Extended First Law for Entanglement Entropy in Lovelock Gravity

David Kastor [1,*], Sourya Ray [2] and Jennie Traschen [1]

[1]   Amherst Center for Fundamental Interactions, Department of Physics, University of Massachusetts, Amherst, MA 01003, USA; traschen@physics.umass.edu

[2]   Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile; ray@uach.cl

[*]   Correspondence: kastor@physics.umass.edu; Tel.: +1-413-545-0545

Abstract: The first law for the holographic entanglement entropy of spheres in a boundary CFT (Conformal Field Theory) with a bulk Lovelock dual is extended to include variations of the bulk Lovelock coupling constants. Such variations in the bulk correspond to perturbations within a family of boundary CFTs. The new contribution to the first law is found to be the product of the variation δa of the "A"-type trace anomaly coefficient for even dimensional CFTs, or more generally its extension δa^* to include odd dimensional boundaries, times the ratio S/a^*. Since a^* is a measure of the number of degrees of freedom N per unit volume of the boundary CFT, this new term has the form $\mu \delta N$, where the chemical potential μ is given by the entanglement entropy per degree of freedom.

1. Introduction

The AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence [1] has been most extensively studied for CFTs that have bulk Einstein duals. However, this does not include the most general CFTs of interest. In four dimensions, for example, the trace anomaly for a general CFT is given by

$$\langle T_a{}^a \rangle = \frac{c}{16\pi^2} C_{abcd} C^{abcd} - \frac{a}{16\pi^2} (R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2) \tag{1}$$

while the holographic calculation of the trace anomaly with Einstein gravity in the bulk [2] yields only the special case $a = c$. Studies including higher curvature interactions in the bulk, which allow for more general boundary CFTs, often focus on Lovelock gravity theories [3], which are better behaved than generic higher curvature theories having, for example, field equations that depend only on the Riemann tensor and not its derivatives (see, for example, the studies of causality constraints on the bulk higher curvature duals of CFTs [4–13]).

Interest in CFTs with bulk Lovelock duals extends to holographic computations of entanglement entropy [14–23]. For theories with bulk Einstein duals, Ryu and Takayanagi [24] proposed that the entanglement entropy S_E associated with the division of the boundary into complementary regions A and B is given by the Bekenstein–Hawking entropy formula

$$S_E = \frac{A_\Sigma}{4G} \tag{2}$$

where A_Σ is the area of a bulk minimal surface Σ that is homologous to the boundary between A and B. For CFTs with bulk Lovelock duals, it has been conjectured [15] that the entanglement entropy will be similarly given by

$$S_E = S_L \tag{3}$$

where S_L denotes the formula for horizon entropy in Lovelock gravity found by Jacobson and Myers [25] evaluated for a surface Σ, homologous to the boundary between the regions *A* and *B*, that minimizes S_L. We will assume that this equality holds below and simply denote the entanglement entropy by *S* in the following. Note that the Wald entropy formula [26,27] for Lovelock gravity differs from the Jacobson–Myers formula by extrinsic curvature terms that vanish for the bifurcation surface of a Killing horizon, but not necessarily for a bulk entangling surface. Note also that the area Equation (2) in the Ryu–Takayanagi proposal is infinite, due to the divergence in the area element in the AdS metric. Of course the entanglement entropy of the boundary region in the dual CFT is also infinite, and much work has been done in demonstrating that the divergences match both for the leading (uninteresting) terms and the (interesting, universal) logarithmic terms. The expressions are rendered finite by a short-distance cutoff. We will omit subscripts such as "renormalized" since there will be sufficient other subscripts.

Entanglement entropy is not a thermal phenomenon. However, it has been shown to obey a first law with respect to variations in the quantum state of the CFT [28,29]. For a spherical entangling surface on the boundary, this first law follows from the bulk gravitational first law associated with the entangling surface Σ [30]. This works because the bulk surface Σ, in this case, turns out to be the bifurcation surface of a Killing horizon. The proof of the first law for stationary black holes [31] then applies in this non-black hole setting as well.

In reference [32], we used the bulk methods of [30] to prove an extension of the entanglement first law [28,29] for CFTs with a bulk Einstein dual, that gives the variation in entanglement entropy with respect to variation in the number of CFT degrees of freedom. As in [30], our construction applies only to spherical entangling surfaces on the boundary, but may well hold more generally. This result relies on the earlier generalization of the bulk first law to include variations in the cosmological constant Λ [33]. For static black holes, this latter result has the form

$$\delta M = \frac{\kappa \delta A}{8\pi G} - \frac{V \delta \Lambda}{8\pi G} \tag{4}$$

where *M* is the ADM mass, *A* the horizon area, κ the surface gravity. The quantity *V* is the "thermodynamic volume" of the black hole, which is conjugate to the cosmological constant Λ, which can be regarded as a pressure. The additional term in the first law then has a $V\delta P$ form. Including this additional part of the black hole phase space has led to a rich phenomenology of phase transitions, see [34] for a review of progress in this area. In the AdS/CFT correspondence, with Einstein gravity in the bulk, the cosmological constant is a measure of the number of CFT degrees of freedom. For example, for AdS_5/CFT_4 [1] the number of degrees of freedom per unit volume of the boundary $\mathcal{N} = 4\, SU(N)$ gauge theory scales as N^2 and is related to the bulk cosmological constant by

$$N^2 = \frac{\pi}{2G_5} \left(-\frac{6}{\Lambda} \right)^{\frac{3}{2}} \tag{5}$$

Similarly in AdS_3/CFT_2, the CFT central charge *c* is a measure of the number of degrees of freedom and is given [35] in terms of the bulk cosmological constant by

$$c = \frac{3}{2G_3 \sqrt{-\Lambda}} \tag{6}$$

Varying the bulk cosmological constant corresponds to varying the number of boundary degrees of freedom, and this forms the basis for the extension of the first law of entanglement entropy in [32]. The quantity conjugate to varying the number of boundary degrees of freedom has a natural interpretation as a chemical potential.

The connection between the bulk gravitational first law and the first law for entanglement entropy, for spherical entangling surfaces, also holds in Lovelock gravity [30]. An extension of the bulk first

law for black holes to include variations in the Lovelock couplings was proved in [36]. In addition to the thermodynamic volume, there are now thermodynamic potentials conjugate to each of the higher curvature couplings. The purpose of this paper is to apply this extended bulk first law to derive an extension of the first law for entanglement entropy in this theory, for the case of spherical entangling surfaces. Variation in the bulk Lovelock coefficients corresponds to variation within a family of boundary CFTs (it is important to note that such a family of CFTs is not a moduli space. Variation between nearby CFTs in a family does not correspond to perturbing by a marginal operator, which would leave the central charges unchanged. Rather one should have in mind the example given above of the family of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang–Mills theory. These form a discrete 1-parameter family of CFTs with different central charges. In the limit of large N, where the bulk dual description is valid, the relevant normalized variation $\delta N / N$ between neighboring theories becomes small). We find that at fixed energy, the variation in CFT entanglement entropy with respect to the bulk Lovelock coupling constants assembles into the simple form

$$\delta S = S \frac{\delta a^*}{a^*} \tag{7}$$

where the quantity a^* can alternatively be viewed as a function of the Lovelock couplings or in terms of its significance in the corresponding boundary CFT. For even dimensional CFTs, a^* is the suitably normalized coefficient of the Euler density term in the trace anomaly, expressed in terms of the bulk Lovelock couplings [37]. This expression can then be continued to odd dimensions as well. It has been argued that a^* is proportional to the density of degrees of freedom in the CFT [17,38], and so the effect of including the variation of the Lovelock couplings is to give a work term in the first law accounting for the change in the number of degrees of freedom, with chemical potential proportional to S/a^*, which can be interpreted as the entanglement entropy per degree of freedom.

This paper is organized as follows. In Section 2 we present the first law in terms of the entanglement entropy and anomaly coefficient. In Section 3 we give the derivation of the first law in terms of the area and Lovelock coefficients, which is the basis for the results of Section 2. In Section 4 we offer some concluding remarks.

2. Extended First Law for Entanglement Entropy

We consider entanglement entropy in CFTs with bulk Lovelock gravity duals. The Lagrangian for Lovelock gravity in D spacetime dimensions is given by

$$\mathcal{L} = \frac{1}{16\pi G} \sum_{k=0}^{k_{max}} b_k \mathcal{L}^{(k)} \tag{8}$$

where $k_{max} = [(D-1)/2]$ and the b_k are real-valued coupling constants. The symbol $\mathcal{L}^{(k)}$ stands for the contraction of k powers of the Riemann tensor given by

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \dots R_{a_k b_k}{}^{c_k d_k} \tag{9}$$

Here the δ-symbol is the totally anti-symmetrized product normalized so that it takes nonzero values ± 1. The term $\mathcal{L}^{(0)}$ gives the cosmological constant term in the Lagrangian, while $\mathcal{L}^{(1)}$ gives the Einstein-Hilbert term and $\mathcal{L}^{(2)}$ the quadratic Gauss-Bonnet term. The upper bound in the sum Equation (8) comes about because $\mathcal{L}^{(k)}$ vanishes identically for $D < 2k$ and turns out to make no contribution to the equations of motion in $D = 2k$. We will fix $b_1 = 1$ and note that $b_0 = -2\Lambda$, where Λ is the cosmological constant.

Let ξ be a Killing field with a bifurcate Killing horizon, and let Σ be the intersection of the horizon with a constant time slice. Then the entropy associated with Σ is a sum of contributions associated with each curvature term in the Lovelock Lagrangian, given by [25]

$$S = \frac{1}{4G} \sum b_k A^{(k)} , \quad A^{(k)} = k \int_\Sigma da \mathcal{L}^{(k-1)}[\gamma] \tag{10}$$

where γ_{ab} is the induced metric on Σ. We have omitted the boundary term that appears in the definition of S in reference [25] since this vanishes when Σ is generated by a Killing field. The $k = 1$ term is just the area, while the $k = 0$ term vanishes, corresponding to the fact that the cosmological constant term in the Lagrangian does not contribute to the entropy. The first law in Lovelock gravity including variations of the Lovelock parameters [36] is given by

$$\delta E = \frac{\kappa}{2\pi} \delta S - \frac{1}{16\pi G} \left(2V\delta\Lambda + \sum_{k=2} \Psi^{(k)} \delta b_k \right) \tag{11}$$

Here V is the thermodynamic volume, the parameter conjugate to Λ [33], the $\Psi^{(k)}$ are potentials conjugate to the higher curvature couplings b_k with $k \geq 2$, and E is the ADM charge associated with the Killing field. One may wonder about the motivation for varying the Lovelock couplings b_k. There are several reasons why this is useful to do. First, to derive Smarr relations from the first law, the first law must include the variations of all the dimensionful parameters, as was done for Einstein gravity with cosmological constant in [33] and for Lovelock gravity in [36]. Second, terms that appear as constants in a lower dimensional theory may indeed be moduli associated with dynamical fields in a higher dimensional theory, as was discussed in [39] for Λ, and similar remarks apply to the Lovelock couplings. Third, as is well appreciated by condensed matter physicists, understanding the effect of varying a parameter can help elucidate the meaning of other quantities, even if that parameter will not be varied in an experiment, as is illustrated by the distinction between internal energy and enthalpy. In fact, in the current work, we will see that varying the couplings is the right thing to do in order to compute the chemical potential for a dual CFT, or equivalently, the change in the entanglement entropy due to a variation in the "A"-type anomaly coefficient.

We will be working in an asymptotically AdS spacetime, such that the metric in the asymptotic regime is given approximately in Poincare coordinates by

$$ds^2 \approx \frac{l^2}{z^2} \left(-dt^2 + dz^2 + dr^2 + r^2 d\Omega_{D-3}^2 \right) \tag{12}$$

where l is the AdS curvature scale. Consider the Killing field used in [30]

$$\xi = -\frac{2\pi t}{r_0}(z\partial_z + r\partial_r) + \frac{\pi}{r_0}(r_0^2 - z^2 - r^2 - t^2)\partial_t \tag{13}$$

At time $t = 0$, the horizon of ξ is given by the surface $\Sigma : z^2 + r^2 = r_0^2$. The surface Σ intersects the AdS boundary at $z = 0$ in a $(D-3)$-sphere $r^2 = r_0^2$ whose interior is, in turn, a $(D-2)$-dimensional ball B on the boundary. Because Σ is a bifurcation surface for the Killing vector ξ, the first law Equation (11) governs perturbations about AdS. On the other hand, since Σ is a minimal surface one can apply holographic conjecture [24] to relate the area of Σ to the entanglement entropy of the boundary sphere. As emphasized in [30], the possibility of applying a first law type construction is special to the spherical boundary, because of the bifurcate Killing horizon property. The argument proceeds as follows.

First we review relevant features of the first law results for Einstein gravity [30] including the extension to include variation in Λ [32]. In this case, the entropy S in the first law reduces to $A/4G$, where A is the area of Σ and one finds that V_{therm} is also proportional to A, given by

$$V = \frac{2\pi l^2}{D-1} A \tag{14}$$

Note that in these and subsequent formulas A denotes the regularized area, obtained by cutting off the area integral at some small value $z = \epsilon$, since the area receives an infinite contribution from the region near the AdS boundary. This divergence of the area A as $\epsilon \to 0$ corresponds to the divergence of the entanglement entropy in the boundary CFT as a cutoff is removed. The surface gravity for the Killing vector ξ is found to be $\kappa = 2\pi$, and the first law Equation (11) with the higher curvature terms set to zero then reduces to

$$\delta E = \frac{\delta A}{4G} - (D-2)\frac{A}{4G}\frac{\delta l}{l} \tag{15}$$

where the ADM charge E arises as the contribution from the AdS boundary to the Gauss' law relation. In reference [30] this was identified as

$$E_\xi = 2\pi \int_B d^{D-2}x \frac{r_0^2 - r^2}{2r_0} T_{tt}^{boundary}(t=0,\vec{x}) \tag{16}$$

where $T_{\mu\nu}^{boundary}$ is the boundary stress tensor. In Equation (15) the cosmological constant Λ has been rewritten in terms of the AdS curvature scale by means of

$$\Lambda = -\frac{(D-1)(D-2)}{2l^2} \tag{17}$$

For AdS$_5$ the dual CFT is given by $\mathcal{N} = 4$ $SU(N)$ Super-Yang–Mills theory [1], where the AdS curvature scale is related to the number of colors N according to $l^8 \sim N^2$. The first law Equation (15) can then be written in terms of variations in N as

$$\delta E = \delta S - (D-2)\frac{S}{N^2}\delta(N^2) \tag{18}$$

The number of degrees of freedom of the CFT is proportional to N^2, and therefore Equation (18) determines the chemical potential for changing the number of degrees of freedom of the boundary CFT to be

$$\mu = -(D-2)\frac{S}{N^2} \tag{19}$$

Hence, including $\delta\Lambda$ in the first law has allowed us to identify the chemical potential μ_{chem}, which is seen to be proportional to the entanglement entropy per degree of freedom. We note that other work has also included a temperature associated with the variation of E with respect to S [40,41].

We find that a similar result holds for boundary CFTs that are dual to Lovelock gravity in the bulk. The derivation of this result will be given in Section 3 below. Here we will focus on the results. A key feature of the calculation given below is that with the constant curvature form of the Riemann tensor for AdS, each of the terms in first law Equation (11) works out to be proportional to the corresponding term in the Einstein case. Consequently, a sum over the Lovelock coupling constants b_k factors out of the entire equation, giving a simple result in terms of the horizon area. The extended first law including variations in the Lovelock couplings will then take a simple form if we rescale the variation of the energy according to

$$\delta\tilde{E} = \frac{(D-1)(D-2)}{s_{(1)}}\delta E \tag{20}$$

where the quantity $s_{(1)}$ is given by the sum over the Lovelock couplings

$$s_{(1)} = -l^2(D-1)! \sum_{k=0} \frac{(-1)^k}{l^{2k}} \frac{k \, b_k}{(D-2k-1)!} \tag{21}$$

Written in terms of rescaled quantity $\delta \tilde{E}$, the extended first law in Lovelock gravity for perturbations about the minimal surface Σ that intersects the AdS boundary in a sphere is then given by

$$\delta \tilde{E} = \frac{\delta A}{4G} - (D-2)\frac{A}{4G}\frac{\delta l}{l} \tag{22}$$

where the AdS curvature scale l is now related to the Lovelock couplings by

$$\sum_{k=0} \frac{(-1)^k}{l^{2k}} \frac{b_k}{(D-2k-1)!} = 0 \tag{23}$$

Note that this result has the same form as the first law Equation (15) with Einstein gravity in the bulk.

In Einstein gravity, the first law written in terms of the horizon area Equation (15) translates directly into a statement Equation (18) about the entropy and its variation. However, in Lovelock gravity such a restatement requires additional steps. The different horizon integrals contributing the entropy Equation (10) for the surface Σ all work out to be proportional to its area, with the $A^{(k)}$ given by

$$A^{(k)} = \frac{A}{4G} \, k \, e_k \,, \qquad e_k = \left(\frac{-1}{l^2}\right)^{k-1} \frac{(D-2)!}{(D-2k)!} \tag{24}$$

Substituting this in Equation (10) we find that the entropy associated with the minimal surface Σ is given by

$$S = \left(\sum_k k e_k b_k\right) \frac{A}{4G} \tag{25}$$

The entropy can be rewritten in terms of the "A"-type anomaly coefficient a^*. In the four-dimensional CFT $\mathcal{N} = 4$ $SU(N)$ Super-Yang–Mills dual to Einstein gravity without higher curvature terms the central charges a and c are equal. However with higher derivative terms in the gravitational Lagrangians this is no longer the case. Both holographic and direct CFT calculations of the entanglement entropy have found that the several anomaly coefficients of the CFT can be distinguished by studying entangling boundaries with different geometries [14,15,42,43]. For example, in Lovelock gravity it was found that the entanglement entropy of a cylinder is proportional to the "c" coefficient, while that of a sphere is proportional to a^*, where

$$a^* = \frac{l^{D-2}}{4G} \left(\sum_k k e_k b_k\right) \tag{26}$$

The "star" indicates that the coefficient has been extended to include CFTs of odd dimensions. Our normalization differs from that in [15] by the factor $4\pi^{(D-1)/2}/[\Gamma(\frac{D-1}{2})]$. Hence the entropy [14,15] can be written as

$$S = \frac{a^* A}{l^{D-2}} \tag{27}$$

and its variation is given by

$$\delta S = \frac{a^* A}{l^{D-2}} \left(\frac{\delta a^*}{a^*} - (D-2)\frac{\delta l}{l}\right) + \frac{a^*}{l^{D-2}}\delta A \tag{28}$$

The prefactor of the first term above is just the entropy, and the last term can be rewritten in terms of the entropy and the change in energy using the first law Equation (22). One of the terms in the first law cancels the δl term, and so all the variations of the Lovelock couplings combine to form δa^*, giving the result

$$\delta S = S\frac{\delta a^*}{a^*} + \frac{4a^*G}{l^{D-2}}\delta\tilde{E} \tag{29}$$

The anomaly coefficient a^* has the interpretation of the number of degrees of freedom per cell in the regulated field theory [43,44] and so the last term is proportional to the change in the number of degrees of freedom in the CFT, as was the case for Einstein-Λ gravity (compare to Equation (18)). Hence the variation of S with respect to a^* at fixed energy, for a spherical boundary, can be thought of as a generalized chemical potential with value

$$\mu = \left.\frac{\partial S}{\partial a^*}\right|_E = \frac{S}{a^*} \tag{30}$$

for a CFT dual to a Lovelock theory. The chemical potential μ is simply the entanglement entropy per degree of freedom.

Again, we note that the area A of the surface that extends to the boundary of AdS is infinite, and so the entropy in the first law refers to the renormalized quantity. For a given linearized perturbation one or both of the resulting quantities δS and δE must also be renormalized. On the other hand, the logarithmic changes $\delta S/S$ and $\delta a^*/a^*$, as well as the ration $\delta E/S$ can be finite. It would be interesting to also be able to probe the effect of varying the "B"-type contributions to the trace anomaly, which derive from the Weyl, Cotton and Bach tensors rather than the Euler densities. Reference [15] computes the logarithmically divergent contributions to the entanglement entropy for several different types of entangling surfaces, illustrating that all the A and B-type coefficients will contribute in general. However, the bulk first law type construction used here is based on the bulk entangling surface being a Killing horizon, and therefore it is not possible to extend the present results to capture the effect of the "B"-type anomalies. One can speculate that more general surfaces might be assembled as composites of basic surfaces, each defined via symmetries, but so far we have not been successful in accomplishing this.

Explicit Formula and an Example in $D = 5$

The relation Equation (29) which gives δS at fixed energy in terms of δa^* is a nice and compact expression. However, it is also useful to have the equivalent expression in terms of the variations of the Lovelock coefficients and the AdS radius. This is given by

$$\delta S|_E = \frac{A}{4G}\sum_{k=0}^{k_{max}}\left(\frac{-1}{l^2}\right)^{k-1}\frac{k(D-2)!}{(D-2k)!}\left[\delta b_k + (D-2k)b_k\frac{\delta l}{l}\right] \tag{31}$$

As an illustration and a check of our work, in this section we start with the entropy in terms of the Lovelock couplings and translate to the conformal field theory coefficients a and c, calculated by other techniques. In $D = 5$ the only nonzero coupling constants are b_0, $b_1 \equiv 1$, and b_2, and Equation (31) reduces to

$$\delta S|_E = -\frac{3A}{4G}\left[\frac{4\delta b_2}{l^2} + \left(\frac{4b_2}{l^2} - 1\right)\frac{\delta l}{l}\right] \tag{32}$$

From reference [14], the coupling constants b_0 and b_2 are related to the 4D CFT trace anomaly coefficients a and c according to

$$b_0 b_2 = \frac{3(a-5c)(a-c)}{2(a-3c)^2} \tag{33}$$

while from Equation (23) the AdS radius l is determined by the equation

$$l^2 b_0 - 12 + \frac{24}{l^2} b_2 = 0$$

These can be combined to obtain expressions for the couplings in terms of a, c and l

$$b_2 = \frac{l^2(a-c)}{4(a-3c)} \qquad \text{and} \qquad b_0 = \frac{6(a-5c)}{l^2(a-3c)}. \tag{34}$$

Additionally the AdS radius is given [14] in terms of the anomaly coefficients and Newton's constant by $l^3 = G(3c-a)/90\pi$.

Hence the variations of the Gauss-Bonnet coupling b_2 and the AdS radius l can be expressed in terms of the variations of the anomaly coefficients by

$$\frac{\delta b_2}{l^2} = \frac{(a-c)}{2(a-3c)} \frac{\delta l}{l} + \frac{a\delta c - c\delta a}{2(a-3c)^2} , \qquad \frac{\delta l}{l} = \frac{3\delta c - \delta a}{3(3c-a)}$$

Plugging these into the expression Equation (32) for the variation of the entanglement entropy gives

$$\delta S|_E = -\frac{A}{2G} \frac{\delta a}{(a-3c)} \tag{35}$$

Note that the terms proportional to δc have cancelled. The unperturbed entropy S is determined by Equation (25) to be

$$S = \frac{A}{4G} \left(1 - \frac{12b_2}{l^2} \right) \tag{36}$$

$$= -\frac{A}{2G} \frac{a}{(a-3c)} \tag{37}$$

Hence the variation of the entanglement entropy is found to be $\delta S|_E = S\frac{\delta a}{a}$, which is in agreement with the general result Equation (29) above, since in this case $a^* = a$.

3. Details of the Derivation

In this section we give the details of the derivation of the extended first law Equation (22) including variations in the Lovelock couplings for the change in the area of the minimal surface Σ that intersects the AdS boundary in a sphere. The extended first law is valid for small perturbations around AdS, as well as in the far field of an AdS black hole. The derivation makes use of the Hamiltonian perturbation theory formalism. Here only give needed details for the derivation at hand, while a more complete treatment can be found in [36].

In the Hamiltonian framework, we start by decomposing the metric as

$$g_{ab} = -n_a n_b + s_{ab} , \quad n_a n^a = -1 , \quad n_a s^a{}_b = 0 \tag{38}$$

In the asymptotically AdS region the timelike normal is simply $n_a = -(l/z)\nabla_a t$. The Killing field used in the first law construction, given explicitly in Equation (13), is decomposed as

$$\xi^a = Fn^a + \beta^a \tag{39}$$

We will take the background metric to be AdS, and denote the perturbation to the spatial metric as h_{ab},

$$s_{ab} = s_{ab}^{AdS} + h_{ab} \tag{40}$$

There is also a perturbation to the gravitational momentum, but it doesn't enter into this calculation.

The extended first law was derived previously for Lovelock black holes [36], with the result given above in Equation (11). That general result was applied in [45] to spherically symmetric Lovelock black holes and used to derive properties relating the mass, entropy, and temperature. In the spherically symmetric case additional progress can be made due to the additional symmetry assumption about the spacetime. In this current work, one can likewise make additional progress since the bulk horizon Σ lies in the asymptotic AdS region, and so the background Riemann tensor has the special constant curvature form. We next show that the individual Lovelock integrals in the sums all reduce to multiplies of a basic integral, times the appropriate b_k, times a numerical coefficient which is the result of careful calculation. In the end, the relative coefficients from the several coefficients give the simple form in Equation (29).

For the geometry of interest here, it is convenient to backtrack to a more "primitive" version of the first law. This amounts to the following integral identity, which holds for solutions about a background solution to the Lovelock equations of motion

$$\int_{\partial V} da_c \sum_k (b_k B^{(k)c} + \delta b_k \beta^{(k)cd} n_d) = 0. \tag{41}$$

Here the boundary vectors $B^{(k)a}$, which depend on the metric perturbation, are given by

$$B^{(k)a} = \frac{k}{2^{k-1}} \delta^{aba_1b_1...a_{k-1}b_{k-1}}_{cdc_1d_1....c_{k-1}d_{k-1}} R_{a_1b_1}{}^{c_1d_1} \dots R_{a_{k-1}b_{k-1}}{}^{c_{k-1}d_{k-1}} \left(F\nabla^c h^d_b - h^d_b \nabla^c F \right) \tag{42}$$

and the Killing–Lovelock potentials $\beta^{(k)ab}$ [46] corresponding to the Killing vector ξ^a are solutions to

$$-\frac{1}{2} \nabla_b \beta^{(k)ba} = \mathcal{G}^{(k)a}{}_b \xi^b$$

Here $\mathcal{G}^{(k)a}{}_b$ is the kth order Lovelock tensor,

$$\mathcal{G}^{(k)a}{}_b = -\frac{1}{2^{k+1}} \delta^{aa_1b_1...a_kb_k}_{bc_1d_1....c_kd_k} R_{a_1b_1}{}^{c_1d_1} \dots R_{a_kb_k}{}^{c_kd_k}. \tag{43}$$

We apply the identity Equation (41) with the Killing vector Equation (13) to the boundary composed of the spherical ball B of radius r_0 at spatial infinity plus the bulk minimal surface Σ, which is the bifurcate Killing horizon of ξ. Since the background is AdS and the Riemann tensor has the simple constant curvature form, it turns out that the various lengthy expressions indexed by k differ only in the multiplicative pre-factors. Explicitly, using $R^{ab}_{cd} = -(1/l^2)\delta^{ab}_{cd}$, one finds the Lovelock tensors Equation (43) are given by

$$\mathcal{G}^{(k)a}{}_b = -\frac{1}{2} \left(\frac{-1}{l^2} \right)^k \frac{(D-1)!}{(D-2k-1)!} \delta^a_b.$$

and that the Killing–Lovelock potentials can be written in terms of the $k = 0$ Killing potential as

$$\beta^{(k)ab} = \left(\frac{-1}{l^2} \right)^k \frac{(D-1)!}{(D-2k-1)!} \beta^{(0)ab} \tag{44}$$

Further, the Killing potential $\beta^{(0)ab}$ can be obtained simply by combining the Ricci identity $\nabla_a \nabla^b \xi^b = -R^b_a \xi^a$ for the Killing vector along with the Ricci tensor $R_{ab} = -((D-1)/l^2)g_{ab}$ of the AdS background which gives

$$\beta^{(0)ab} = \frac{l^2}{D-1} \nabla^a \xi^b.$$

For the Killing vector Equation (13), this gives

$$\beta^{(k)} = \frac{1}{2} \beta^{(k)ab} \partial_a \wedge \partial_b \tag{45}$$

$$= \left(\frac{-1}{l^2} \right)^k \frac{\pi z (D-2)!}{r_0 (D-2k-1)!} \left\{ (r_0^2 + z^2 - t^2 - r^2)\partial_t \wedge \partial_z + 2tx^k \partial_z \wedge \partial_k + 2zx^k \partial_t \wedge \partial_k \right\} \tag{46}$$

Similarly, the boundary terms $B^{(k)a}$ corresponding to each order k can be expressed in terms of the boundary term corresponding to the Einstein-Hilbert term (*i.e.*, $k = 1$) as

$$B^{(k)a} = k \left(\frac{-1}{l^2} \right)^{k-1} \frac{(D-3)!}{(D-2k-1)!} B^{(1)a} \tag{47}$$

$$B^{(1)a} = F(D^a h - D_b h^{ab}) - h D^a F + h^{ab} D_b F \tag{48}$$

The weighted sum of boundary vectors can be expressed more compactly as

$$\sum_k b_k B^{(k)a} = \frac{s_{(1)}}{(D-1)(D-2)} B^{(1)a} \tag{49}$$

where the sum $s_{(1)}$ was defined in Equation (21).

3.1. Boundary at Infinity

We are now ready to evaluate the integral Equation (41) on the boundary at spatial infinity. We will find that at infinity the terms arising due to variations in the coupling constants b_k are separately divergent but sum to zero, leaving only the ADM energy corresponding to the Killing field ξ^a. This cancellation works essentially in the same way as in the calculations in [32,36]. First, we analyze the boundary vector $B^{(1)a}$ in Equation (48), which depends on h_{ab}. The metric perturbation can be divided into a contribution with l held fixed, and a contribution from a change in l. The second portion is simply $h_{ab} = (2\delta l/l)s_{ab}$. The normal component F of the Killing field on the boundary is $F = (\pi l/r_0 z)(r_0^2 - r^2)$.

$$da_z B^{(1)z} = \left(\frac{l^2}{z^2} r dr d\Omega \right) \left[B^{(1)z} \big|_{\delta l=0} + \frac{2(D-2)\pi \delta l}{r_0 l^2}(r_0^2 - r^2) \right]$$

We now integrate the sum of boundary vectors in Equation (49) on the boundary at infinity using the above expression. The integral of the term corresponding to the perturbation with the AdS length l held fixed gives the variation in the ADM charge associated with the Killing field ξ^a [45], and hence we obtain

$$\int_B da_c \sum_k b_k B^{(k)c} = -16\pi G \delta E_\xi + \delta l \frac{2\pi s_{(1)}}{(D-1)} \frac{K}{z^2} \tag{50}$$

where

$$K = \frac{1}{r_0} \int_B \frac{r dr d\Omega}{z^2}(r_0^2 - r^2) = \Omega_{D-3} \frac{r_0^3}{4} \tag{51}$$

The last term in Equation (50) diverges as $z \to 0$, which is to be expected from the way l enters the metric. However, this will be cancelled by the contribution from the Lovelock-Killing potentials, which we evaluate next. The relevant contribution is from the components $\beta^{(k)tz}$, which from Equation (45) gives

$$\sum_{k=0} \delta b_k \beta^{(k)zt} n_t = \left(\sum_{k=0} \left(\frac{-1}{l^2} \right)^k \frac{(D-2)! \delta b_k}{(D-2k-1)!} \right) \frac{\pi l}{r_0} (r_0^2 - r^2)$$

The sum inside the parenthesis on the right hand side involving δb_k's can be expressed in terms of δl by taking the variation of Equation (23),

$$\sum_{k=0} \left(\frac{-1}{l^2} \right)^k \frac{(D-1)! \delta b_k}{(D-2k-1)!} = -s_{(1)} \frac{2\delta l}{l^3}. \tag{52}$$

Finally, integrating over the boundary at infinity, we get

$$\int_B da_c \sum_{k=0} \delta b_k \beta^{(k)cd} n_d = -\delta l \frac{2\pi s_{(1)}}{(D-1)r_0} \frac{K}{z^2} \tag{53}$$

which precisely cancels the diverging contribution in Equation (50).

3.2. Boundary in the Interior

Let us now evaluate the integral in Equation (41) on the bulk minimal surface Σ in the interior. Since Σ is a bifurcate Killing horizon, the integral of the boundary vector $B^{(1)a}$ over this surface is $-2\kappa \delta A$, where A is the area of the minimal surface and $\kappa = 2\pi$ for the Killing vector ξ^a. Using Equation (49), we then have

$$\int_\Sigma da_c \sum_k b_k B^{(k)c} = -\frac{4\pi s_{(1)} \delta A}{(D-1)(D-2)} \tag{54}$$

There remains the integral of the Killing-potential terms on Σ. Since the Killing potentials of different orders differ only in the multiplicative factors as displayed in Equation (44), each of the integrals is proportional to the thermodynamic volume defined by

$$V_{ther} = -\int_\Sigma da_c \beta^{(0)cb} n_b$$

which, noting that the unit outward normal to Σ is $m = -l(zdz + \vec{x} \cdot d\vec{x})/(zr_0)$, has the value

$$V_{ther} = \frac{2\pi l^2}{D-1} A \tag{55}$$

Using Equation (45), we then obtain the sum of all contributions of the Killing potentials on the bulk minimal surface in terms of the area

$$\int_\Sigma da_c \sum_k \delta b_k \beta^{(k)cd} n_d = \frac{4\pi s_{(1)} A}{D-1} \frac{\delta l}{l} \tag{56}$$

In the last line we have expressed the sum involving the δb_k in terms of δl using Equation (52). Adding the contributions on the boundary Σ, Equations (54) and (56), gives

$$\int_\Sigma da_c \sum_k \left(b_k B^{(k)c} + \delta b_k \beta^{(k)cd} n_d \right) = -\frac{4\pi s_{(1)}}{(D-1)(D-2)} \left[\delta A - (D-2) A \frac{\delta l}{l} \right] \tag{57}$$

Combining the results from the previous subsections we have the following extension of the first law for an entangling surface that intersects the AdS boundary in a sphere,

$$\delta E_\xi = \frac{s_{(1)}}{4G(D-1)(D-2)}\left[\delta A - (D-2)A\frac{\delta l}{l}\right]$$

which completes the derivation for Equation (22) which forms the input in Section 2 leading to our main result Equation (29).

4. Conclusions

To summarize, by making use of standard Hamiltonian perturbation theory methods we have derived an extended form of the first law for entanglement entropy, for spherical entangling surfaces, in CFTs with a bulk Lovelock dual. This extension gives the variation of the holographic entanglement entropy as the bulk Lovelock coupling constants are varied, corresponding to variation within a family of boundary CFTs. We have shown that variations of the bulk Lovelock couplings impact the entanglement entropy through their contributions to the variation of the "A"-type trace anomaly coefficient a of the boundary CFT, or its generalization a^* for odd dimensional boundaries. At constant energy, we find that the logarithmic change in S is equal to the logarithmic change in a^*. Given that a^* is a measure of the number of degrees of freedom of the CFT, we can regard the quantity S/a^* as a chemical potential for increasing the number of degrees of freedom within a family of boundary CFTs.

One natural question is whether the variation in holographic entanglement entropy with respect to variations in the bulk Lovelock couplings is linked to the variation in the trace anomaly coefficients flow for more generally shaped regions. The restriction to spherical boundary regions arises because the construction is based on the background AdS isometry Equation (13), but progress could be made by studying small deformations of spheres. A second, more speculative question is whether there might be connection between a bulk second law associated with entangling surfaces and renormalization group flow between UV and IR CFTs. A great deal of work has been done on the issue of a higher dimensional version of the c-theorem [37,43,44,47–55]. References [43,44] showed that with an energy condition, $(a^*)_{UV} \geq (a^*)_{IR}$, and so based on Equation (7) one can speculate that an entropy increase property is connected to the anomaly flow. In this context, interesting work on entropy increase has appeared in [55–57].

Author Contributions: The authors all contributed to each stage of this research work; including the initial formulation of the project, the calculations and analysis, and the composition and editing of the paper. All of the authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Maldacena, J. *The Large N Limit of Superconformal Field Theories and Supergravity*; AIP Publishing: Melville, NY, USA, 1999.
2. Henningson, M.; Skenderis, K. The Holographic Weyl anomaly. *J. High Energy Phys.* **1998**, *1998*, doi:10.1088/1126-6708/1998/07/023.
3. Lovelock, D. The Einstein tensor and its generalizations. *J. Math. Phys.* **1971**, *12*, doi:10.1063/1.1665613.
4. Brigante, M.; Liu, H.; Myers, R.; Shenker, S.; Yaida, S. Viscosity Bound Violation in Higher Derivative Gravity. *Phys. Rev. D* **2008**, *77*, doi:10.1103/PhysRevD.77.126006.
5. Brigante, M.; Liu, H.; Myers, R.; Shenker, S.; Yaida, S. Viscosity Bound and Causality Violation. *Phys. Rev. Lett.* **2008**, *100*, doi:10.1103/PhysRevLett.100.191601.

6. Buchel, A.; Myers, R. Causality of Holographic Hydrodynamics. *J. High Energy Phys.* **2009**, *8*, doi:10.1088/1126-6708/2009/08/016.

7. Hofman, D. Higher Derivative Gravity, Causality and Positivity of Energy in a UV complete QFT. *Nucl. Phys. B* **2009**, *823*, 174–194.

8. De Boer, J.; Kulaxizi, M.; Parnachev, A. AdS$_7$/CFT$_6$, Gauss-Bonnet Gravity, and Viscosity Bound. *J. High Energy Phys.* **2010**, *2010*, doi:10.1007/JHEP03(2010)087.

9. Camanho, X.; Edelstein, J. Causality constraints in AdS/CFT from conformal collider physics and Gauss-Bonnet gravity. *J. High Energy Phys.* **2010**, *2010*, doi:10.1007/JHEP04(2010)007.

10. Buchel, A.; Escobedo, J.; Myers, R.; Paulos, M.; Sinha, A.; Smolkin, M. Holographic GB gravity in arbitrary dimensions. *J. High Energy Phys.* **2010**, *2010*, doi:10.1007/JHEP03(2010)111.

11. De Boer, J.; Kulaxizi, M.; Parnachev, A. Holographic Lovelock Gravities and Black Holes. *J. High Energy Phys.* **2010**, *2010*, doi:10.1007/JHEP06(2010)008.

12. Camanho, X.; Edelstein, J. Causality in AdS/CFT and Lovelock theory. *J. High Energy Phys.* **2010**, *2010*, doi:10.1007/JHEP06(2010)099.

13. Camanho, X.; Edelstein, J.; de Santos, J. Lovelock theory and the AdS/CFT correspondence. *Gen. Rel. Grav.* **2014**, *46*, 1–27.

14. De Boer, J.; Kulaxizi, M.; Parnachev, A. Holographic Entanglement Entropy in Lovelock Gravities. *J. High Energy Phys.* **2011**, *2011*, doi:10.1007/JHEP07(2011)109.

15. Hung, L.; Myers, R.; Smolkin, M. On Holographic Entanglement Entropy and Higher Curvature Gravity. *J. High Energy Phys.* **2011**, *2011*, doi:10.1007/JHEP04(2011)025.

16. Ogawa, N.; Takayanagi, T. Higher Derivative Corrections to Holographic Entanglement Entropy for AdS Solitons. *J. High Energy Phys.* **2011**, *2011*, doi:10.1007/JHEP10(2011)147.

17. Myers, R.; Singh, A. Comments on Holographic Entanglement Entropy and RG Flows. *J. High Energy Phys.* **2012**, *2012*, doi:10.1007/JHEP04(2012)122.

18. Bhattacharyya, A.; Kaviraj, A.; Sinha, A. Entanglement entropy in higher derivative holography. *J. High Energy Phys.* **2013**, *2013*, doi:10.1007/JHEP08(2013)012.

19. Chen, B.; Zhang, J. Note on generalized gravitational entropy in Lovelock gravity. *J. High Energy Phys.* **2013**, *2013*, doi:10.1007/JHEP07(2013)185.

20. Dong, X. Holographic Entanglement Entropy for General Higher Derivative Gravity. *J. High Energy Phys.* **2014**, *2014*, doi:10.1007/JHEP01(2014)044.

21. Camps, J. Generalized entropy and higher derivative Gravity. *J. High Energy Phys.* **2014**, *2014*, doi:10.1007/JHEP03(2014)070.

22. Pal, S. Extremal Surfaces And Entanglement Entropy. *Nucl. Phys. B* **2014**, *882*, 352–385.

23. Banerjee, S.; Bhattacharyya, A.; Kaviraj, A.; Sen, K.; Sinha, A. Constraining gravity using entanglement in AdS/CFT. *J. High Energy Phys.* **2014**, *2014*, doi:10.1007/JHEP05(2014)029.

24. Ryu, S.; Takayanagi, T. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.* **2006**, *96*, doi:10.1103/PhysRevLett.96.181602.

25. Jacobson, T.; Myers, R. Black Hole Entropy And Higher Curvature Interactions. *Phys. Rev. Lett.* **1993**, *70*, doi:10.1103/PhysRevLett.70.3684.

26. Wald, R.M. Black hole entropy is the Noether charge. *Phys. Rev. D* **1993**, *48*, doi:10.1103/PhysRevD.48.R3427.

27. Iyer, V.; Wald, R.M. Some properties of Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D* **1994**, *50*, doi:10.1103/PhysRevD.50.846.

28. Blanco, D.; Casini, H.; Hung, L.; Myers, R. Relative Entropy and Holography. *J. High Energy Phys.* **2013**, *2013*, doi:10.1007/JHEP08(2013)060.

29. Wong, G.; Klich, I.; Zayas, L.; Vaman, D. Entanglement Temperature and Entanglement Entropy of Excited States. *J. High Energy Phys.* **2013**, *2013*, doi:10.1007/JHEP12(2013)020.

30. Faulkner, T.; Guica, M.; Hartman, T.; Myers, R.; van Raamsdonk, M. Gravitation from Entanglement in Holographic CFTs. *J. High Energy Phys.* **2014**, *2014*, doi:10.1007/JHEP03(2014)051.

31. Sudarsky, D.; Wald, R.M. Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang–Mills equations. *Phys. Rev. D* **1992**, *46*, 1453–1474.

32. Kastor, D.; Ray, S.; Traschen, J. Chemical Potential in the First Law for Holographic Entanglement Entropy. *J. High Energy Phys.* **2014**, *2014*, doi:10.1007/JHEP11(2014)120.

33. Kastor, D.; Ray, S.; Traschen, J. Enthalpy and the Mechanics of AdS Black Holes. *Class. Quant. Grav.* **2009**, *26*, doi:10.1088/0264-9381/26/19/195011.

34. Kubiznak, D.; Mann, R. Black hole chemistry. *Can. J. Phys.* **2015**, *93*, doi:10.1139/cjp-2014-0465.

35. Brown, J.; Henneaux, M. Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.* **1986**, *104*, doi:10.1007/BF01211590.

36. Kastor, D.; Ray, S.; Traschen, J. Smarr Formula and an Extended First Law for Lovelock Gravity. *Class. Quant. Grav.* **2010**, *27*, doi:10.1088/0264-9381/27/23/235014.

37. Liu, J.T.; Sabra, W.; Zhao, Z. Holographic *c*-theorems and higher derivative gravity. *Phys. Rev. D* **2012**, *85*, doi:10.1103/PhysRevD.85.126004.

38. Liu, H.; Mezei, M. A Refinement of entanglement entropy and the number of degrees of freedom. *J. High Energy Phys.* **2013**, *2013*, doi:10.1007/JHEP04(2013)162.

39. Cvetic, M.; Gibbons, G.; Kubiznak, D.; Pope, C. Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume. *Phys. Rev. D* **2011**, *84*, doi:10.1103/PhysRevD.84.024037.

40. Bhattacharya, J.; Nozaki, M.; Takayanagi, T.; Ugajin, T. Thermodynamical Property of Entanglement Entropy for Excited States. *Phys. Rev. Lett.* **2013**, *110*, doi:10.1103/PhysRevLett.110.091602.

41. Guo, W.; He, S.; Tao, J. Note on Entanglement Temperature for Low Thermal Excited States in Higher Derivative Gravity. *J. High Energy Phys.* **2013**, *2013*, doi:10.1007/JHEP08(2013)050.

42. Solodukhin, S. Entanglement entropy, conformal invariance and extrinsic geometry. *Phys. Lett. B* **2008**, *665*, 305–309.

43. Myers, R.; Sinha, A. Holographic c-theorems in arbitrary dimensions. *J. High Energy Phys.* **2011**, *2011*, doi:10.1007/JHEP01(2011)125.

44. Myers, R.; Sinha, A. Seeing a c-theorem with holography. *Phys. Rev. D* **2010**, *82*, doi:10.1103/PhysRevD.82.046006.

45. Kastor, D.; Ray, S.; Traschen, J. Mass and Free Energy of Lovelock Black Holes. *Class. Quant. Grav.* **2011**, *28*, doi:10.1088/0264-9381/28/19/195022.

46. Kastor, D. Komar Integrals in Higher (and Lower) Derivative Gravity. *Class. Quant. Grav.* **2008**, *25*, doi:10.1088/0264-9381/25/17/175007.

47. Zamolodchikov, A.B. Irreversibility of the flux of the renormalization group in a 2D field theory. *JETP Lett.* **1986**, *43*, 730–732.

48. Freedman, D.; Gubser, S.; Pilch, K.; Warner, N. Renormalization group flows from holography supersymmetry and a c theorem. *Adv. Theor. Math. Phys.* **1999**, *3*, 363–417.

49. Solodukhin, S. Entanglement entropy and the Ricci flow. *Phys. Lett. B* **2007**, *646*, doi:10.1016/j.physletb.2007.01.031.

50. Liu, J.T.; Zhao, Z. A holographic c-theorem for higher derivative gravity. **2011**, arXiv:1108.5179.

51. Casini, H.; Huerta, M. On the RG running of the entanglement entropy of a circle. *Phys. Rev. D* **2012**, *85*, doi:10.1103/PhysRevD.85.125016.

52. Komargodski, Z. The Constraints of Conformal Symmetry on RG Flows. *J. High Energy Phys.* **2012**, *2012*, doi:10.1007/JHEP07(2012)069.

53. Komargodski, Z.; Schwimmer, A. On Renormalization Group Flows in Four Dimensions. *J. High Energy Phys.* **2011**, *2011*, doi:10.1007/JHEP12(2011)099.

54. Solodukhin, S. The a-theorem and entanglement entropy. **2013**, arXiv:1304.4411.

55. Bhattacharjee, S.; Bhattacharyya, A.; Sarkar, S.; Sinha, A. Entropy functionals and c-theorems from the second law. **2015**, arXiv:1508.01658.

56. Bhattacharjee, S.; Sarkar, S.; Wall, A. Holographic entropy increases in quadratic curvature gravity. *Phys. Rev. D* **2015**, *92*, doi:10.1103/PhysRevD.92.064006.

57. He, S.; Sun, J.; Zhang, H. On Holographic Entanglement Entropy with Second Order Excitations. **2014**, arXiv:1411.6213.