t) + \widetilde{\sigma}_6(v_i,x_i)\;H(t)\dot{H}(t)\ddot{H}(t) \\ &+ \widetilde{\sigma}_7(v_i,x_i)\ddot{H}(t)^2 + \widetilde{\sigma}_8(v_i,x_i)\;H(t)^2H^{(3)}(t) + \widetilde{\sigma}_9(v_i,x_i)\;\dot{H}(t)H^{(3)}(t)\Bigg). \end{aligned}$$

To find squares, we use the following general procedure that is useful for FLRW space-time and necessary for spherical symmetry: Take the higher order perfect square $\ddot{H}(t)^2$. In the expansion of our square, each term will be multiplied by $\ddot{H}(t)$, so that the only terms that can enter inside are those $K_i(t)$ for which $K_i(t)\ddot{H}(t)$ and $K_i(t)K_j(t)$ exist in the expansion of order six scalars. Because of this, we need to impose the conditions: $\widetilde{\sigma}_9 = 0$, $\widetilde{\sigma}_8 = 0$ and $\widetilde{\sigma}_4 = 0$, which give $x_1 = x_3 = 0$ and $v_7 = -v_4 - 5v_6/12$. Therefore, $\widetilde{\sigma}_5 = 0$, and there are only two possible forms of squares made of order six scalars:

$$\sum_{i,j}\left(v_i\mathcal{L}_i + x_j\mathscr{L}_j\right) = \left(\delta H(t)\dot{H}(t) + \gamma H(t)^3\right)^2,$$

and:

$$\sum_{i,j}\left(v_i\mathcal{L}_i + x_j\mathscr{L}_j\right) = \left(\xi\ddot{H}(t) + \delta H(t)\dot{H}(t)\right)^2.$$

This means that all of their square-roots can be decomposed in the basis $\left(H(t)^3,\ H(t)\dot{H}(t),\ \ddot{H}(t)\right)$. Moreover, the general Lagrangian density:

$$\sqrt{\sum_{i,j}\left(v_i\mathcal{L}_i + x_j\mathscr{L}_j\right)} = \xi\ddot{H}(t) + \delta H(t)\dot{H}(t) + \gamma H(t)^3\,, \tag{15}$$

leads to second order differential equations for all $\left(\xi,\delta,\gamma\right)$, so we do not need to impose additional conditions on these coefficients, and there are then only three independent second order corrections that we can find in this way.

Now, let us see what are the actual perfect squares that one can find. Solving the natural conditions for respectively the first and the second kind of perfect squares, $\widetilde{\sigma}_2^2 = 4\,\widetilde{\sigma}_3\,\widetilde{\sigma}_1$, $\widetilde{\sigma}_6 = \widetilde{\sigma}_7 = 0$ and $\widetilde{\sigma}_6^2 = 4\,\widetilde{\sigma}_3\,\widetilde{\sigma}_7$, $\widetilde{\sigma}_1 = \widetilde{\sigma}_2 = 0$, one finds the following ones:

$$\begin{aligned}
J_2 &= 2\mathcal{L}_4 + 12\mathcal{L}_6 - 7\mathcal{L}_7 = -72\left(4H^3 + 3\dot{H}H\right)^2,\\
J_3 &= \left(6\mathcal{L}_4 - 12\mathcal{L}_6 - \mathcal{L}_7\right)\alpha\left(5\alpha + 18\beta\right) + 6\left(\alpha + 3\beta\right)\left(\alpha\mathscr{L}_5 + \beta\mathscr{L}_8\right)\\
&= -72\left(3\left(\alpha + 4\beta\right)H(t)\dot{H}(t) + \left(\alpha + 3\beta\right)\ddot{H}(t)\right)^2.
\end{aligned} \tag{16}$$

the last one being a general formula that gives perfect squares for any value of $\left(\alpha,\beta\right)$. As we just saw, there are only three independent square-roots of these squares, so from the general expression J_3, we can choose the two last to be the squares given by $\left(\alpha = 0, \beta = \sqrt{2}/6\right)$ and $\left(\alpha = 1, \beta = 0\right)$:

$$J_{3,1} = \mathscr{L}_8 = -36\left(4H(t)\dot{H}(t) + \ddot{H}(t)\right)^2, \tag{17}$$

and

$$J_{3,2} = \left(6\mathscr{L}_5 + 5\left(6\mathcal{L}_4 - 12\mathcal{L}_6 - \mathcal{L}_7\right)\right) = -72\left(3H(t)\dot{H}(t) + \ddot{H}(t)\right)^2. \tag{18}$$

Indeed, one can check that the following combination, with ϵ_i the signs inside the squares,

$$\begin{aligned}
\nu_0\,\epsilon_0\sqrt{-J_3} + \nu_1\,\epsilon_1\sqrt{-J_{3,1}} + \nu_2\,\epsilon_2\sqrt{-J_{3,2}} =&6\left(3\sqrt{2}\left(\alpha + 4\,\beta\right)\nu_0\,\epsilon_0 + 4\,\nu_1\,\epsilon_1 + 3\,\sqrt{2}\,\nu_2\,\epsilon_2\right)H(t)\dot{H}(t)\\
&+ 6\left(\sqrt{2}\left(\alpha + 3\,\beta\right)\nu_0\,\epsilon_0 + \,\nu_1\,\epsilon_1 \,+ \sqrt{2}\,\nu_2\,\epsilon_2\right)\ddot{H}(t),
\end{aligned}$$

vanishes for $\nu_2 = -\alpha\nu_0\epsilon_0/\epsilon_2$ and $\nu_1 = -3\sqrt{2}\beta\nu_0\epsilon_0/\epsilon_1$, and so, the relation becomes explicitly, for all value of (α, β):

$$\sqrt{-\left[\Big(6\mathcal{L}_4 - 12\mathcal{L}_6 - \mathcal{L}_7\Big)\alpha\big(5\alpha + 18\beta\big) + 6\big(\alpha + 3\beta\big)\Big(\alpha\mathscr{L}_5 + \beta\mathscr{L}_8\Big)\right]}$$
$$- \frac{3\beta\epsilon_0\sqrt{2}}{\epsilon_1}\sqrt{-\mathscr{L}_8} - \frac{\epsilon_0\alpha}{\epsilon_2}\sqrt{-\Big(6\mathscr{L}_5 + 5\big(6\mathcal{L}_4 - 12\mathcal{L}_6 - \mathcal{L}_7\big)\Big)} = 0.$$

Therefore, this general formula for perfect squares depends only on $\sqrt{-J_{3,1}}$ and $\sqrt{-J_{3,2}}$, as we said.

We note here that an interesting property coming from the existence of an infinite number $J_3\big(\alpha, \beta\big)$ of perfect squares for which the square-root can be decomposed in a small basis is that it gives some non-linear algebraic relations between the scalars of the FKWC basis that reduce it in a non-trivial way and allow one to have a very small number of independent corrections. Indeed, solving the previous equation for $\mathcal{L}_4$, we find:

$$\mathcal{L}_4 = \frac{1}{54}\Big(- 9\mathscr{L}_5 + 2\mathscr{L}_8 + 108\mathcal{L}_6 + 9\mathcal{L}_7 - \sqrt{\mathscr{L}_8\big(18\mathscr{L}_5 - 5\mathscr{L}_8\big)}\epsilon_0\epsilon_1\Big). \tag{19}$$

We can now calculate the equations of motion for the three Lagrangian densities $\sqrt{-J_2}$, $\sqrt{-J_{3,1}}$ and $\sqrt{-J_{3,2}}$. First, one can check that the last one is in fact a topological term that does not bring any contribution to the equation of motion. Moreover, the first two Lagrangian densities give the same equation of motion, $54\,\dot{a}(t)\ddot{a}(t) = 0$ for the first one and $18\sqrt{2}\,\dot{a}(t)\ddot{a}(t) = 0$ for the second one. This means that they are equal up to an invariant scalar U for which $\sqrt{-gU}$ is a total derivative,

$$\sqrt{-\Big(2\mathcal{L}_4 + 12\mathcal{L}_6 - 7\mathcal{L}_7\Big)} = \frac{3}{\sqrt{2}}\sqrt{-\mathscr{L}_8} + U\ , \tag{20}$$

such that we can in fact consider a unique scalar (let us choose $\sqrt{-\mathscr{L}_8}$) made of order six scalars that leads to non-vanishing second order differential equations. We can note that U cannot be equal to $\sqrt{-J_{3,2}}$ because $\sqrt{-J_2}$ contains H^3 terms. Therefore, U could be found considering higher order derivative scalars and other perfect powers; in this case, powers of four for order 12 scalars, for example. There is then another non-linear relation between order six scalars and (possibly) order 12 ones.

Finally, to recapitulate this part, we have found that it is natural to consider only one perfect square made of order six scalars: $\mathscr{L}_8 = \nabla^\sigma R \nabla_\sigma R$. As a result, the action:

$$S_2 = \int d^4x\sqrt{-g}\,\left(\frac{1}{16\pi}\left[R + \nu\sqrt{-\nabla^\sigma R\nabla_\sigma R}\,\right] + \mathscr{L}_m\right), \tag{21}$$

leads to a unique second order H^3-correction to the Friedmann equation, as is easy to see using the same reasoning as in the previous section.

3.2. Order Eight

Now, let us study the linear combination and squares made of order eight scalars that lead to second order equations of motion. We do not copy all of the FKWC basis for the general metric, but we name the scalars according to their position in [18], where this basis is fully written. The reduced FKWC basis for order eight scalars in FLRW space-time is the following:

$$\begin{cases} \mathcal{K}_1 = R^4, \quad \mathcal{K}_{10} = R^{\mu\nu} R^{\alpha\beta} R^{\sigma\rho}{}_{\mu\alpha} R_{\sigma\rho\nu\beta}, \quad \mathcal{K}_{11} = R\, R^{\mu\nu\alpha\beta} R_{\mu\ \alpha}^{\ \sigma\ \rho} R_{\nu\sigma\beta\rho} = R\big(\tfrac{1}{4}\mathcal{L}_1 - \mathcal{L}_2\big) \\ \mathcal{K}_{12} = T^2, \quad \mathcal{M}_1 = R\,\Box^2 R, \quad \mathcal{M}_2 = R_{\mu\nu}\nabla^\mu\nabla^\nu\Box R, \quad \mathcal{M}_3 = R^{\mu\nu}\Box^2 R_{\mu\nu} \\ \mathcal{M}_5 = \nabla^\mu \Box R \nabla_\mu R, \;\; \mathcal{M}_6 = \nabla_\mu\nabla_\nu\nabla_\alpha R \nabla^\mu R^{\nu\alpha}, \;\; \mathcal{M}_{10} = \big(\Box R\big)^2, \;\; \mathcal{M}_{11} = \nabla_\mu\nabla_\nu R \nabla^\mu\nabla^\nu R \\ \mathcal{M}_{12} = \nabla^\mu\nabla^\nu R \Box R_{\mu\nu}, \quad \mathcal{M}_{14} = \nabla_\mu\nabla_\nu R_{\alpha\beta}\nabla^\mu\nabla^\nu R^{\alpha\beta}, \quad \mathcal{M}_{18} = R\,\mathscr{L}_1, \quad \mathcal{M}_{19} = R\,\mathscr{L}_4 \\ \mathcal{M}_{20} = S\Box R, \quad \mathcal{M}_{33} = R\,\mathscr{L}_8 \end{cases}$$

We also introduce the definitions $\mathcal{K}_9 = R^{\mu\nu} R^{\alpha}_{\ \mu} R_{\nu}^{\ \beta\sigma\rho} R_{\rho\sigma\beta\alpha}$, $\mathcal{M}_{13} = \Box R_{\mu\nu}\Box R^{\mu\nu}$ and $\mathcal{M}_{16} = \nabla_\mu\nabla_\nu R_{\alpha\beta}\nabla^\beta\nabla^\alpha R^{\nu\mu}$ that will be useful later for static spherically-symmetric space-times.

3.2.1. Linear Combination: H^8 Correction

Consider the sum of all independent order eight scalars for FLRW space-time:

$$J = \sum v_i \mathcal{K}_i + \sum x_j \mathcal{M}_j$$

Here, we follow exactly what we did for order six scalars. We derive the equation of motion associated with the previous sum and see what conditions on (v_i, x_j) cancel the equation, such that we find the 10 equivalence relations that exist between the scalars of the reduced basis. Therefore, we can consider only the following independent scalars with respect to the equation of motion, $\big(\mathcal{K}_1, \mathcal{K}_{10}, \mathcal{K}_{11}, \mathcal{K}_{12}, \mathcal{M}_1, \mathcal{M}_{11}, \mathcal{M}_{12}\big)$ and the reduced sum:

$$J = \sum_{i=1,10,11,12} v_i \mathcal{K}_i + \sum_{j=1,11,12} x_j \mathcal{M}_j \, .$$

We derive its associated equation of motion and see what a simultaneous cancellation of all of the higher order terms implies for the coefficients (v_i, x_j): we find that the unique linear combination of order eight scalars for FLRW space-time that leads to the second order equation is:

$$J_4 = \mathcal{K}_1 - 48\,\mathcal{K}_{11} - 9\,\mathcal{K}_{12} = 1728\left(H^8 + 2\dot{H}H^6 \right). \tag{22}$$

Therefore, one may consider the action:

$$S_3 = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}\left[R + \nu\Big(R^4 - 48\,R\,R^{\mu\nu\alpha\beta} R_{\mu\ \alpha}^{\ \sigma\ \rho} R_{\nu\sigma\beta\rho} - 9\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right)^2 \Big) \right] + \mathscr{L}_m \right), \tag{23}$$

and see that it brings an H^8 correction to the Friedmann equation. We note that this correction involves only contraction of curvature tensors, like for the order six case.

3.2.2. Non-Polynomial Gravity: H^4 Correction

Correction to the Einstein–Hilbert Action: To find non-polynomial second order models from order eight scalars, we follow exactly what we did for the order six and find the same kind of result: there are only two classes of perfect squares in this case. Those for which the square-roots give topological scalars, which does not give any contribution to the equation of motion, and a class of equivalent scalars with respect to the equation of motion, up to the topological scalars of the first class. Therefore, in this case also, we can consider a unique perfect square that contributes to the dynamics.

To begin, the more general square made of order eight scalars has the form:

$$\big(\alpha H(t)^4 + \beta H(t)^2\dot{H}(t) + \gamma\dot{H}(t)^2 + \delta H(t)\ddot{H}(t) + \sigma H^{(3)}(t)\big)^2.$$

Additionally, its square-root gives the following equation of motion:

$$\begin{aligned}&3\big(\alpha-\beta+\gamma+2\delta-6\sigma\big)\,\dot{a}(t)^2\left(\dot{a}(t)^2-4a(t)\ddot{a}(t)\right)\\&+\big(\gamma-\delta+3\sigma\big)\,a(t)^2\left(3\ddot{a}(t)^2+4\dot{a}(t)a^{(3)}(t)+2a(t)a^{(4)}(t)\right)=0.\end{aligned}$$

Following the section concerning order six, we can say that there are five independent contributions made of square-roots of order eight scalars, but in this case, there is also one condition to impose on $\big(\alpha,\beta,\gamma,\delta,\sigma\big)$ in order to have second order equations of motion. Therefore, there will be only four independent contributions, and we can choose them to be:

$$\begin{aligned}\sqrt{J_5}&=\sqrt{-38\mathcal{K}_1-2448\mathcal{K}_{10}+2400\mathcal{K}_{11}+1086\mathcal{K}_{12}-143\mathcal{M}_{10}-220\mathcal{M}_{11}+792\mathcal{M}_{12}+88\mathcal{M}_{18}-352\mathcal{M}_{19}}\\&=6\sqrt{33}\left(3H(t)\ddot{H}(t)+H^{(3)}(t)\right),\\\mathcal{E}_4&=\sqrt{\big(\mathcal{E}_4\big)^2}=\frac{1}{\sqrt{33}}\sqrt{\mathcal{K}_1-144\mathcal{K}_{10}+48\mathcal{K}_{11}+27\mathcal{K}_{12}}=24\,H(t)^2\left(H(t)^2+\dot{H}(t)\right),\\\Box R&=\sqrt{\mathcal{M}_{10}}=-6\left(12H(t)^2\dot{H}(t)+4\dot{H}(t)^2+7H(t)\ddot{H}(t)+H^{(3)}(t)\right),\\\sqrt{J_6}&=\sqrt{-\Big(5\mathcal{K}_1+9\big(8\mathcal{K}_{10}-32\mathcal{K}_{11}-7\mathcal{K}_{12}\big)\Big)}=12\sqrt{66}H(t)^2\dot{H}(t),\end{aligned}\tag{24}$$

where only the last one gives a non-vanishing contribution to the equations of motion. We note that, of course, because there are much more perfect squares than the previous ones, the fact that they form a basis gives once more some non-linear algebraic relations between the order eight scalars for FLRW space-time and reduces the already reduced FKWC basis. However, it is not the aim of this work to find all of these relations. We focus here on the unique modified action with respect to order eight scalars:

$$S_4=\int d^4x\sqrt{-g}\left(\frac{1}{16\pi}\left[R+\nu\sqrt{-5R^4-9\Big(8R^{\mu\nu}R^{\alpha\beta}R^{\sigma\rho}{}_{\mu\alpha}R_{\sigma\rho\nu\beta}-32R\,R^{\mu\nu\alpha\beta}R_{\mu}{}^{\sigma}{}_{\alpha}{}^{\rho}R_{\nu\sigma\beta\rho}-7\big(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\big)^2\Big)}\,\right]+\mathscr{L}_m\right).\tag{25}$$

Correction to the Friedmann Equation: From this action, we get the acceleration equation:

$$3H(t)^2+2\dot{H}(t)-6\nu\sqrt{66}H(t)^2\left(3H(t)^2+4\dot{H}(t)\right)=-8\pi p.\tag{26}$$

In analogy to Equations (12) to (14), this last modification leads to an H^4 modification of the Friedmann equation. Now, we are going to see that this unique correction is very interesting regarding the problem of the Big Bang singularity, as was shown in [17]. It is convenient to introduce the dimensional quantity ρ_c, critical energy density of the universe and the constant parameter ϵ, such that:

$$\frac{\epsilon}{8\pi\rho_c} = -2\nu\sqrt{66}\,. \tag{27}$$

Thus, the modified Friedmann equation induced by the action S_4 is:

$$3H(t)^2 + \epsilon\,\frac{9H(t)^4}{8\pi\rho_c} = 8\pi\rho. \tag{28}$$

The solution $H(t)^2$ that reduces to the standard Friedmann equation in the limit when ρ_c goes to infinity is:

$$3H(t)^2 = \frac{4\pi\rho_c}{\epsilon}\left(-1 + \sqrt{\frac{4\epsilon\rho}{\rho_c} + 1}\right). \tag{29}$$

Furthermore, if the energy density of the universe is small compared to the critical one, $\rho \ll \rho_c$, this equation becomes:

$$H(t)^2 = \frac{8\pi\rho}{3}\left(1 - \epsilon\,\frac{\rho}{\rho_c}\right) + O(\rho^3). \tag{30}$$

Choosing $\epsilon = 1$ gives the loop quantum cosmology correction to the Friedmann equation [20], and choosing $\epsilon = -\frac{1}{2}$ gives the Randall–Sundrum brane world model [21]. These two corrections are regular cosmological solutions and allow one to avoid the Big-Bang singularity [17,20]. One should also note that the above equation may be obtained within other approaches (see [22,23]).

4. Static Spherically-Symmetric Space-Times

In this section, we study static spherically-symmetric space-times, defined by the general metric:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2. \tag{31}$$

First, following exactly the same procedure as for the FLRW case, it is possible to show that there is no second order linear combination made of order six scalars. The same conclusion is valid for the classes $\mathcal{R}^0_{2,2}$ and $\mathcal{R}^0_{8,4}$.

Now, for order four scalars (and more generally, for all orders, considering only monomials of the curvature tensor), the result of Deser and Ryzhov [19] shows that the most general second order action is:

$$S_5 = \int d^4x\sqrt{-g}\left(R + \sqrt{3}\,\sigma\sqrt{W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta}}\right). \tag{32}$$

Moreover, there is no other perfect squares than $W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta}$ and R^2. Concerning order four, all of the scalars that are perfect squares lead to second order equations of motion inside the square.

Now, starting from order six scalars, it is possible to show that there are again only two perfect squares. Thus, we can consider the action:

$$S_6 = \int d^4x\sqrt{-g}\left(\delta\sqrt{\nabla_\sigma R \nabla^\sigma R} + \sqrt{3}\,\gamma\sqrt{C^{\mu\nu\alpha}C_{\mu\nu\alpha}}\right), \tag{33}$$

here $C_{\mu\nu\alpha}$ is the Cotton tensor, expressed in terms of the Weyl tensor as $C_{\mu\nu\alpha} = -2\nabla^\sigma W_{\mu\nu\alpha\sigma}$. Its square may be written in our basis as $C^{\mu\nu\alpha}C_{\mu\nu\alpha} = 2\big(\mathscr{L}_5 - \mathscr{L}_6\big) - \frac{1}{6}\mathscr{L}_8$.

In our search for second order differential equations, it is interesting to note that the property of order four perfect squares is preserved here: these two terms are such that their higher order terms cancel perfectly, which makes their associated equations of motion second order. For example, the term:

$$\begin{aligned}\sqrt{-g}\,\sqrt{C^{\mu\nu\alpha}C_{\mu\nu\alpha}} = \frac{\sqrt{3}}{6}\frac{\sqrt{B(r)}}{rB(r)^3A(r)^3}\Bigg(&\Sigma\Big(r, B(r), A(r), B'(r), A'(r), B''(r), A''(r)\Big)\\ &- 3r^3B(r)^2A(r)A'(r)B''(r) - r^3B(r)^2A(r)B'(r)A''(r) + 2r^3B(r)^2A(r)^2B^{(3)}(r)\Bigg),\end{aligned} \tag{34}$$

where $\Sigma\Big(r, B(r), A(r), B'(r), A'(r), B''(r), A''(r)\Big)$ is a sum of 15 first order terms (that lead trivially to second order differential equations), is equivalent, up to boundary terms, to the following first order expression:

$$\begin{aligned}\sqrt{-g}\,\sqrt{C^{\mu\nu\alpha}C_{\mu\nu\alpha}} \equiv \frac{\sqrt{3}}{6}\frac{\sqrt{B(r)}}{rB(r)^3A(r)^3}\Bigg(&4B(r)^3A(r)^2 - 4B(r)^3A(r)^3 - 5rB(r)^2A(r)^2B'(r)\\ &+ 2r^2B(r)A(r)^2B'(r)^2 - \frac{1}{4}r^3A(r)^2B'(r)^3\Bigg).\end{aligned} \tag{35}$$

Therefore, we have shown first that, up to order six, all of the perfect squares that one can build for static spherically-symmetric space-times are also perfect squares in FLRW, as we have seen for the case of $\nabla_\sigma R\nabla^\sigma R$, because in this space-time, $W_{\mu\nu\alpha\sigma} = 0$; and secondly, that they also share the property that their square-root leads to second order equations of motion for the metric field.

These equations, coming from the action S_6 for $A(r)$ and $B(r)$, are respectively:

$$\begin{aligned}&16\big(\gamma + 2\delta\big)B(r)^3 + 4r\big(-5\gamma + 2\delta\big)B(r)^2B'(r)\\ &\quad + 4r^2\big(2\gamma + \delta\big)B(r)B'(r)^2 + r^3\big(-\gamma + \delta\big)B'(r)^3 = 0,\end{aligned} \tag{36}$$

and,

$$\begin{aligned}&8\big(\gamma + 2\delta\big)\Big(-1 + A(r)\Big)A(r)B(r)^3 + 4r\big(5\gamma - 2\delta\big)B(r)^3A'(r) - 18r^2\gamma A(r)B(r)B'(r)^2\\ &\quad + 8r\big(2\gamma + \delta\big)B(r)^2\Big(-rA'(r)B'(r) + A(r)\left(B'(r) + rB''(r)\right)\Big)\\ &\quad + r^3\big(\gamma - \delta\big)B'(r)\Big(5A(r)B'(r)^2 + 3B(r)\left(A'(r)B'(r) - 2A(r)B''(r)\right)\Big) = 0.\end{aligned} \tag{37}$$

The first one provides solutions for $B(r)$, so the two equations decouple. We have found real exact vacuum solutions for three couples $(\gamma\,,\,\delta)$ and have computed their associated Kretschmann scalars $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ to see if they suffer from singularities at $r = 0$.

- For $\gamma = 1$ and $\delta = 0$:

$$ds^2 = -k\,r^2\,dt^2 + \frac{r^2}{p + r^2}\,dr^2 + r^2d\Omega^2 \qquad \text{and} \qquad R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \to \frac{p^2}{r^8}, \tag{38}$$

- For $\gamma = 0$ and $\delta = 1$:

$$ds^2 = -\frac{k}{r^4}\, dt^2 + \frac{3}{1+p\,r^2}\, dr^2 + r^2 d\Omega^2 \qquad \text{and} \qquad R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \to \frac{15p^2-2p+3}{r^4}, \tag{39}$$

- For $\gamma = 1$ and $\delta = -\frac{1}{2}$:

$$ds^2 = -k\, dt^2 + p\, dr^2 + r^2 d\Omega^2 \qquad \text{and} \qquad R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \to \frac{(p-1)^2}{p^2\, r^4}, \tag{40}$$

where p and k are integration constants. Note that k can be set equal to one by the right choice of the time coordinate.

Recall that the Kretschmann scalar of the Schwarzschild solution of the Einstein equations diverges as $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \to 1/r^6$. Therefore, we see that our first solution has a worse divergence. In the last case, choosing the initial condition $p = 1$ provides $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = 0$ for all r, so there is no singularity of curvature, because the space-time is flat. However, in the second case, as for p real, $15p^2 - 2p + 3 > 0$, we cannot "cancel" the singularity by a proper choice of the initial condition, but still, the divergence is milder than the standard one.

Now, let us note that considering only the two order eight classes $\mathcal{R}^0_{2,2}$ and $\mathcal{R}^0_{8,4}$, we have found a unique perfect square $\Big[2K_{10} - 2K_9 - M_{11} + 2M_{12} - M_{13} + 4M_{14} - 4M_{16}\Big]$ (that is zero in the FLRW case, so possibly again related to Cotton and Weyl "conformal" tensors), but its square-root does not lead to second order equations of motion, such that we can conjecture that this property is true only for the three scalars $W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta}$, $C^{\mu\nu\alpha}C_{\mu\nu\alpha}$ and $\nabla_\sigma R \nabla^\sigma R$, without counting the scalars R^2, $(\Box^i R)^2$, $(\mathcal{E}_4)^2$, *etc.*, for which the square-root always leads to second order equations.

To conclude this section, we point out that it would be interesting to find black hole solutions, for some values of σ, γ and δ, and starting from the following action:

$$S_7 = \int d^4x\sqrt{-g}\,\Big(R + \sqrt{3}\,\sigma\,\sqrt{W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta}} + \sqrt{3}\,\gamma\,\sqrt{C^{\mu\nu\alpha}C_{\mu\nu\alpha}} + \delta\,\sqrt{\nabla_\sigma R\nabla^\sigma R}\,\Big), \tag{41}$$

where the two last terms would be considered as high energy corrections to the standard Einstein–Hilbert action. In this action, the last two terms are very similar to the first two, present in S_5. And we recall that starting from S_5, one can find exact black hole solutions [16].

After finding black hole solutions for some class of the parameters, it would be possible to reduce the number of physically-relevant values of $(\sigma,\, \gamma,\, \delta)$, for example by calculating the Wald entropy associated with the action and the class of solutions, as is done in [24], by imposing the positivity of the entropy to find some constraint on the parameters.

5. Conclusions

Considering the order six and eight FKWC basis of all independent invariant scalars built from the metric field and its derivatives, we have found modified gravity models admitting second order equations of motion for FLRW and static spherically-symmetric space-times.

For FLRW, we have shown that there are unique second order polynomial corrections to the Einstein–Hilbert action for both order six and eight. They involve only contractions of curvature tensors, which is what one could expect to follow from quantum field theory. Yet, concerning static spherical symmetry, we have seen that there is none.

Some non-polynomial corrections, which are square-roots of the specific combinations that are perfect squares, have also been studied for FLRW and spherical symmetry for both order six and eight. We have shown that up to order six, all of the perfect squares lead to second order equations of motion in this way. However, for order eight, there are squares that do not have this property, for example the one we have found for spherical symmetry that involves scalars of the $\mathcal{R}^0_{2,2}$ and $\mathcal{R}^0_{8,4}$ order eight classes.

Moreover, concerning FLRW, we have seen a "mechanism" that provides unique corrections with non-vanishing contributions to the equations of motion: there are lots of squares, but their square-roots can be decomposed into a very small basis, such that there exist some non-linear algebraic relations between order six scalars and order eight ones. Taking this into account, we can choose the independent "square-root" scalars, such that for both orders, there are topological scalars and only one that is not.

For order six, the one we chose turned out to be the same as for spherical symmetry. The other perfect squares for this last space-time are squares of the Weyl and Cotton tensors, which are identically zeros for FLRW, such that all of the squares of static spherical symmetry are present in FLRW. Let us say also that the resemblance between S_6 and S_5, which admits exact black hole solutions, could suggest that it might be possible to find also these kinds of solutions from the first. It could also be interesting to understand better why the second order corrections specific to spherical symmetry involve "conformal" tensors.

We note here that in addition, we have checked if this common non-vanishing square, that is $\nabla_\sigma R \nabla^\sigma R$, is also a square for another physically-relevant spacetime, that is Bianchi I, defined by the following metric and describing an anisotropic universe as the early universe was [29,30]:

$$ds^2 = -dt^2 + \alpha(t)dx^2 + \beta(t)dy^2 + \gamma(t)dz^2. \tag{42}$$

It turns out that this scalar is also a perfect square in this case and that it leads also to second order equations of motion. These results are just copied in the Appendix A.3 and make the results concerning the action:

$$S = \int d^4x\sqrt{-g}\left(R + \nu\sqrt{-\nabla^\sigma R \nabla_\sigma R}\right), \tag{43}$$

our main ones.

The study of non-polynomial gravity with order eight scalars represents also an interesting result in the sense that we can choose the four independent square-root scalars to be three topological ones and one that contributes to the equations of motion and gives the well-known H^4 correction to the Friedmann equation [27,28], which leads to a unique physically-relevant solution for which the limit of small density allows one to reproduce the loop quantum cosmology result and the one coming from the Randall–Sundrum brane world model. In these two, the big-bang singularity is absent, and the cosmological solutions are regular ones.

To finish, we note that our study was done on fixed backgrounds, but general relativity is a background-independent theory, so it could be important to check for the general background if the scalar $\nabla_\sigma R \nabla^\sigma R$ is still a perfect square and, if that is the case, if its square-root leads in the general case to second order equations of motion for the metric field. Moreover, to our knowledge, there is no proof that, considering all of the FKWC bases for all orders, it is impossible to find linear combinations of invariant scalars that lead to second order partial differential equations for a general metric. Indeed, the Lovelock papers [11,31,32] have only restricted their studies to Lagrangians of the general form: $L\Big(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}\Big)$, *i.e.*, involving curvature tensors, but not explicit covariant derivatives of them. Therefore, if this remark is true, one could search for second order equations of motion coming from the general Lagrangian $L\Big(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, ..., \partial_\alpha ... \partial_\beta g_{\mu\nu}\Big)$. We know from our results that there is no such combination for order six and eight scalars, because there are none for static spherically-symmetric space-time. However, as the order grows, even if there are more and more terms in the expansions of the scalars, the number of independent scalars inside the same class grows, as well, such that for some "high" order in the FKWC basis, it might be possible to cancel all of the higher order derivatives for a general metric.

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Author Contributions

Sergio Zerbini designed the research, and made contributions to all calculations. Aimeric Colléaux made large part of calculations. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

A. Appendix

A.1. Relations for the Reduced FKWC Basis

First, we copy the following relations from [25]:

$$\mathcal{E}_6 = 2\mathcal{L}_1 + 8\mathcal{L}_2 + 24\mathcal{L}_3 + 2\mathcal{L}_4 + 24\mathcal{L}_5 + 16\mathcal{L}_6 - 12\mathcal{L}_7 + \mathcal{L}_8 \tag{44}$$

is the order six Euler density,

$$W^2 = W_{\mu\nu\alpha\beta} W^{\mu\nu\alpha\beta} = \frac{1}{3} R^2 - 2S + T \tag{45}$$

is the square of the Weyl tensor,

$$W_1^3 = W^{\mu\nu\alpha\beta} W_{\alpha\beta\sigma\rho} W^{\sigma\rho}{}_{\mu\nu} \tag{46}$$

and:

$$W_2^3 = W^{\mu\nu\alpha\beta} W_{\mu\sigma\rho\beta} W^{\sigma}{}_{\nu\alpha}{}^{\rho} \tag{47}$$

are the two independent cubic contractions of the Weyl tensor in four dimensions.

From the relations of [26], we have found the following geometrical identities:

$$L_{\mu\nu}R^{\mu\nu} = -(2\mathcal{L}_3 + \frac{1}{2}\mathcal{L}_4 + 4\mathcal{L}_5 + 4\mathcal{L}_6 - 4\mathcal{L}_7 + \frac{1}{2}\mathcal{L}_8) \tag{48}$$

$$W_{\mu\nu\alpha\beta}\nabla^\nu\nabla^\beta R^{\mu\alpha} = \mathscr{L}_3 - \frac{1}{2}\mathscr{L}_2 + \frac{1}{12}\mathscr{L}_1 + \frac{1}{2}\mathcal{L}_6 - \frac{1}{2}\mathcal{L}_5 \tag{49}$$

$$\nabla^\rho R^{\mu\nu\alpha\beta}\nabla_\rho W_{\mu\nu\alpha\beta} = \mathscr{L}_7 - 2\mathscr{L}_5 + \frac{1}{3}\mathscr{L}_8 \tag{50}$$

$$R^{\mu\alpha}\nabla^\beta\nabla^\nu W_{\mu\nu\alpha\beta} = \frac{1}{2}\mathscr{L}_2 - \frac{1}{12}\mathscr{L}_1 - \frac{1}{6}\mathscr{L}_4 - \frac{1}{2}L_6 + \frac{1}{2}L_5 \tag{51}$$

$$\nabla^\mu W_{\mu\nu\alpha\beta}\nabla^\alpha R^{\nu\beta} = \frac{1}{2}\mathscr{L}_5 - \frac{1}{2}\mathscr{L}_6 - \frac{1}{24}\mathscr{L}_8 \tag{52}$$

Note that $\nabla^\mu\nabla^\nu L_{\mu\nu} = 0$ identically, such that there are no new relations coming from this term.

Because in four dimensions, $\mathcal{E}_6 = 0$ and $L_{\mu\nu} = 0$, leading to $L_{\mu\nu}R^{\mu\nu} = 0$ for order six scalars, the FKWC basis, which does not take into account these relations, is then reduced.

As for conformally-invariant space-times, their metrics verify $W_{\mu\nu\alpha\beta} = 0$, which provides again new relations between the scalars coming from:

$RW^2 = 0$, $W_1^3 = 0$, $W_2^3 = 0$ (note that because $\mathcal{E}_6$ is a linear combination of W_1^3 and W_2^3 [25], the relation $\mathcal{E}_6 = 0$ becomes redundant), $W_{\mu\nu\alpha\beta}\nabla^\nu\nabla^\beta R^{\mu\alpha} = 0$, $\nabla^\rho R^{\mu\nu\alpha\beta}\nabla_\rho W_{\mu\nu\alpha\beta} = 0$, $R^{\mu\alpha}\nabla^\beta\nabla^\nu W_{\mu\nu\alpha\beta} = 0$ and $\nabla^\mu W_{\mu\nu\alpha\beta}\nabla^\alpha R^{\nu\beta} = 0$. Therefore, for conformally-invariant space-times, there is one relation coming from the corollary of the Lovelock theorem and seven coming from $W_{\mu\nu\alpha\beta} = 0$, so that there are eight scalars less in the reduced basis, *i.e.*, eight scalars left.

A.2. General Order Six Linear Combination for FLRW

The sum of all order six independent scalars, expressed in terms of $a(t)$ and its derivatives, is:

$$\begin{aligned} J =& \sum\left(v_i\mathcal{L}_i + x_i\mathscr{L}_i\right) \\ =& \frac{3}{a(t)^6}\Big(\sigma_1(v_i,x_i)\,\dot{a}(t)^6 + \sigma_2(v_i,x_i)\,a(t)\,\dot{a}(t)^4\,\ddot{a}(t) + \sigma_3(v_i,x_i)\,a(t)^2\,\dot{a}(t)^2\,\ddot{a}(t)^2 \\ &+ \sigma_4(v_i,x_i)\,a(t)^3\,\ddot{a}(t)^3 + \sigma_5(v_i,x_i)\,a(t)^2\,\dot{a}(t)^3\,a^{(3)}(t) + \sigma_6(v_i,x_i)\,a(t)^3\,\dot{a}(t)\,\ddot{a}(t)\,a^{(3)}(t) \\ &+ \sigma_7(v_i,x_i)\,a(t)^4\,a^{(3)}(t)^2 + \sigma_8(v_i,x_i)\,a(t)^3\,\dot{a}(t)^2\,a^{(4)}(t) + \sigma_9(v_i,x_i)\,a(t)^4\,\ddot{a}(t)\,a^{(4)}(t)\Big) \end{aligned}$$

with,

$$\begin{cases} \sigma_1 = 4(2v_1 - 2v_3 + 6v_4 + 2v_5 + 2v_6 + 6v_7 + 18v_8 + 2x_2 + x_3 + 6x_4 \\ \qquad -6x_5 - 5x_6 - 8x_7 - 12x_8) \\ \sigma_2 = 2\big(- 2v_3 + 12v_4 + 4v_5 + 6v_6 + 24v_7 + 108v_8 + 30x_1 + x_2 \\ \qquad -3x_3 - 18x_4 + 20x_5 + 18x_6 + 32x_7 + 24x_8\big) \\ \sigma_3 = \big[- 6v_2 - 4v_3 + 24v_4 + 14v_5 + 6v_6 + 48v_7 + 216v_8 + 48x_1 \\ \qquad +14x_2 + 7x_3 + 42x_4 - 20x_5 - 19x_6 - 36x_7 - 12x_8\big] \\ \sigma_4 = 8v_1 + 2v_2 - 8v_3 + 24v_4 + 6v_5 + 10v_6 + 24v_7 + 72v_8 - 12x_1 - x_3 - 6x_4 \\ \sigma_5 = -2\left(18x_1 + 5x_2 + x_3 + 6x_4 - 4x_5 - 2x_6 - 24x_8\right) \\ \sigma_6 = -\left(36x_1 + 8x_2 + x_3 + 6x_4 - 2x_6 - 8x_7 + 24x_8\right) \\ \sigma_7 = -\left(4x_5 + 3x_6 + 4\left(x_7 + 3x_8\right)\right) \\ \sigma_8 = -2\left(6x_1 + x_2\right) \\ \sigma_9 = -\left(12x_1 + 4x_2 + x_3 + 6x_4\right) \end{cases}$$

Now, in terms of $H(t)$ and its derivatives, considering only the scalars of the reduced FKWC basis:

$$\begin{aligned} J = &\sum_{i=4,6,7} v_i \mathcal{L}_i + \sum_{j=1,3,5,8} x_j \mathscr{L}_j \\ =&3\Bigg(\widetilde{\sigma}_1(v_i,x_i)\ H(t)^6 + \widetilde{\sigma}_2(v_i,x_i)\ H(t)^4\dot{H}(t) + \widetilde{\sigma}_3(v_i,x_i)\ H(t)^2\dot{H}(t)^2 \\ &+ \widetilde{\sigma}_4(v_i,x_i)\ \dot{H}(t)^3 + \widetilde{\sigma}_5(v_i,x_i)\ H(t)^3\ddot{H}(t) + \widetilde{\sigma}_6(v_i,x_i)\ H(t)\dot{H}(t)\ddot{H}(t) \\ &+ \widetilde{\sigma}_7(v_i,x_i)\ddot{H}(t)^2 + \widetilde{\sigma}_8(v_i,x_i)\ H(t)^2H^{(3)}(t) + \widetilde{\sigma}_9(v_i,x_i)\ \dot{H}(t)H^{(3)}(t)\Bigg) \end{aligned}$$

with,

$$\begin{cases} \widetilde{\sigma}_1 = 12\left(8v_4 + 3\left(v_6 + 4v_7\right)\right) \\ \widetilde{\sigma}_2 = 6\left(24v_4 + 9v_6 + 36v_7 - 48x_1 - 2x_3\right) \\ \widetilde{\sigma}_3 = 4\left(24v_4 + 9v_6 + 30v_7 - 60x_1 - 2x_3 - 14x_5 - 48x_8\right) \\ \widetilde{\sigma}_4 = 2\left(12v_4 + 5v_6 + 12v_7 - 24x_1 - 2x_3\right) \\ \widetilde{\sigma}_5 = -7\left(24x_1 + x_3\right) \\ \widetilde{\sigma}_6 = -\left(84x_1 + 5x_3 + 24\left(x_5 + 4x_8\right)\right) \\ \widetilde{\sigma}_7 = -4\left(x_5 + 3x_8\right) \\ \widetilde{\sigma}_8 = -\left(24x_1 + x_3\right) \\ \widetilde{\sigma}_9 = -\left(12x_1 + x_3\right) \end{cases}$$

A.3. $\nabla_\sigma R \nabla^\sigma R$ in Bianchi I Spacetime

We reproduce only here the value of the particular scalar $\nabla_\sigma R \nabla^\sigma R$ for Bianchi I spacetime and the associated equations of motion of its square-root.

$$\begin{aligned}\nabla_\sigma R \nabla^\sigma R = &-4\Bigg\{\alpha(t)\gamma(t)\dot\beta(t)\Big(\alpha(t)\dot\beta(t)\dot\gamma(t)+\gamma(t)\big(\dot\alpha(t)\dot\beta(t)+\alpha(t)\ddot\beta(t)\big)\Big)\\&+\beta(t)\Big(\alpha(t)^2\dot\beta(t)\dot\gamma(t)^2-\alpha(t)^2\gamma(t)\big(\dot\gamma(t)\ddot\beta(t)+\dot\beta(t)\ddot\gamma(t)\big)\Big)\\&+\gamma(t)^2\Big(\dot\alpha(t)^2\dot\beta(t)-\alpha(t)\dot\alpha(t)\ddot\beta(t)-\alpha(t)\big(\dot\beta(t)\ddot\alpha(t)+\alpha(t)\beta^{(3)}(t)\big)\big)\Big)\\&+\beta(t)^2\Bigg[\alpha(t)\dot\gamma(t)\Big(\dot\alpha(t)\dot\gamma(t)+\alpha(t)\ddot\gamma(t)\Big)+\gamma(t)^2\Big(\dot\alpha(t)\ddot\alpha(t)-\alpha(t)\alpha^{(3)}(t)\Big)\\&+\gamma(t)\Big(\dot\alpha(t)^2\dot\gamma(t)-\alpha(t)\dot\alpha(t)\ddot\gamma(t)-\alpha(t)\Big(\dot\gamma(t)\ddot\alpha(t)+\alpha(t)\gamma^{(3)}(t)\Big)\Big)\Bigg]\Bigg\}^2\Bigg/\Bigg\{\alpha(t)^4\beta(t)^4\gamma(t)^4\Bigg\}\end{aligned}\tag{53}$$

The associated equations of motion, which are very complicated, yet second order, for respectively $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are:

$$\begin{aligned}0 = &-\alpha(t)^3\gamma(t)^3\dot\beta(t)^3+\alpha(t)^3\beta(t)\gamma(t)^3\dot\beta(t)\ddot\beta(t)\\&+\alpha(t)\beta(t)^2\gamma(t)^2\Bigg[2\alpha(t)\dot\alpha(t)\dot\beta(t)\dot\gamma(t)+\gamma(t)\Big(\dot\alpha(t)^2\dot\beta(t)+\alpha(t)\dot\beta(t)\ddot\alpha(t)+\alpha(t)\dot\alpha(t)\ddot\beta(t)\Big)\Bigg]\\&+\beta(t)^3\Bigg[-\alpha(t)^3\dot\gamma(t)^3+\gamma(t)^3\Big(-2\dot\alpha(t)^3+3\alpha(t)\dot\alpha(t)\ddot\alpha(t)\Big)+\alpha(t)^3\gamma(t)\dot\gamma(t)\ddot\gamma(t)\\&+\alpha(t)\gamma(t)^2\Big(\dot\alpha(t)^2\dot\gamma(t)+\alpha(t)\dot\gamma(t)\ddot\alpha(t)+\alpha(t)\dot\alpha(t)\ddot\gamma(t)\Big)\Bigg]\end{aligned}\tag{54}$$

$$\begin{aligned}0 = &-2\alpha(t)^3\gamma(t)^3\dot\beta(t)^3+\alpha(t)^2\beta(t)\gamma(t)^2\dot\beta(t)\Bigg[\alpha(t)\dot\beta(t)\dot\gamma(t)+\gamma(t)\Big(\dot\alpha(t)\dot\beta(t)+3\alpha(t)\ddot\beta(t)\Big)\Bigg]\\&+\alpha(t)^2\beta(t)^2\gamma(t)^2\Bigg[\gamma(t)\dot\beta(t)\ddot\alpha(t)+\alpha(t)\dot\gamma(t)\ddot\beta(t)+\dot\alpha(t)\Big(2\dot\beta(t)\dot\gamma(t)+\gamma(t)\ddot\beta(t)\Big)+\alpha(t)\dot\beta(t)\ddot\gamma(t)\Bigg]\\&+\beta(t)^3\Bigg[-\alpha(t)^3\dot\gamma(t)^3+\gamma(t)^3\Big(-\dot\alpha(t)^3+\alpha(t)\dot\alpha(t)\ddot\alpha(t)\Big)+\alpha(t)^3\gamma(t)\dot\gamma(t)\ddot\gamma(t)\Bigg]\end{aligned}\tag{55}$$

$$\begin{aligned}0 = &-\alpha(t)^3\gamma(t)^3\dot\beta(t)^3+\alpha(t)^3\beta(t)\gamma(t)^3\dot\beta(t)\ddot\beta(t)\\&+\beta(t)^3\Bigg[-2\alpha(t)^3\dot\gamma(t)^3+\gamma(t)^3\Big(-\dot\alpha(t)^3+\alpha(t)\dot\alpha(t)\ddot\alpha(t)\Big)\\&+\alpha(t)^2\gamma(t)\dot\gamma(t)\Big(\dot\alpha(t)\dot\gamma(t)+3\alpha(t)\ddot\gamma(t)\Big)+\alpha(t)^2\gamma(t)^2\Big(\dot\gamma(t)\ddot\alpha(t)+\dot\alpha(t)\ddot\gamma(t)\Big)\Bigg]\\&+\alpha(t)^2\beta(t)^2\gamma(t)\Bigg[\alpha(t)\dot\beta(t)\dot\gamma(t)^2+\gamma(t)\Big(2\dot\alpha(t)\dot\beta(t)\dot\gamma(t)+\alpha(t)\dot\gamma(t)\ddot\beta(t)+\alpha(t)\dot\beta(t)\ddot\gamma(t)\Big)\Bigg]\end{aligned}\tag{56}$$

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